Communications for Statistical Applications and Methods
2013, Vol. 20, No. 4, 329–337

DOI: http://dx.doi.org/10.5351/CSAM.2013.20.4.329

An Improvement of the James-Stein Estimator with Some Shrinkage Points using the Stein Variance Estimator

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Abstract

Consider a p-variate($p \geq 3$) normal distribution with mean $\boldsymbol{\theta}$ and covariance matrix $\Sigma = \sigma^2 \boldsymbol{I}_p$ for any unknown scalar σ^2. In this paper we improve the James-Stein estimator of $\boldsymbol{\theta}$ in cases of shrinking toward some vectors using the Stein variance estimator. It is also shown that this domination does not hold for the positive part versions of these estimators.

Keywords: Shrinkage points, James-Stein estimator, Stein variance estimator, domination.

1. Introduction

There has been considerable interest in the problem of simultaneous estimation of the means of three or more independent normal populations with common unknown variance. The "usual" estimator in this problem is the maximum likelihood estimator, however, as demonstrated in James and Stein (1961) and Lindley (1962), these estimators are dominated by a shrinkage estimator that incorporates the minimum risk equivariant estimator of the common variance. Baranchik (1970), Strawderman (1973), Berry (1994) and Maruyama (1996) have demonstrated (amongst others) that there is a large class of shrinkage estimators that dominate the usual estimator. Much work in this area is concerned with the domination of the usual estimator even though this estimator is known to be dominated, e.g., by the traditional James-Stein estimator with some shrinkage points.

In this same context, Stein (1964) provides a variance estimator that dominates the minimum risk equivariant variance estimator. This improved variance estimator uses information contained in the usual estimator of the mean vector to provide improvement. George (1990) suggested that it might be possible to use the improved variance estimator to improve the James-Stein estimator with some shrinkage points. Kim *et al.* (1995) investigated the behavior of risks of Stein-type estimators that shrink the usual estimator toward the mean of observations. Maruyama (1996) considered a class of generalized Bayes estimators dominating the James-Stein estimator. Baek and Han (2004) produced a sequence of smooth estimators dominating the Lindley-type estimator, Park and Baek (2011) considered the generalized Bayes estimators dominating the Lindley-type estimator.

This paper improves the James-Stein estimator with some shrinkage points. In Section 2 the James-Stein estimator shrinking toward $\boldsymbol{\mu}$ is improved and the Lindley-type estimator is improved in Section 3. It is shown that these dominations do not hold comparing the positive part versions of these estimators and some concluding remarks are given in Section 4.

This work was supported by the Wonkwang University Research Fund 2012.
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2. Improving the James-Stein Estimator in Case of Shrinking toward $\boldsymbol{\mu}$

This section will demonstrate that the estimator that incorporates the improved variance estimator dominates the traditional James-Stein estimator shrinking toward $\boldsymbol{\mu}$ where $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$ is a constant vector.

Let $\boldsymbol{X}$ and S be independently distributed with $\boldsymbol{X} \sim N_p(\boldsymbol{\theta}, \sigma^2 \boldsymbol{I})$ and $S/\sigma^2 \sim \chi^2_n$, where $p \geq 3$. Let $T = ||\boldsymbol{X} - \boldsymbol{\mu}||^2$ and let $F = T/S$. Then $T/\sigma^2 \sim \chi^2(\lambda)$ independently of S, where $\chi^2(\lambda)$ denotes a noncentral chi-square distribution with noncentrality parameter $\lambda = ||\boldsymbol{\theta} - \boldsymbol{\mu}||^2/\sigma^2$. Our interest is to estimate $\boldsymbol{\theta}$ with respect to the scaled squared error loss function $L((\boldsymbol{\theta}, \sigma^2), \delta) = (||\delta - \boldsymbol{\theta}||^2)/\sigma^2$. Consider estimators of the form

$$\delta(\boldsymbol{X}) = \boldsymbol{\mu} + g(F)(\boldsymbol{X} - \boldsymbol{\mu}) = \boldsymbol{\mu} + \left(1 - \frac{r(F)}{F}\right)(\boldsymbol{X} - \boldsymbol{\mu}).$$

The application of Baranchik's method (1970) shows that the estimators of this form are minimax, *i.e.*, they dominate $\boldsymbol{X}$, provided $r(F)$ is monotone, nondecreasing, and $0 \leq r(F) \leq 2(p-2)/(n+2)$. The risk function of $\delta(\boldsymbol{X})$ depends on $(\boldsymbol{\theta}, \sigma^2)$ only through $\lambda = ||\boldsymbol{\theta} - \boldsymbol{\mu}||^2/\sigma^2$, hence, without loss of generality let $\sigma^2 = 1$.

Consider the estimators $\delta_1(\boldsymbol{X}) = \boldsymbol{\mu} + g_1(F)(\boldsymbol{X} - \boldsymbol{\mu})$ and $\delta_2(\boldsymbol{X}) = \boldsymbol{\mu} + g_2(F)(\boldsymbol{X} - \boldsymbol{\mu})$ determined by

$$r_1(F) = \frac{p-2}{n+2}$$

and

$$r_2(F) = \begin{cases} \dfrac{p-2}{n+2}, & \text{if } F \geq \dfrac{p}{n+2}, \\ \dfrac{p-2}{n+p+2}(1+F), & \text{if } F < \dfrac{p}{n+2}. \end{cases}$$

Both of these estimators satisfy Baranchik's conditions and dominate $\boldsymbol{X}$. Notice that $\delta_1(\boldsymbol{X})$ is the James-Stein estimator with shrinking toward $\boldsymbol{\mu}$, where σ^2 is estimated by the minimum risk equivariant estimator $S/(n+2)$. The other estimator, $\delta_2(\boldsymbol{X})$, is obtained from $\delta_1(\boldsymbol{X})$ by replacing $S/(n+2)$ by the improved variance estimator of Stein (1964).

Let the difference in risk for these estimators

$$\Delta = R(\lambda, \delta_1(\boldsymbol{X})) - R(\lambda, \delta_2(\boldsymbol{X})).$$

Then

$$\Delta = E\left[2(\boldsymbol{X} - \boldsymbol{\theta})'\left(\frac{r_2(F) - r_1(F)}{F}\right)(\boldsymbol{X} - \boldsymbol{\mu})\right] + E\left[\left(\frac{r_1^2(F) - r_2^2(F)}{F^2}\right)||\boldsymbol{X} - \boldsymbol{\mu}||^2\right].$$

Subject to mild conditions on the function h, it is shown in Stein (1981) that

$$E\left[(\boldsymbol{X} - \boldsymbol{\theta})' h(\boldsymbol{X})\right] = E\left[\nabla \bullet h(\boldsymbol{X})\right]. \tag{2.1}$$

Let $\mathbf{1}[\,\bullet\,]$ denote the indicator function for the expression in square brackets. In the current context

$$\begin{aligned} h(\boldsymbol{X}) &= \left(\frac{r_2(F) - r_1(F)}{F}\right)(\boldsymbol{X} - \boldsymbol{\mu}) \\ &= \left(\frac{p-2}{n+p+2}\right)\left(1 - \frac{p}{n+2}\frac{S}{||\boldsymbol{X} - \boldsymbol{\mu}||^2}\right)(\boldsymbol{X} - \boldsymbol{\mu})\, \mathbf{1}\left[F < \frac{p}{n+2}\right]. \end{aligned}$$

Then using the similar method of Berry (1994)

$$\nabla \bullet h(\boldsymbol{X}) = \sum_{i=1}^{p} \frac{\partial}{\partial x_i} h_i(\boldsymbol{x}) = \left(\frac{p(p-2)}{n+p+2}\right)\left\{1 - \frac{p-2}{n+2}\left(\frac{1}{F}\right)\right\} \mathbf{1}\left[F < \frac{p}{n+2}\right]$$

for fixed S.

Applying this expectation identity conditionally on S yields

$$\begin{aligned}\Delta = E&\left[\left(\frac{2p(p-2)}{n+p+2}\right)\left\{1 - \frac{p-2}{n+2}\left(\frac{S}{T}\right)\right\} \mathbf{1}\left[\frac{T}{S} < \frac{p}{n+2}\right]\right] \\ &+ E\left[\frac{(p-2)^2}{(n+p+2)^2}\left\{\frac{p(p+2n+4)}{(n+2)^2}\left(\frac{S^2}{T}\right) - 2S - T\right\} \mathbf{1}\left[\frac{T}{S} < \frac{p}{n+2}\right]\right].\end{aligned}$$

Let $K \sim \text{Poisson}(\lambda/2)$ independently of S such that the conditional distribution of T, given S and K, is central chi-square with $p + 2K$ degrees of freedom, then $\Delta = E[E[\Delta_k]]$, where Δ_k is the risk difference conditional on $K = k$. If $Y \sim \chi^2_\nu$, then $E[Yh(Y)] = \nu E[h(Y^*)]$, where $Y^* \sim \chi^2_{\nu+2}$. Applying this chi-square identity to each of S and T, while conditioning on the other variable yields

$$\begin{aligned}\left(\frac{(n+p+2)^2}{(p-2)^2}\right)\Delta_k = &\left(\frac{2p(n+p+2)}{p-2}\right)E\left[\mathbf{1}\left[\frac{T}{S} < \frac{p}{n+2}\right]\right] - \left(\frac{2p(n+p+2)}{n+2}\right)E\left[\left(\frac{S}{T}\right)\mathbf{1}\left[\frac{T}{S} < \frac{p}{n+2}\right]\right] \\ &+ \left(\frac{np(p+2n+4)}{(n+2)^2}\right)E\left[\left(\frac{S^*}{T}\right)\mathbf{1}\left[\frac{T}{S*} < \frac{p}{n+2}\right]\right] \\ &- 2nE\left[\mathbf{1}\left[\frac{T}{S*} < \frac{p}{n+2}\right]\right] - (p+2k)E\left[\mathbf{1}\left[\frac{T^*}{S} < \frac{p}{n+2}\right]\right],\end{aligned}$$

where $S^* \sim \chi^2_{(n+2)}$ and $T^* \sim \chi^2_{(p+2k+2)}$.

Let $G^{\nu_1}_{\nu_2}$ denote the distribution of a ratio of two independent chi-square random variables with ν_1 and ν_2 degrees of freedom, respectively. The corresponding cumulative distribution function is given by

$$P\left[G^{\nu_1}_{\nu_2} \le c\right] = I_{\frac{c}{(1+c)}}\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right), \tag{2.2}$$

where $I_r(a,b)$ denotes the incomplete beta ratio function, *i.e.*, the probability that a beta random variable with parameters a and b does not exceed r. If $Y \sim G^{\nu_1}_{\nu_2}$, then

$$E\left[\frac{h(Y)}{Y}\right] = \left(\frac{\nu_2}{\nu_1 - 2}\right)E\left[h(Y^*)\right], \tag{2.3}$$

where $Y^* \sim G^{\nu_1-2}_{\nu_2+2}$. Two applications of this expectation identity yield

$$\begin{aligned}\left(\frac{(n+p+2)^2}{(p-2)^2}\right)\Delta_k = &\left(\frac{2p(n+p+2)}{p-2}\right)I(a,b) - \left(\frac{2np(n+p+2)}{(n+2)(p+2k-2)}\right)I(a-1,b+1) \\ &+ \left(\frac{np(p+2n+4)}{(n+2)(p+2k-2)}\right)I(a-1,b+2) - 2nI(a,b+1) - (p+2k)I(a+1,b),\end{aligned}$$

where

$$I(a,b) = I_r(a,b), \quad \text{for } r = \frac{p}{n+p+2},\ a = \frac{p+2k}{2},\ b = \frac{n}{2}.$$

Application of the incomplete beta ratio function identities (Abramowitz and Stegun, 1964)

$$(a+b)I(a,b) = aI(a+1,b) + bI(a,b+1) \tag{2.4}$$

and

$$I(a,b) = rI(a-1,b) + (1-r)I(a,b-1) \tag{2.5}$$

along with some algebra yields

$$\begin{aligned}&\left(\frac{(n+p+2)^2(n+2)(p+2k-2)}{2(p-2)}\right)\Delta_k\\ &= \left[n(n+p+2)(k-1)(p-2)\right] I(a,b+1) - \Big[2k^2(p-2)(n+2)\\ &\quad +k\left\{n^2(p-6) + n(p^2-10p-8) + 8 - 20p\right\} - (p-2)(n+p)(3n+p+6)\Big] I(a,b).\end{aligned}$$

Notice that the coefficient of $I(a,b+1)$ in the preceding expression is negative for $k = 0$, zero for $k = 1$, and positive for $k \geq 2$. It can be verified that

$$\left(\frac{(p+2k+n)(n+2)}{n(n+p+2)}\right) I(a,b) < I(a,b+1) < \left(\frac{p+2k+n}{n}\right) I(a,b).$$

Insertion of the lower bound, for $k \geq 1$, yields

$$\begin{aligned}&\left(\frac{(n+p+2)^2(n+2)(p+2k-2)}{2(p-2)}\right)\Delta_k\\ &\geq \left[k\left(4n^2 + 8np + 8n + 12p + 2p^2\right) + (2n+p+4)(p-2)(n+p)\right] I(a,b) > 0,\end{aligned}$$

for all $p \geq 3$ and $n \geq 1$. Insertion of the upper bound, when $k = 0$, yields

$$\left(\frac{(n+p+2)^2(n+2)(p-2)}{2(p-2)}\right)\Delta_0 > 2(p-2)(n+p)(n+2)I(a,b) > 0,$$

for all $p \geq 3$ and $n \geq 1$. Hence, $\Delta_k > 0$ for all $k = 0, 1, \ldots,$ for all $p \geq 3$ and for all $n \geq 1$. Hence $\Delta > 0$ for all $p \geq 3$ and for all $n \geq 1$, since $\Delta = E[E[\Delta_k]]$. Therefore, we can obtain the main theorem of this section from above results.

Theorem 1. *$\delta_2(\boldsymbol{X})$ dominates $\delta_1(\boldsymbol{X})$ for all $p \geq 3$ and $n \geq 1$.*

In case $\boldsymbol{\mu} = \mathbf{0}$, the result of Theorem 1 coincides with that of Berry (1994).

3. Improving the Lindley-Type Estimator

In this section, it is considered the shrinkage estimator of the form that shrinks $\boldsymbol{X}$ toward $\bar{X}\mathbf{1}$, where $\bar{X} = (1/p)\sum_{i=1}^{p} X_i$ and $\mathbf{1}$ is the column vector of 1's.

Let $\boldsymbol{X}$ and S be independently distributed with $\boldsymbol{X} \sim N_p(\boldsymbol{\theta}, \sigma^2 \boldsymbol{I})$ and $S/\sigma^2 \sim \chi^2_n$, where $p \geq 4$. Let $T = \|\boldsymbol{X} - \bar{X}\boldsymbol{1}\|^2$ and let $F = T/S$. Then $T/\sigma^2 \sim \chi^2(\lambda)$ independently of S, where $\chi^2(\lambda)$ denotes a noncentral chi-square distribution with noncentrality parameter $\lambda = \|\boldsymbol{\theta} - \bar{\theta}\boldsymbol{1}\|^2/\sigma^2$ with $\bar{\theta} = (1/p)\sum_{i=1}^{p} \theta_i$. Our interest is to estimate $\boldsymbol{\theta}$ with respect to the scaled squared error loss function $L((\boldsymbol{\theta}, \sigma^2), \delta) = (\|\delta - \boldsymbol{\theta}\|^2)/\sigma^2$. Now consider the estimators of the form

$$\delta^*(\boldsymbol{X}) = \bar{X}\boldsymbol{1} + g(F)\left(\boldsymbol{X} - \bar{X}\boldsymbol{1}\right) = \bar{X}\boldsymbol{1} + \left(1 - \frac{r^*(F)}{F}\right)\left(\boldsymbol{X} - \bar{X}\boldsymbol{1}\right). \tag{3.1}$$

The application of Baranchik's method (1970) shows that estimators of this form are minimax, *i.e.*, they dominate $\boldsymbol{X}$, provided $r(F)$ is monotone, nondecreasing, and $0 \leq r(F) \leq 2(p-3)/(n+2)$. The risk function of $\delta(\boldsymbol{X})$ depends on $(\boldsymbol{\theta}, \sigma^2)$ only through $\lambda = \|\boldsymbol{\theta} - \bar{\theta}\boldsymbol{1}\|^2/\sigma^2$, hence, without loss of generality let $\sigma^2 = 1$.

Consider the estimators

$$\delta_1^*(\boldsymbol{X}) = \bar{X}\boldsymbol{1} + g_1(F)\left(\boldsymbol{X} - \bar{X}\boldsymbol{1}\right) = \bar{X}\boldsymbol{1} + \left(\frac{1 - r_1^*(F)}{F}\right)\left(\boldsymbol{X} - \bar{X}\boldsymbol{1}\right)$$

and

$$\delta_2^*(\boldsymbol{X}) = \bar{X}\boldsymbol{1} + g_2(F)\left(\boldsymbol{X} - \bar{X}\boldsymbol{1}\right) = \bar{X}\boldsymbol{1} + \left(\frac{1 - r_2^*(F)}{F}\right)\left(\boldsymbol{X} - \bar{X}\boldsymbol{1}\right)$$

determined by

$$r_1^*(F) = \frac{p-3}{n+2}$$

and

$$r_2^*(F) = \begin{cases} \dfrac{p-3}{n+2}, & \text{if } F \geq \dfrac{p-1}{n+2}, \\ \dfrac{p-3}{n+p+1}(1+F), & \text{if } F < \dfrac{p-1}{n+2}. \end{cases}$$

Notice that $\delta_1^*(\boldsymbol{X})$ is the Lindley-type estimator, where σ^2 is estimated by the minimum risk equivariant estimator $S/(n+2)$. The other estimator, $\delta_2^*(\boldsymbol{X})$ is obtained from $\delta_1^*(\boldsymbol{X})$ by replacing $S/(n+2)$ by the improved variance estimator of Stein (1964).

The difference in risk for these estimators is given by

$$\Delta = E\left[2(\boldsymbol{X} - \boldsymbol{\theta})'\left(\frac{r_2^*(F) - r_1^*(F)}{F}\right)\left(\boldsymbol{X} - \bar{X}\boldsymbol{1}\right)\right] + E\left[\left(\frac{r_1^{*2}(F) - r_2^{*2}(F)}{F^2}\right)\left\|\boldsymbol{X} - \bar{X}\boldsymbol{1}\right\|^2\right]$$

and in the expectation identity (2.1)

$$\begin{aligned} h(\boldsymbol{X}) &= \left(\frac{r_2^*(F) - r_1^*(F)}{F}\right)\left(\boldsymbol{X} - \bar{X}\boldsymbol{1}\right) \\ &= \left(\frac{p-3}{n+p+1}\right)\left(1 - \frac{p-1}{n+2}\frac{S}{\left\|\boldsymbol{X} - \bar{X}\boldsymbol{1}\right\|^2}\right)\left(\boldsymbol{X} - \bar{X}\boldsymbol{1}\right)\boldsymbol{1}\left[F < \frac{p-1}{n+2}\right]. \end{aligned}$$

Hence it can be proved the following Lemma.

Lemma 1. *For fixed S,*

$$\nabla \bullet h(\boldsymbol{X}) = \sum_{i=1}^{p} \frac{\partial}{\partial x_i} h_i(\boldsymbol{x}) = \frac{(p-1)(p-3)}{n+p+1} \left\{1 - \frac{p-3}{n+2}\left(\frac{1}{F}\right)\right\} \mathbf{1}\left[F < \frac{p-1}{n+2}\right].$$

Proof: Since

$$\frac{\partial}{\partial x_i} h_i(\boldsymbol{x}) = \left(\frac{p-3}{n+p+1}\right)\left\{\left(1-\frac{1}{p}\right) - \frac{p-1}{n+2}\left(\frac{(1-1/p)\,\|\boldsymbol{x}-\bar{x}\mathbf{1}\|^2 - 2(x_i-\bar{x})^2}{\|\boldsymbol{x}-\bar{x}\mathbf{1}\|^4}\right) S\right\} \mathbf{1}\left[F < \frac{p-1}{n+2}\right],$$

$$\begin{aligned}
\nabla \bullet h(\boldsymbol{X}) &= \sum_{i=1}^{p} \frac{\partial}{\partial x_i} h_i(\boldsymbol{x}) \\
&= \frac{p-3}{n+p+1}\left\{p\left(1-\frac{1}{p}\right) - \frac{p-1}{n+2}\left(\left(1-\frac{1}{p}\right)\frac{pS}{\left\|\boldsymbol{X}-\bar{X}\mathbf{1}\right\|^2} - \frac{2S}{\left\|\boldsymbol{X}-\bar{X}\mathbf{1}\right\|^2}\right)\right\} \mathbf{1}\left[F < \frac{p-1}{n+2}\right] \\
&= \frac{(p-1)(p-3)}{n+p+1}\left\{1 - \frac{p-3}{n+2}\left(\frac{1}{F}\right)\right\} \mathbf{1}\left[F < \frac{p-1}{n+2}\right].
\end{aligned}$$

Applying this expectation identity (2.1) conditionally on S yields

$$\begin{aligned}
\Delta &= E\left[\left(\frac{2(p-1)(p-3)}{n+p+1}\right)\left\{1 - \frac{p-3}{n+2}\left(\frac{1}{F}\right)\right\} \mathbf{1}\left[F < \frac{p-1}{n+2}\right]\right] \\
&\quad + E\left[\frac{(p-3)^2}{(n+p+1)^2}\left\{\frac{(p-1)(p+2n+3)}{(n+2)^2}\left(\frac{1}{F^2}\right) - \frac{2}{F} - 1\right\}\left\|\boldsymbol{X}-\bar{X}\mathbf{1}\right\|^2 \mathbf{1}\left[F < \frac{p-1}{n+2}\right]\right] \\
&= E\left[\left(\frac{2(p-1)(p-3)}{n+p+1}\right)\left\{1 - \frac{p-3}{n+2}\left(\frac{S}{T}\right)\right\} \mathbf{1}\left[\frac{T}{S} < \frac{p-1}{n+2}\right]\right] \\
&\quad + E\left[\frac{(p-3)^2}{(n+p+1)^2}\left\{\frac{(p-1)(p+2n+3)}{(n+2)^2}\left(\frac{S^2}{T}\right) - 2S - T\right\} \mathbf{1}\left[\frac{T}{S} < \frac{p-1}{n+2}\right]\right].
\end{aligned}$$

Let $K \sim \text{Poisson}(\lambda/2)$ independently of S such that the conditional distribution of T, given S and K, is central chi-square with $p+2K-1$ degrees of freedom, then $\Delta = E[E[\Delta_k]]$, where Δ_k is the risk difference conditional on $K = k$. If $Y \sim \chi^2_\nu$, then $E[Yh(Y)] = \nu E[h(Y^*)]$, where $Y^* \sim \chi^2_{\nu+2}$. The application of this chi-square identity to each of S and T while conditioning on the other variable yields

$$\begin{aligned}
&\left(\frac{(n+p+1)^2}{(p-3)^2}\right)\Delta_k \\
&= \left(\frac{2(p-1)(n+p+1)}{p-3}\right) E\left[\mathbf{1}\left[\frac{T}{S} < \frac{p-1}{n+2}\right]\right] - \left(\frac{2(p-1)(n+p+1)}{n+2}\right) E\left[\left(\frac{S}{T}\right)\mathbf{1}\left[\frac{T}{S} < \frac{p-1}{n+2}\right]\right] \\
&\quad + \left(\frac{n(p-1)(p+2n+3)}{(n+2)^2}\right) E\left[\left(\frac{S^*}{T}\right)\mathbf{1}\left[\frac{T}{S^*} < \frac{p-1}{n+2}\right]\right] \\
&\quad - 2nE\left[\mathbf{1}\left[\frac{T}{S^*} < \frac{p-1}{n+2}\right]\right] - (p+2k-1)E\left[\mathbf{1}\left[\frac{T^*}{S} < \frac{p-1}{n+2}\right]\right],
\end{aligned}$$

where $S^* \sim \chi^2_{(n+2)}$ and $T^* \sim \chi^2_{(p+2k+1)}$.

Applying (2.2) and (2.3) to this identity yields

$$\left(\frac{(n+p+1)^2}{(p-3)^2}\right)\Delta_k = \left(\frac{2(p-1)(n+p+1)}{p-3}\right)I(a,b) - \left(\frac{2n(p-1)(n+p+1)}{(n+2)(p+2k-3)}\right)I(a-1,b+1)$$
$$+ \left(\frac{n(p-1)(p+2n+3)}{(n+2)(p+2k-3)}\right)I(a-1,b+2) - 2nI(a,b+1) - (p+2k-1)I(a+1,b),$$

where

$$I(a,b) = I_r(a,b), \quad \text{for } r = \frac{p-1}{n+p+1},\ a = \frac{p+2k-1}{2},\ b = \frac{n}{2}.$$

The application of the incomplete beta ratio function identities (2.4) and (2.5) along with some algebra yields

$$\left(\frac{(n+p+1)^2(n+2)(p+2k-3)}{2(p-3)}\right)\Delta_k$$
$$= \left[n(n+p+1)(k-1)(p-3)\right] I(a,b+1)$$
$$- \Big[2k^2(p-3)(n+2) + k\left\{n^2(p-7) + n(p^2-12p+3) + 28 - 20p\right\}$$
$$-(p-3)(n+p-1)(3n+p+5)\Big] I(a,b).$$

Notice that the coefficient of $I(a,b+1)$ in the preceding expression is negative for $k = 0$, zero for $k = 1$, and positive for $k \geq 2$. It can be verified that

$$\left(\frac{(p+2k+n-1)(n+2)}{n(n+p+1)}\right)I(a,b) < I(a,b+1) < \left(\frac{p+2k+n-1}{n}\right)I(a,b).$$

Insertion of the lower bound, for $k \geq 1$, yields

$$\left(\frac{(n+p+1)^2(n+2)(p+2k-3)}{2(p-3)}\right)\Delta_k$$
$$\geq [k\left\{4n^2 + 8n(p-1) + 8n + 12(p-1) + 2(p-1)^2\right\} + (2n+p+3)(p-3)(n+p-1)]I(a,b) > 0,$$

for all $p \geq 4$ and $n \geq 1$. Insertion of the upper bound, when $k = 0$, yields

$$\left(\frac{(n+p+1)^2(n+2)(p-3)}{2(p-3)}\right)\Delta_0 > 2(p-3)(n+p-1)(n+2)I(a,b) > 0,$$

for all $p \geq 4$ and $n \geq 1$. Hence, $\Delta_k > 0$ for all $k = 0, 1, \ldots,$ for all $p \geq 4$ and for all $n \geq 1$. And since $\Delta = E[E[\Delta_k]]$, $\Delta > 0$ for all $p \geq 4$ and for all $n \geq 1$. Therefore, we can obtain the main theorem of this section from above results. □

Theorem 2. *$\delta_2^*(\boldsymbol{X})$ dominates $\delta_1^*(\boldsymbol{X})$ for all $p \geq 4$ and $n \geq 1$.*

4. Concluding Remarks

Graphing the positive part versions of the multipliers $g_1^+(F)$ and $g_2^+(F)$ it is readily seen that δ_2^+ does not dominate δ_1^+. For $F < (p-2)/(n+4)$ and for $F > p/(n+2)$, $\delta_1^+ = \delta_2^+$. For $\boldsymbol{\theta} = \mathbf{0}$ and $(p-2)/(n+4) < F < p/(n+2)$, the risk difference $\Delta = R(\mathbf{0}, \delta_1^+) - R(\mathbf{0}, \delta_2^+) < 0$. Thus δ_2^+ does not dominate δ_1^+. Similarly, for $F < (p-3)/(n+4)$ and for $F > (p-1)/(n+2)$, $\delta_1^{*+} = \delta_2^{*+}$. For $\boldsymbol{\theta} = \bar{\theta}\mathbf{1}$ and $(p-3)/(n+4) < F < (p-1)/(n+2)$, $\Delta = R(\bar{\theta}\mathbf{1}, \delta_1^{*+}) - R(\bar{\theta}\mathbf{1}, \delta_2^{*+}) < 0$, *i.e.*, the contribution to the risk difference is in favor of δ_1^{*+}, thus δ_2^{*+} does not dominate δ_1^{*+}.

We can represent $\bar{X}\mathbf{1}$ as $(1/p)J\boldsymbol{X}$, where J is the $p \times p$ matrix all entries are 1's and make the general form of the estimator in (3.1) replacing $\bar{X}\mathbf{1}$ by $P_V\boldsymbol{X}$, where P_V is the $p \times p$ projection matrix (Kim *et al*, 2002; Lehmann and Casella, 1999). The estimators in Berry (1994) and (3.1) are the cases of $P_V = O_{p\times p}$ and $P_V = (1/p)J$, respectively. It is left to further research to determine the estimators shrinking toward $P_V\boldsymbol{X}$ with a more general projection matrix P_V.

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Received May 20, 2013; Revised July 1, 2013; Accepted July 1, 2013