

## CONVOLUTION SUMS AND THEIR RELATIONS TO EISENSTEIN SERIES

DAEYEOUL KIM, AERAN KIM, AND AYYADURAI SANKARANARAYANAN

ABSTRACT. In this paper, we consider several convolution sums, namely,  $\mathcal{A}_i(m, n; N)$  ( $i = 1, 2, 3, 4$ ),  $\mathcal{B}_j(m, n; N)$  ( $j = 1, 2, 3$ ), and  $\mathcal{C}_k(m, n; N)$  ( $k = 1, 2, 3, \dots, 12$ ), and establish certain identities involving their finite products. Then we extend these types of product convolution identities to products involving Faulhaber sums. As an application, an identity involving the Weierstrass  $\wp$ -function, its derivative and certain linear combination of Eisenstein series is established.

### 1. Introduction

Let  $\sigma_s(N)$  denote the sum of the  $s$ th powers of the positive divisors of  $N$ , and let  $\sigma_s(0) = \frac{1}{2}\zeta(-s)$ , where  $\zeta(s)$  is the Riemann Zeta-function. In his celebrated paper [12], Srinivasa Ramanujan considered the sums of the type

$$(1) \quad \sigma_r(0)\sigma_s(N) + \sigma_r(1)\sigma_s(N-1) + \cdots + \sigma_r(N)\sigma_s(0) = \sum_{k=0}^N \sigma_r(k)\sigma_s(N-k).$$

Using elementary arguments, Ramanujan established nine identities of the convolution sums of type (1). He also introduced ([11, pp. 136–162], [12]) three interesting Eisenstein series  $P(q)$ ,  $Q(q)$  and  $R(q)$  defined for  $|q| < 1$  by

$$\begin{aligned} P(q) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n, \\ Q(q) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \\ R(q) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n \end{aligned}$$

---

Received December 10, 2012; Revised January 3, 2013.

2010 *Mathematics Subject Classification*. Primary 33E05, 11A25, 11B65; Secondary 11Y35, 11Y70.

*Key words and phrases*. sum of divisor functions, convolution sums, Faulhaber sums, Eisenstein series, elliptic function.

and proved that these series satisfy the differential equations ([11, (30)] and [11, p. 142])

$$\begin{aligned} q \frac{dP(q)}{dq} &= \frac{P^2(q) - Q(q)}{12}, \\ q \frac{dQ(q)}{dq} &= \frac{P(q)Q(q) - R(q)}{3}, \\ \text{and } q \frac{dR(q)}{dq} &= \frac{P(q)R(q) - Q^2(q)}{2}, \end{aligned}$$

respectively.

In [10, Theorem 2], G. Melfi established seven identities of the type

$$(2) \quad \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \sigma_r(k) \sigma_s(n - mk) = A \sigma_{r+s+1}(n) + B n \sigma_{r+s-1}(n)$$

using the theory of modular forms. (2) holds for every  $n$  satisfying some suitable congruences for integers  $m \geq 2$  and  $r, s = 1$  or  $3$ . The coefficients  $A$  and  $B$  are rational numbers.

In [2, Theorem 4.1], B. Berndt and A. J. Yee considered the series

$$\frac{1}{P(q)} := \sum_{n=0}^{\infty} \alpha_n q^n, \quad |q| < 1$$

and proved that

$$\alpha_n \equiv 0 \pmod{3^4} \quad \text{for } n \equiv 2 \pmod{3}.$$

They also established similar congruences for several quotients of Eisenstein series expansions of  $P(q)$ ,  $Q(q)$ , and  $R(q)$ .

One of the important motivations in studying these convolution sums is their connection with the Eisenstein series. Their relation can very well be explained by the following two illustrations. Let the full modular group  $SL(2, \mathbb{Z})$  be defined by

$$SL(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Let  $\mathcal{H}$  denote the complex upper-half plane, i.e.,

$$\mathcal{H} := \{ \tau \in \mathbb{C} : \text{Im } \tau > 0 \}.$$

A meromorphic function  $f : \mathcal{H} \rightarrow \mathbb{C}$  is said to be *weakly modular of weight  $k$*  for  $SL(2, \mathbb{Z})$  if  $f$  satisfies the transformation formula

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

The function  $f$  is called a *modular form of weight  $k$*  for  $SL(2, \mathbb{Z})$  if

- (i)  $f$  is weakly modular of weight  $k$  for  $SL(2, \mathbb{Z})$ ,
- (ii)  $f$  is holomorphic on  $\mathcal{H}$ ,

(iii)  $f$  is holomorphic at  $\infty$ , i.e.,  $f$  has the Fourier expansion

$$f(q) = \sum_{n=0}^{\infty} a_n q^n$$

with  $q = e^{2\pi i\tau}$ ,  $\tau \in \mathcal{H}$ .

Let  $M_{2k}(SL(2, \mathbb{Z}))$  denote the set of all modular forms of weight  $2k$  for  $SL(2, \mathbb{Z})$ . It is well-known that  $\dim(M_{2k}) = \lfloor \frac{k}{6} \rfloor + \alpha_{2k}$  ( $\lfloor y \rfloor$  denotes the greatest integer  $\leq y$ ) with

$$\alpha_{2k} = \begin{cases} 1 & \text{if } 2k \not\equiv 2 \pmod{12}, \\ 0 & \text{if } 2k \equiv 2 \pmod{12}. \end{cases}$$

For any integer  $k \geq 1$ , the normalized Eisenstein series  $E_{2k}(\tau)$  is defined by

$$E_{2k}(\tau) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where  $B_{2k}$  is the  $2k$ -th Bernoulli number. Note that  $E_2 = P$ ,  $E_4 = Q$  and  $E_6 = R$ .

We focus our attention to two identities (see [13], [14]), namely,

$$(3) \quad \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{1}{120} (\sigma_7(n) - \sigma_3(n))$$

and

$$(4) \quad \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m) = \frac{65}{174132} \sigma_{11}(n) + \frac{1}{252} \sigma_5(n) - \frac{3}{691} \tau(n),$$

where  $\tau(n)$  is the Ramanujan's tau function defined by

$$\Delta(q_1) := q_1 \prod_{n=1}^{\infty} (1 - q_1^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q_1^n, \quad q_1 \in \mathbb{C}, \quad |q_1| < 1.$$

The classical way of deriving the identities (3) and (4) is by the theory of modular forms. We observe here that both the Eisenstein series  $Q^2 (= E_4^2)$  and  $E_8$  are modular forms of weight 8 and that  $\dim(M_8) = 1$ . This implies that there exists  $c \in \mathbb{C}$  such that  $E_8 = cQ^2 (= cE_4^2)$ . By equating the constant terms, we obtain  $c = 1$ . Then equating the coefficients of  $q^n$  ( $n \in \mathbb{N}$ ), we indeed obtain the identity in (3).

To prove (4), observe that

$$Q^3 (= E_4^3), \quad R^2 (= E_6^2), \quad E_{12} \in M_{12}(SL(2, \mathbb{Z})),$$

and hence

$$\Delta := \frac{1}{1728} (Q^3 - R^2) \left( = \frac{1}{1728} (E_4^3 - E_6^2) \right) \in M_{12}(SL(2, \mathbb{Z})).$$

We note that  $\dim(M_{12}) = 2$  and that the quantities  $E_{12}$  and  $\Delta$  are linearly independent over  $\mathbb{C}$ . Hence the set  $\{E_{12}, \Delta\}$  forms a basis for  $M_{12} (SL(2, \mathbb{Z}))$ . Therefore, there exist  $c, d \in \mathbb{C}$  such that

$$R^2 (= E_6^2) = cE_{12} + d\Delta.$$

Taking  $q = 0$ , we obtain  $c = 1$ . Equating the coefficients of  $q$ , we get  $d = -\frac{762048}{691}$ . Now, equating the coefficients of  $q^n$  ( $n \in \mathbb{N}$ ), indeed we obtain the identity in (4).

Likewise, convolution sums and identities involving them are interesting objects in number theory. They have been studied extensively by several authors (see [1, 3, 5, 9, 10, 14]).

Inspired by all these works, in this paper, we consider certain convolution sums (involving the generalized divisor functions) of classes, namely,  $\mathcal{A}_i(m, n; N)$  ( $i = 1, 2, 3, 4$ ),  $\mathcal{B}_j(m, n; N)$  ( $j = 1, 2, 3$ ), and  $\mathcal{C}_k(m, n; N)$  ( $k = 1, 2, 3, \dots, 12$ ). Using elementary techniques, the goal is to establish first some product relation identities among these functions  $\mathcal{A}_i(m, n; N)$ ,  $\mathcal{B}_j(m, n; N)$ ,  $\mathcal{C}_k(m, n; N)$ . As an interesting consequence, we derive that if

$$f \in \{\mathcal{A}_i(m, n; p), \mathcal{B}_j(m, n; p), \mathcal{C}_k(m, n; N), 1 \leq i \leq 4, 1 \leq j \leq 3, 1 \leq k \leq 12\},$$

then  $f$  is a linear combination of Faulhaber sums  $S_1(q)$ ,  $S_2(q)$ ,  $S_3(q)$  and  $S_4(q)$  whenever  $p = 2q + 1$  is an odd prime.

Finally, we obtain an identity involving Weierstrass  $\wp$ -function and its derivatives along with certain linear combinations of Eisenstein series as an application to this general theory. Before stating the result, we introduce some notations.

For  $N, m, r, s, d \in \mathbb{Z}^+$  with  $d, s > 0$  and  $r \geq 0$ , we define below some necessary divisor functions and convolution sums for later use, which also appear in many areas of number theory.

$$\begin{aligned} \sigma_s(N) &:= \sum_{d|N} d^s, & \tilde{\sigma}_s(n) &:= \sum_{d|n} (-1)^{d-1} d^s, & \sigma_{s,r}(N; m) &:= \sum_{\substack{d|N, \\ d \equiv r \pmod m}} d^s, \\ \sigma_s^*(N) &:= \sum_{\substack{d|N, \\ \frac{N}{d} \text{ odd}}} d^s, & \sigma_s^*(N; m) &:= \sum_{\substack{d|N, \\ \frac{N}{d} \not\equiv 0 \pmod m}} d^s, \\ \text{Faulhaber sums } S_i(N) &:= \sum_{k=1}^N k^i. \end{aligned}$$

More precisely, we consider

$$\begin{aligned} \mathcal{A}_i(m, n; N) &:= \sum_{k=1}^{N-1} \alpha_i(2^m k) \beta_i(2^n(N-k)) - \sum_{k=1}^{N-1} \alpha_i(k) \beta_i(2^n(N-k)), \\ \mathcal{B}_j(m, n; N) &:= \sum_{k=1}^{N-1} \gamma_j(2^m k) \delta_j(2^n(N-k)) - \sum_{k=1}^{N-1} \gamma_j(k) \delta_j(2^n(N-k)), \end{aligned}$$

TABLE 1. Functions of  $\alpha_i, \beta_i, \gamma_j, \delta_j, \omega_k$  and  $\rho_k$

$i$	$\alpha_i$	$\beta_i$	$j$	$\gamma_j$	$\delta_j$	$k$	$\omega_k$	$\rho_k$	$k$	$\omega_k$	$\rho_k$	$k$	$\omega_k$	$\rho_k$
1	$\sigma_3$	$\sigma_1$	1	$\sigma_1$	$\sigma_3$	1	$\sigma_1$	$\sigma_1$	5	$\sigma_{1,0}$	$\tilde{\sigma}_1$	9	$\tilde{\sigma}_1$	$\sigma_{1,0}$
2	$\sigma_3$	$\sigma_{1,0}$	2	$\sigma_{1,0}$	$\sigma_3$	2	$\sigma_{1,0}$	$\sigma_1$	6	$\tilde{\sigma}_1$	$\tilde{\sigma}_1$	10	$\sigma_{1,0}$	$\sigma_{1,1}$
3	$\tilde{\sigma}_3$	$\tilde{\sigma}_1$	3	$\tilde{\sigma}_1$	$\tilde{\sigma}_3$	3	$\tilde{\sigma}_1$	$\sigma_1$	7	$\sigma_{1,0}$	$\sigma_{1,0}$	11	$\sigma_1$	$\sigma_{1,1}$
4	$\sigma_3$	$\sigma_{1,1}$				4	$\sigma_1$	$\tilde{\sigma}_1$	8	$\sigma_1$	$\sigma_{1,0}$	12	$\tilde{\sigma}_1$	$\sigma_{1,1}$

$$\mathcal{C}_k(m, n; N) := \sum_{k=1}^{N-1} \omega_k(2^m k) \rho_k(2^n(N-k)) - \sum_{k=1}^{N-1} \omega_k(k) \rho_k(2^n(N-k)),$$

where  $\alpha_i, \beta_i, \gamma_j, \delta_j, \omega_k$  and  $\rho_k$  are as in Table 1. We prove :

**Theorem 1.1.** *Let  $1 \leq k, l \leq n$  be integers with  $1 \leq i, j \leq 4, 1 \leq i', j' \leq 3$  and  $1 \leq i'', j'' \leq 12$ . Then we have*

(a)

$$\left( \prod_{m=1}^k \frac{\mathcal{A}_j(m, n; N)}{\mathcal{A}_i(m, n; N)} \right) \left( \prod_{m'=1}^l \frac{\mathcal{A}_j(m', n; N)}{\mathcal{A}_i(m', n; N)} \right) = \prod_{m=1}^{k+l} \frac{\mathcal{A}_j(m, n; N)}{\mathcal{A}_i(m, n; N)},$$

(b)

$$\left( \prod_{m=1}^k \frac{\mathcal{B}_{j'}(m, n; N)}{\mathcal{B}_{i'}(m, n; N)} \right) \left( \prod_{m'=1}^l \frac{\mathcal{B}_{j'}(m', n; N)}{\mathcal{B}_{i'}(m', n; N)} \right) = \prod_{m=1}^{k+l} \frac{\mathcal{B}_{j'}(m, n; N)}{\mathcal{B}_{i'}(m, n; N)},$$

(c)

$$\left( \prod_{m=1}^k \frac{\mathcal{C}_{j''}(m, n; N)}{\mathcal{C}_{i''}(m, n; N)} \right) \left( \prod_{m'=1}^l \frac{\mathcal{C}_{j''}(m', n; N)}{\mathcal{C}_{i''}(m', n; N)} \right) = \prod_{m=1}^{k+l} \frac{\mathcal{C}_{j''}(m, n; N)}{\mathcal{C}_{i''}(m, n; N)},$$

(d)

$$\left( \prod_{m=1}^k \frac{\mathcal{B}_{i'}(m, n; N)}{\mathcal{C}_{i''}(m, n; N)} \right) \left( \prod_{m'=1}^l \frac{\mathcal{B}_{i'}(m', n; N)}{\mathcal{C}_{i''}(m', n; N)} \right) = \prod_{m=1}^{k+l} \frac{\mathcal{B}_{i'}(m, n; N)}{\mathcal{C}_{i''}(m, n; N)}.$$

The paper is organized as follows. In §2, we present some convolution formulas and state some basic comments. In §3, we obtain a generalized result (see Theorem 3.2) of Theorem 1.1 and give an interesting example (Example 3.3). In §4, we give a relation between Faulhaber sums and convolution sums. Finally in §5, as applications, we establish an identity involving Weierstrass  $\wp$ -function, its derivatives and linear combinations of certain Eisenstein series, and as a corollary we obtain the  $q$ -expansion of certain Eisenstein series. We also prove some congruence relations on the coefficients related to the  $q$ -series expansion of  $-\frac{3\wp(\frac{\tau}{2})}{\pi^2}$ .

**2. The convolution sums**

In this section, we first prove some convolution formulas on  $m, n$  for various convolution sums. Then we establish explicit expressions for  $\mathcal{A}_i, \mathcal{B}_i$  and  $\mathcal{C}_i$  involving Faulhaber sums.

**Proposition 2.1.** *Let  $N$  be any positive integer. If  $m, n \in \mathbb{N} \cup \{0\}$  with  $0 \leq m \leq n$ , then we have*

TABLE 2. Convolution formulas for  $m$  and  $n$

<i>Convolution sums</i>	<i>Convolution formula</i>
$\sum_{k=1}^{N-1} \sigma_3(2^m k) \sigma_1(2^n(N-k))$	$\frac{1}{1680} [(3 - 5 \cdot 2^{3m+4} - 2^{n+4} + 15 \cdot 2^{n+3m+4}) \sigma_5(N) + 16(9 + 5 \cdot 2^{3m} + 2^n - 15 \cdot 2^{n+3m}) \sigma_5(\frac{N}{2}) - 10(2^{3(m+1)} - 1)(3 \cdot 2^n N - 1) \sigma_3(N) + 80(2^{3m} - 1)(3 \cdot 2^n N - 1) \sigma_3(\frac{N}{2}) - 7(2^{n+1} - 1) \sigma_1(N) + 14(2^n - 1) \sigma_1(\frac{N}{2})]$
$\sum_{k=1}^{N-1} \sigma_1(2^m k) \sigma_3(2^n(N-k))$	$\frac{1}{1680} [(15 \cdot 2^{3n+m+4} - 2^{m+4} - 5 \cdot 2^{3n+4} + 3) \sigma_5(N) - 16(15 \cdot 2^{3n+m} - 5 \cdot 8^n - 2^m - 9) \sigma_5(\frac{N}{2}) - 10(8^{n+1} - 1)(3 \cdot 2^m N - 1) \sigma_3(N) + 80(8^n - 1)(3 \cdot 2^m N - 1) \sigma_3(\frac{N}{2}) - 7(2^{m+1} - 1) \sigma_1(N) + 14(2^m - 1) \sigma_1(\frac{N}{2})]$
$\sum_{k=1}^{N-1} \sigma_3(2^m k) \sigma_{1,0}(2^n(N-k); 2)$	$\frac{1}{840} \{ (15 \cdot 2^{n+3m+3} - 5 \cdot 2^{3m+4} - 2^{n+3} + 3) \sigma_5(N) - 8(15 \cdot 2^{n+3m} - 5 \cdot 2^{3m+1} - 2^n - 18) \sigma_5(\frac{N}{2}) - 5(8^{m+1} - 1)(3 \cdot 2^n N - 2) \sigma_3(N) + 40(8^m - 1)(3 \cdot 2^n N - 2) \sigma_3(\frac{N}{2}) - 7(2^n - 1) \sigma_1(N) + 7(2^n - 2) \sigma_1(\frac{N}{2}) \}$
$\sum_{k=1}^{N-1} \sigma_{1,0}(2^m k; 2) \sigma_3(2^n(N-k))$	$\frac{1}{840} \{ (15 \cdot 2^{3n+m+3} - 2^{m+3} - 5 \cdot 2^{3n+4} + 3) \sigma_5(N) + 8(-15 \cdot 2^{3n+m} + 2^m + 5 \cdot 2^{3n+1} + 18) \sigma_5(\frac{N}{2}) - 5(8^{n+1} - 1)(3 \cdot 2^m N - 2) \sigma_3(N) + 40(8^n - 1)(3 \cdot 2^m N - 2) \sigma_3(\frac{N}{2}) - 7(2^m - 1) \sigma_1(N) + 7(2^m - 2) \sigma_1(\frac{N}{2}) \}$
$\sum_{k=1}^{N-1} \tilde{\sigma}_3(2^m k) \tilde{\sigma}_1(2^n(N-k))$	$\frac{1}{112} \{ (2^{n+3m+4} - 2^{n+4} - 2^{3m+4} + 9) \sigma_5(N) + 16(-2^{n+3m} + 2^n + 2^{3m} + 27) \sigma_5(\frac{N}{2}) - 2(2^{3m+3} - 15)(2^n N - 1) \sigma_3(N) + 16(2^{3m} - 15)(2^n N - 1) \sigma_3(\frac{N}{2}) - 7(2^{n+1} - 3) \sigma_1(N) + 14(2^n - 3) \sigma_1(\frac{N}{2}) \}$
$\sum_{k=1}^{N-1} \tilde{\sigma}_1(2^m k) \tilde{\sigma}_3(2^n(N-k))$	$\frac{1}{112} \{ (2^{3n+m+4} - 2^{m+4} - 2^{3n+4} + 9) \sigma_5(N) + 16(8^n - 2^{3n+m} + 2^m + 27) \sigma_5(\frac{N}{2}) - 2(8^{n+1} - 15)(2^m N - 1) \sigma_3(N) + 16(8^n - 15)(2^m N - 1) \sigma_3(\frac{N}{2}) - 7(2^{m+1} - 3) \sigma_1(N) + 14(2^m - 3) \sigma_1(\frac{N}{2}) \}$
$\sum_{k=1}^{N-1} \sigma_3(2^m k) \sigma_{1,1}(2^n(N-k); 2)$	$\frac{1}{1680} \{ (5 \cdot 2^{3m+4} - 3) \sigma_5(N) - 16(5 \cdot 8^m + 9) \sigma_5(\frac{N}{2}) - 10(8^{m+1} - 1) \sigma_3(N) + 80(8^m - 1) \sigma_3(\frac{N}{2}) - 7 \sigma_1(N) + 14 \sigma_1(\frac{N}{2}) \}$

<i>Convolution sums</i>	<i>Convolution formula</i>
$\sum_{k=1}^{N-1} \sigma_1(2^m k) \sigma_1(2^n(N-k))$	$\frac{1}{24}[(3 \cdot 2^{n+m} - 2^m - 2^n - 1)\sigma_3(2N) - (3 \cdot 2^{n+m} - 2^m - 2^n - 11)\sigma_3(N) + \{2^n + 2^m - 2 - 6(2^{n+m+1} - 2^m - 2^n)N\}\sigma_1(2N) - \{2^n + 2^m - 4 - 12(2^{n+m} - 2^m - 2^n)N\}\sigma_1(N)]$
$\sum_{k=1}^{N-1} \sigma_{1,0}(2^m k; 2) \sigma_1(2^n(N-k))$	$\frac{1}{24}[(3 \cdot 2^{n+m} - 2^m - 2^{n+1} - 2)\sigma_3(2N) - (3 \cdot 2^{n+m} - 2^m - 2^{n+1} - 22)\sigma_3(N) + \{2^m + 2^{n+1} - 4 - 6(2^{n+m+1} - 2^m - 2^{n+1})N\}\sigma_1(2N) - \{2^m + 2^{n+1} - 8 - 12(2^{n+m} - 2^m - 2^{n+1})N\}\sigma_1(N)]$
$\sum_{k=1}^{N-1} \tilde{\sigma}_1(2^m k) \sigma_1(2^n(N-k))$	$\frac{1}{24}[-(3 \cdot 2^{n+m} - 2^m - 3 \cdot 2^n - 3)\sigma_3(2N) + (3 \cdot 2^{n+m} - 2^m - 3 \cdot 2^n - 33)\sigma_3(N) - \{2^m + 3 \cdot 2^n - 6 - 6(2^{n+m+1} - 2^m - 3 \cdot 2^n)N\}\sigma_1(2N) + \{2^m + 3 \cdot 2^n - 12 - 12(2^{n+m} - 2^m - 3 \cdot 2^n)N\}\sigma_1(N)]$
$\sum_{k=1}^{N-1} \sigma_1(2^m k) \tilde{\sigma}_1(2^n(N-k))$	$\frac{1}{24}[-(3 \cdot 2^{n+m} - 3 \cdot 2^m - 2^n - 3)\sigma_3(2N) + (3 \cdot 2^{n+m} - 3 \cdot 2^m - 2^n - 33)\sigma_3(N) - \{3 \cdot 2^m + 2^n - 6 - 6(2^{n+m+1} - 3 \cdot 2^m - 2^n)N\}\sigma_1(2N) + \{3 \cdot 2^m + 2^n - 12 - 12(2^{n+m} - 3 \cdot 2^m - 2^n)N\}\sigma_1(N)]$
$\sum_{k=1}^{N-1} \sigma_{1,0}(2^m k; 2) \tilde{\sigma}_1(2^n(N-k))$	$\frac{1}{24}[-(3 \cdot 2^{n+m} - 3 \cdot 2^m - 2^{n+1} - 6)\sigma_3(2N) + (3 \cdot 2^{n+m} - 3 \cdot 2^m - 2^{n+1} - 66)\sigma_3(N) - \{3 \cdot 2^m + 2^{n+1} - 12 - 6(2^{n+m+1} - 3 \cdot 2^m - 2^{n+1})N\}\sigma_1(2N) + \{3 \cdot 2^m + 2^{n+1} - 24 - 12(2^{n+m} - 3 \cdot 2^m - 2^{n+1})N\}\sigma_1(N)]$
$\sum_{k=1}^{N-1} \tilde{\sigma}_1(2^m k) \tilde{\sigma}_1(2^n(N-k))$	$\frac{1}{8}[(2^{n+m} - 2^m - 2^n - 3)\sigma_3(2N) - (2^{n+m} - 2^m - 2^n - 33)\sigma_3(N) + \{2^n + 2^m - 6 - 2(2^{n+m+1} - 3 \cdot 2^m - 3 \cdot 2^n)N\}\sigma_1(2N) - \{2^n + 2^m - 12 - 4(2^{n+m} - 3 \cdot 2^m - 3 \cdot 2^n)N\}\sigma_1(N)]$
$\sum_{k=1}^{N-1} \sigma_{1,0}(2^m k; 2) \sigma_{1,0}(2^n(N-k); 2)$	$\frac{1}{24}[(3 \cdot 2^{n+m} - 2^{m+1} - 2^{n+1} - 4)\sigma_3(2N) - (3 \cdot 2^{n+m} - 2^{m+1} - 2^{n+1} - 44)\sigma_3(N) + 2\{2^n + 2^m - 4 - 6(2^{n+m} - 2^m - 2^n)N\}\sigma_1(2N) - 2\{2^n + 2^m - 8 - 6(2^{n+m} - 2^{m+1} - 2^{n+1})N\}\sigma_1(N)]$
$\sum_{k=1}^{N-1} \sigma_1(2^m k) \sigma_{1,0}(2^n(N-k); 2)$	$\frac{1}{24}[(3 \cdot 2^{n+m} - 2^{m+1} - 2^n - 2)\sigma_3(2N) - (3 \cdot 2^{n+m} - 2^{m+1} - 2^n - 22)\sigma_3(N) + \{2^{m+1} + 2^n - 4 - 6(2^{n+m+1} - 2^{m+1} - 2^n)N\}\sigma_1(2N) - \{2^{m+1} + 2^n - 8 - 12(2^{n+m} - 2^{m+1} - 2^n)N\}\sigma_1(N)]$
$\sum_{k=1}^{N-1} \tilde{\sigma}_1(2^m k) \sigma_{1,0}(2^n(N-k); 2)$	$\frac{1}{24}[-(3 \cdot 2^{n+m} - 2^{m+1} - 3 \cdot 2^n - 6)\sigma_3(2N) + (3 \cdot 2^{n+m} - 2^{m+1} - 3 \cdot 2^n - 66)\sigma_3(N) - \{2^{m+1} + 3 \cdot 2^n - 12 - 6(2^{n+m+1} - 2^{m+1} - 3 \cdot 2^n)N\}\sigma_1(2N) + \{2^{m+1} + 3 \cdot 2^n - 24 - 12(2^{n+m} - 2^{m+1} - 3 \cdot 2^n)N\}\sigma_1(N)]$
$\sum_{k=1}^{N-1} \sigma_{1,0}(2^m k; 2) \sigma_{1,1}(2^n(N-k); 2)$	$\frac{1}{24}[(2^m + 2)\sigma_3(2N) - (2^m + 22)\sigma_3(N) - (3 \cdot 2^{m+1}N + 2^m - 4)\sigma_1(2N) + (3 \cdot 2^{m+2}N + 2^m - 8)\sigma_1(N)]$
$\sum_{k=1}^{N-1} \sigma_1(2^m k) \sigma_{1,1}(2^n(N-k); 2)$	$\frac{1}{24}[(2^m + 1)\sigma_3(2N) - (2^m + 11)\sigma_3(N) - (3 \cdot 2^{m+1}N + 2^m - 2)\sigma_1(2N) + (3 \cdot 2^{m+2}N + 2^m - 4)\sigma_1(N)]$
$\sum_{k=1}^{N-1} \tilde{\sigma}_1(2^m k) \sigma_{1,1}(2^n(N-k); 2)$	$\frac{1}{24}[-(2^m + 3)\sigma_3(2N) + (2^m + 33)\sigma_3(N) + (3 \cdot 2^{m+1}N + 2^m - 6)\sigma_1(2N) - (3 \cdot 2^{m+2}N + 2^m - 12)\sigma_1(N)]$

*Proof.* (Refer to [6]) Since the proofs are very similar, we prove only

$$\sum_{k=1}^{N-1} \sigma_3(2^m k) \sigma_1(2^n(N-k)) \quad \text{and} \quad \sum_{k=1}^{N-1} \tilde{\sigma}_3(2^m k) \tilde{\sigma}_1(2^n(N-k)).$$

Considering  $\sum_{k=1}^{N-1} \sigma_3(2^m k) \sigma_1(2^n(N-k))$ , we use induction on  $m$  to obtain the general formula for  $\sigma_3(2^m k)$ . If  $m = 1$ , then

$$\sigma_3(2k) = 9\sigma_3(k) - 8\sigma_3\left(\frac{k}{2}\right).$$

By expanding  $m$ , we obtain

$$\sigma_3(2^m k) = \frac{8^{m+1} - 1}{7} \sigma_3(k) + \frac{8 - 8^{m+1}}{7} \sigma_3\left(\frac{k}{2}\right).$$

Similarly, we get

$$\sigma_1(2^n(N-k)) = (2^{n+1} - 1)\sigma_1(N-k) + (2 - 2^{n+1})\sigma_1\left(\frac{N-k}{2}\right).$$

Therefore, we observe that

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_3(2^m k) \sigma_1(2^n(N-k)) \\ &= \sum_{k=1}^{N-1} \left\{ \frac{8^{m+1} - 1}{7} \sigma_3(k) + \frac{8 - 8^{m+1}}{7} \sigma_3\left(\frac{k}{2}\right) \right\} \\ & \quad \cdot \left\{ (2^{n+1} - 1)\sigma_1(N-k) + (2 - 2^{n+1})\sigma_1\left(\frac{N-k}{2}\right) \right\} \\ (5) \quad &= \frac{8^{m+1} - 1}{7} \cdot (2^{n+1} - 1) \sum_{k=1}^{N-1} \sigma_3(k) \sigma_1(N-k) \\ & \quad + \frac{8^{m+1} - 1}{7} \cdot (2 - 2^{n+1}) \sum_{k=1}^{N-1} \sigma_3(k) \sigma_1\left(\frac{N-k}{2}\right) \\ & \quad + \frac{8 - 8^{m+1}}{7} \cdot (2^{n+1} - 1) \sum_{k=1}^{N-1} \sigma_3\left(\frac{k}{2}\right) \sigma_1(N-k) \\ & \quad + \frac{8 - 8^{m+1}}{7} \cdot (2 - 2^{n+1}) \sum_{k=1}^{N-1} \sigma_3\left(\frac{k}{2}\right) \sigma_1\left(\frac{N-k}{2}\right). \end{aligned}$$

Now we note that

$$\begin{aligned} \sum_{k=1}^{N-1} \sigma_3(k) \sigma_1(N-k) &= \sum_{t=1}^{N-1} \sigma_3(N-t) \sigma_1(t), \\ \sum_{k=1}^{N-1} \sigma_3(k) \sigma_1\left(\frac{N-k}{2}\right) &= \sum_{t=1}^{N-1} \sigma_3(N-2t) \sigma_1(t), \\ \sum_{k=1}^{N-1} \sigma_3\left(\frac{k}{2}\right) \sigma_1(N-k) &= \sum_{t=1}^{N-1} \sigma_3(t) \sigma_1(N-2t), \end{aligned}$$



$$\sum_{k=1}^{N-1} \sigma_3\left(\frac{k}{2}\right)\sigma_1\left(\frac{N-k}{2}\right) = \sum_{t=1}^{N-1} \sigma_3\left(\frac{N}{2} - t\right)\sigma_1(t).$$

$\sum_{k=1}^{N-1} \sigma_3(2^m k)\sigma_1(2^n(N-k))$  is also defined and found in [6, Theorem 4.11]. Next, consider  $\sum_{k=1}^{N-1} \tilde{\sigma}_3(2^m k)\tilde{\sigma}_1(2^n(N-k))$ . First, we note (from [4, (1.12)]) that  $\tilde{\sigma}_s(n) = \sigma_s(n) - 2^{s+1}\sigma_s(n/2)$ . Therefore, we get

$$\begin{aligned} & \sum_{k=1}^{N-1} \tilde{\sigma}_3(2^m k)\tilde{\sigma}_1(2^n(N-k)) \\ (6) \quad &= \sum_{k=1}^{N-1} \left(\sigma_3(2^m k) - 16\sigma_3(2^{m-1}k)\right) \left(\sigma_1(2^n(N-k)) - 4\sigma_1(2^{n-1}(N-k))\right). \end{aligned}$$

Then the right-hand side of (6) can be written as

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_3(2^m k)\sigma_1(2^n(N-k)) - 4 \sum_{k=1}^{N-1} \sigma_3(2^m k)\sigma_1(2^{n-1}(N-k)) \\ (7) \quad & - 16 \sum_{k=1}^{N-1} \sigma_3(2^{m-1}k)\sigma_1(2^n(N-k)) + 64 \sum_{k=1}^{N-1} \sigma_3(2^{m-1}k)\sigma_1(2^{n-1}(N-k)). \end{aligned}$$

Now if  $m = 0$ , then for any positive integer  $n$ , Eq.(7) becomes

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_3(k)\sigma_1(2^n(N-k)) - 4 \sum_{k=1}^{N-1} \sigma_3(k)\sigma_1(2^{n-1}(N-k)) \\ & - 16 \sum_{k=1}^{N-1} \sigma_3\left(\frac{1}{2}k\right)\sigma_1(2^n(N-k)) + 64 \sum_{k=1}^{N-1} \sigma_3\left(\frac{1}{2}k\right)\sigma_1(2^{n-1}(N-k)) \\ (8) \quad &= -17 \sum_{k=1}^{N-1} \sigma_3(k)\sigma_1(2^n(N-k)) + 68 \sum_{k=1}^{N-1} \sigma_3(k)\sigma_1(2^{n-1}(N-k)) \\ & + 2 \sum_{k=1}^{N-1} \sigma_3(2k)\sigma_1(2^n(N-k)) - 8 \sum_{k=1}^{N-1} \sigma_3(2k)\sigma_1(2^{n-1}(N-k)), \end{aligned}$$

by using the elementary identity  $\sigma_3(2n) = 9\sigma_3(n) - 8\sigma_3(n/2)$ . Next we assume  $1 \leq m < n$ . Then Eq.(7) is enough to calculate. Finally, we consider the case when  $n = m$ . Then Eq.(7) becomes

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_3(2^n k)\sigma_1(2^n(N-k)) - 4 \sum_{k=1}^{N-1} \sigma_3(2^n k)\sigma_1(2^{n-1}(N-k)) \\ (9) \quad & - 16 \sum_{k=1}^{N-1} \sigma_3(2^{n-1}k)\sigma_1(2^n(N-k)) + 64 \sum_{k=1}^{N-1} \sigma_3(2^{n-1}k)\sigma_1(2^{n-1}(N-k)). \end{aligned}$$

We apply  $\sigma_3(2n) = 9\sigma_3(n) - 8\sigma_3(n/2)$  again to the second term in Eq.(9), thereby obtaining

$$(10) \quad \sum_{k=1}^{N-1} \sigma_3(2^n k) \sigma_1(2^n(N-k)) + 28 \sum_{k=1}^{N-1} \sigma_3(2^{n-1} k) \sigma_1(2^{n-1}(N-k)) \\ - 16 \sum_{k=1}^{N-1} \sigma_3(2^{n-1} k) \sigma_1(2^n(N-k)) + 32 \sum_{k=1}^{N-1} \sigma_3(2^{n-2} k) \sigma_1(2^{n-1}(N-k)).$$

From Eq.(10), our claim follows. □

**Proposition 2.2.** *Let  $k, N \in \mathbb{N}$ . Then we have*

$$\sigma_s^*(N; p) = \sigma_s(N) - \sigma_s(N/p).$$

*In particular, we have*

$$\sigma_s^*(N) = \sigma_s(N) - \sigma_s(N/2),$$

*which can be seen in [14, p. 27].*

*Proof.* We observe that

$$\sigma_s^*(N; p) := \sum_{\substack{d|N, \\ \frac{N}{d} \not\equiv 0 \pmod p}} d^s = \sum_{d|N} d^s - \sum_{\substack{d|N, \\ \frac{N}{d} \equiv 0 \pmod p}} d^s = \sigma_s(N) - \sum_{d|\frac{N}{p}} d^s \\ = \sigma_s(N) - \sigma_s(N/p).$$

Note that  $\sigma_s^*(N) := \sigma_s^*(N; 2)$ . This proves the proposition. □

To prove Theorem 1.1, we need the following lemma.

**Lemma 2.3.** *Let  $N$  be any positive integer. If  $m, n \in \mathbb{N} \cup \{0\}$  with  $0 \leq m \leq n$ , then we have*

(a)

$$\mathcal{A}_1(m, n; N) = \frac{1}{21}(8^m - 1) \left( (3 \cdot 2^n - 1) \sigma_5^*(N) - (3 \cdot 2^n N - 1) \sigma_3^*(N) \right),$$

(b)

$$\mathcal{B}_1(m, n; N) = \frac{1}{840}(2^m - 1) \left( 8(15 \cdot 8^n - 1) \sigma_5^*(N) - 120 \cdot 8^n N \sigma_3^*(N) \right. \\ \left. + 15N \sigma_{3,1}(N; 2) - 7 \sigma_1^*(N) \right),$$

(c)

$$\mathcal{A}_2(m, n; N) = \frac{1}{21}(8^m - 1) \left( (3 \cdot 2^n - 2) \sigma_5^*(N) - (3 \cdot 2^n N - 2) \sigma_3^*(N) \right),$$

(d)

$$\mathcal{B}_2(m, n; N) = \frac{1}{840}(2^m - 1) \left( 8(15 \cdot 8^n - 1)\sigma_5^*(N) - 120 \cdot 8^n N \sigma_3^*(N) + 15N\sigma_{3,1}(N; 2) - 7\sigma_1^*(N) \right),$$

(e)

$$\mathcal{A}_3(m, n; N) = \frac{1}{7}(8^m - 1) \left( (2^n - 1)\sigma_5^*(N) - (2^n N - 1)\sigma_3^*(N) \right),$$

(f)

$$\mathcal{B}_3(m, n; N) = \frac{1}{56}(2^m - 1) \left( 8(8^n - 1)\sigma_5^*(N) - 8^{n+1} N \sigma_3^*(N) + 15N\sigma_{3,1}(N; 2) - 7\sigma_1^*(N) \right),$$

(g)

$$\mathcal{A}_4(m, n; N) = \frac{1}{21}(8^m - 1) \left( \sigma_5^*(N) - \sigma_3^*(N) \right),$$

(h)

$$\begin{aligned} \mathcal{C}_1(m, n; N) &= \mathcal{C}_2(m, n; N) = -\mathcal{C}_3(m, n; N) \\ &= \frac{1}{24}(2^m - 1) \left( (3 \cdot 2^n - 1)\sigma_3^*(2N) + 6N\sigma_{1,1}(N; 2) - (3 \cdot 2^{n+2}N - 1)\sigma_1^*(2N) \right), \end{aligned}$$

(i)

$$\begin{aligned} \mathcal{C}_4(m, n; N) &= \mathcal{C}_5(m, n; N) = -\mathcal{C}_6(m, n; N) \\ &= -\frac{1}{8}(2^m - 1) \left( (2^n - 1)\sigma_3^*(2N) + 6N\sigma_{1,1}(N; 2) - (2^{n+2}N - 1)\sigma_1^*(2N) \right), \end{aligned}$$

(j)

$$\begin{aligned} \mathcal{C}_7(m, n; N) &= \mathcal{C}_8(m, n; N) = -\mathcal{C}_9(m, n; N) \\ &= \frac{1}{24}(2^m - 1) \left( (3 \cdot 2^n - 2)\sigma_3^*(2N) - (3 \cdot 2^{n+2}N - 2)\sigma_1^*(2N) + 12N\sigma_{1,1}(N; 2) \right), \end{aligned}$$

(k)

$$\begin{aligned} \mathcal{C}_{10}(m, n; N) &= \mathcal{C}_{11}(m, n; N) = -\mathcal{C}_{12}(m, n; N) \\ &= \frac{1}{24}(2^m - 1) \left( \sigma_3^*(2N) - 6N\sigma_{1,1}(N; 2) - \sigma_1^*(2N) \right). \end{aligned}$$

In particular, if we let

$$\mathcal{F}_l(m, n; p) := e_l \left( a_l S_4(q) + b_l S_3(q) + c_l S_2(q) + d_l S_1(q) \right)$$

with  $p = 2q + 1$ , where  $\mathcal{F}_l(m, n; p) \in \{\mathcal{A}_i(m, n; p), \mathcal{B}_j(m, n; p), \mathcal{C}_k(m, n; p)\}$  ( $i = 1, 2, 3, 4, j = 1, 2, 3, k = 1, 4, 7, 10$ ), then we obtain Table 3.

TABLE 3. Coefficients of  $\mathcal{F}_l(m, n; p)$  for  $p = 2q + 1$

$\mathcal{F}_l(m, n; p)$	$e_l$	$a_l$	$b_l$	$c_l$	$d_l$
$\mathcal{A}_1(m, n; p)$	$\frac{8}{21}(8^m - 1)$	$20(3 \cdot 2^n - 1)$	$-24 \cdot 2^n$	$15 \cdot 2^{n+1} - 7$	$-6 \cdot 2^n$
$\mathcal{B}_1(m, n; p)$	$\frac{2}{21}(2^m - 1)$	$16(15 \cdot 2^{3n} - 1)$	$-12(2^{3n+3} - 1)$	$8(15 \cdot 2^{3n} - 1)$	$-3(2^{3n+3} - 1)$
$\mathcal{A}_2(m, n; p)$	$\frac{16}{21}(8^m - 1)$	$10(3 \cdot 2^n - 2)$	$-12 \cdot 2^n$	$15 \cdot 2^n - 7$	$-3 \cdot 2^n$
$\mathcal{B}_2(m, n; p)$	$\frac{2}{21}(2^m - 1)$	$16(15 \cdot 2^{3n} - 1)$	$-12(2^{3n+3} - 1)$	$8(15 \cdot 2^{3n} - 1)$	$-3(2^{3n+3} - 1)$
$\mathcal{A}_3(m, n; p)$	$\frac{1}{7}(8^m - 1)$	$160(2^n - 1)$	$-2^{n+6}$	$8(5 \cdot 2^{n+1} - 7)$	$-2^{n+4}$
$\mathcal{B}_3(m, n; p)$	$\frac{2}{7}(2^m - 1)$	$80(2^{3n} - 1)$	$-4(2^{3n+3} - 15)$	$40(2^{3n} - 1)$	$-(2^{3n+3} - 15)$
$\mathcal{A}_4(m, n; p)$	$\frac{8}{21}(8^m - 1)$	20	0	7	0
$\mathcal{C}_1(m, n; p)$	$2(2^m - 1)$	0	0	$4(3 \cdot 2^n - 1)$	$-(2^{n+2} - 1)$
$\mathcal{C}_4(m, n; p)$	$-2(2^m - 1)$	0	0	$12(2^n - 1)$	$-(2^{n+2} - 3)$
$\mathcal{C}_7(m, n; p)$	$4(2^m - 1)$	0	0	$2(3 \cdot 2^n - 2)$	$-(2^{n+1} - 1)$
$\mathcal{C}_{10}(m, n; p)$	$2(2^m - 1)$	0	0	4	-1

*Proof.* Since the proofs are similar, we prove only (a). By the definition of  $\mathcal{A}_1(m, n; N)$  and Proposition 2.1(a), we can obtain

$$\begin{aligned} \mathcal{A}_1(m, n; N) &= \sum_{k=1}^{N-1} \sigma_3(2^m k) \sigma_1(2^n(N-k)) - \sum_{k=1}^{N-1} \sigma_3(k) \sigma_1(2^n(N-k)) \\ &= \frac{1}{1680} \left[ (3 - 5 \cdot 2^{3m+4} - 2^{n+4} + 15 \cdot 2^{n+3m+4}) \sigma_5(N) \right. \\ &\quad + 16(9 + 5 \cdot 2^{3m} + 2^n - 15 \cdot 2^{n+3m}) \sigma_5\left(\frac{N}{2}\right) \\ &\quad - 10(2^{3(m+1)} - 1)(3 \cdot 2^n N - 1) \sigma_3(N) \\ &\quad + 80(2^{3m} - 1)(3 \cdot 2^n N - 1) \sigma_3\left(\frac{N}{2}\right) \\ &\quad \left. - 7(2^{n+1} - 1) \sigma_1(N) + 14(2^n - 1) \sigma_1\left(\frac{N}{2}\right) \right] \\ &\quad - \frac{1}{240} \left\{ (2^{n+5} - 11) \sigma_5(N) - 32(2^n - 1) \sigma_5\left(\frac{N}{2}\right) \right. \\ &\quad \left. - 10(3 \cdot 2^n N - 1) \sigma_3(N) - (2^{n+1} - 1) \sigma_1(N) + 2(2^n - 1) \sigma_1\left(\frac{N}{2}\right) \right\} \\ &= \frac{1}{21} (8^m - 1) \left\{ (3 \cdot 2^n - 1) (\sigma_5(N) - \sigma_5\left(\frac{N}{2}\right)) \right. \\ &\quad \left. - (3 \cdot 2^n N - 1) (\sigma_3(N) - \sigma_3\left(\frac{N}{2}\right)) \right\}. \end{aligned}$$

Also, by Proposition 2.2, we can write

$$\mathcal{A}_1(m, n; N) = \frac{1}{21}(8^m - 1) \left( (3 \cdot 2^n - 1)\sigma_5^*(N) - (3 \cdot 2^n N - 1)\sigma_3^*(N) \right).$$

Let  $N = 2q + 1$  be an odd prime number. From  $\sigma_5^*(N) = 1 + (2q + 1)^5$  and  $\sigma_3^*(N) = 1 + (2q + 1)^3$ , we derive that

$$\begin{aligned} \mathcal{A}_1(m, n; 2q + 1) &= \frac{8(8^m - 1)}{21} \left( 20(3 \cdot 2^n - 1)S_4(q) - 24 \cdot 2^n S_3(q) \right. \\ &\quad \left. + (15 \cdot 2^{n+1} - 7)S_2(q) - 6 \cdot 2^n S_1(q) \right) \end{aligned}$$

in Table 3. □

*Proof of Theorem 1.1.* From Lemma 2.3, we see that

$$\mathcal{A}_i(m, n; N) \text{ and } \mathcal{B}_{i'}(m, n; N)$$

are composed of the terms  $(8^m - 1)$  and  $(2^m - 1)$ , respectively. This means that in the product  $\prod_{m=1}^k \frac{\mathcal{A}_j(m, n; N)}{\mathcal{A}_i(m, n; N)}$ , the factor  $\prod_{m=1}^k (8^m - 1)$  gets eliminated, and similarly for  $\mathcal{B}_{i'}(m, n; N)$ . Then there exist constants  $\omega, \zeta \in \mathbb{Q}$  such that

$$\begin{aligned} &\left( \prod_{m=1}^k \frac{\mathcal{A}_j(m, n; N)}{\mathcal{A}_i(m, n; N)} \right) \left( \prod_{m'=1}^l \frac{\mathcal{A}_j(m', n; N)}{\mathcal{A}_i(m', n; N)} \right) \\ &= \left( \prod_{m=1}^k \frac{8^m - 1}{8^m - 1} \right) \omega^k \left( \prod_{m'=1}^l \frac{8^{m'} - 1}{8^{m'} - 1} \right) \omega^l = \omega^{k+l} \\ &= \left( \prod_{m=1}^{k+l} \frac{8^m - 1}{8^m - 1} \right) \omega^{k+l} = \prod_{m=1}^{k+l} \frac{\mathcal{A}_j(m, n; N)}{\mathcal{A}_i(m, n; N)} \end{aligned}$$

and

$$\begin{aligned} &\left( \prod_{m=1}^k \frac{\mathcal{B}_{j'}(m, n; N)}{\mathcal{B}_{i'}(m, n; N)} \right) \left( \prod_{m'=1}^l \frac{\mathcal{B}_{j'}(m', n; N)}{\mathcal{B}_{i'}(m', n; N)} \right) \\ &= \left( \prod_{m=1}^k \frac{2^m - 1}{2^m - 1} \right) \zeta^k \left( \prod_{m'=1}^l \frac{2^{m'} - 1}{2^{m'} - 1} \right) \zeta^l = \zeta^{k+l} \\ &= \left( \prod_{m=1}^{k+l} \frac{2^m - 1}{2^m - 1} \right) \zeta^{k+l} = \prod_{m=1}^{k+l} \frac{\mathcal{B}_{j'}(m, n; N)}{\mathcal{B}_{i'}(m, n; N)}. \end{aligned}$$

(c) and (d) follow in a similar way. □

### 3. Faulhaber sums and convolution sums

**Theorem 3.1.** *Let  $i, j, m, n \in \mathbb{N} \cup \{0\}$  and let*

$$\mathcal{P}_{i,j}(p, m, n; N) = \sum_{k=1}^{N-1} \sigma_i(p^m k) \sigma_j(p^n(N - k)) - \sum_{k=1}^{N-1} \sigma_i(k) \sigma_j(p^n(N - k))$$

with  $p$  prime. Then there exists a constant  $\eta \in \mathbb{Q}$  such that

$$\mathcal{P}_{i,j}(p, m, n; N) = \frac{p^{i(m+1)} - p^i}{p^i - 1} \eta.$$

*Proof.* First, we claim that

$$\sigma_i(p^m n) = \frac{p^{i(m+1)} - 1}{p^i - 1} \sigma_i(n) + \frac{p^i - p^{i(m+1)}}{p^i - 1} \sigma_i\left(\frac{n}{p}\right).$$

To show this, we consider the elementary identity  $\sigma_i(pn) - (p^i + 1)\sigma_i(n) + p^i \sigma_i\left(\frac{n}{p}\right) = 0$  with a prime  $p$ . When  $m = 2$ ,

$$\begin{aligned} \sigma_i(p^2 n) &= (p^i + 1)\sigma_i(pn) - p^i \sigma_i(n) \\ &= (p^i + 1) \left\{ (p^i + 1)\sigma_i(n) - p^i \sigma_i\left(\frac{n}{p}\right) \right\} - p^i \sigma_i(n) \\ &= \left( (p^i + 1)^2 - p^i \right) \sigma_i(n) - (p^i + 1) p^i \sigma_i\left(\frac{n}{p}\right) \\ &= \frac{p^{i(2+1)} - 1}{p^i - 1} \sigma_i(n) + \frac{p^i - p^{i(2+1)}}{p^i - 1} \sigma_i\left(\frac{n}{p}\right) \end{aligned}$$

is obtained. Continuing with such induction on  $m \in \mathbb{N} \cup \{0\}$ , we have the desired claim. Second, we note that

$$\begin{aligned} \mathcal{P}_{i,j}(p, m, n; N) &:= \sum_{k=1}^{N-1} \sigma_i(p^m k) \sigma_j(p^n(N-k)) - \sum_{k=1}^{N-1} \sigma_i(k) \sigma_j(p^n(N-k)) \\ &= \sum_{k=1}^{N-1} \left\{ \frac{p^{i(m+1)} - 1}{p^i - 1} \sigma_i(k) + \frac{p^i - p^{i(m+1)}}{p^i - 1} \sigma_i\left(\frac{k}{p}\right) \right\} \\ &\quad \cdot \left\{ \frac{p^{j(n+1)} - 1}{p^j - 1} \sigma_j(N-k) + \frac{p^j - p^{j(n+1)}}{p^j - 1} \sigma_j\left(\frac{N-k}{p}\right) \right\} \\ &\quad - \sum_{k=1}^{N-1} \sigma_i(k) \left\{ \frac{p^{j(n+1)} - 1}{p^j - 1} \sigma_j(N-k) \right. \\ &\quad \quad \left. + \frac{p^j - p^{j(n+1)}}{p^j - 1} \sigma_j\left(\frac{N-k}{p}\right) \right\} \\ &= \frac{p^{i(m+1)} - p^i}{p^i - 1} \left\{ \frac{p^{j(n+1)} - 1}{p^j - 1} \sum_{k=1}^{N-1} \sigma_i(k) \sigma_j(N-k) \right. \\ &\quad \quad \left. + \frac{p^j - p^{j(n+1)}}{p^j - 1} \sum_{k=1}^{N-1} \sigma_i(k) \sigma_j\left(\frac{N-k}{p}\right) \right. \\ &\quad \quad \left. - \frac{p^{j(n+1)} - 1}{p^j - 1} \sum_{k=1}^{N-1} \sigma_i\left(\frac{k}{p}\right) \sigma_j(N-k) \right\} \end{aligned}$$

$$-\left. \frac{p^j - p^{j(n+1)}}{p^j - 1} \sum_{k=1}^{N-1} \sigma_i\left(\frac{k}{p}\right) \sigma_j\left(\frac{N-k}{p}\right) \right\}.$$

Taking

$$\begin{aligned} \eta := & \frac{p^{j(n+1)} - 1}{p^j - 1} \sum_{k=1}^{N-1} \sigma_i(k) \sigma_j(N - k) + \frac{p^j - p^{j(n+1)}}{p^j - 1} \sum_{k=1}^{N-1} \sigma_i(k) \sigma_j\left(\frac{N - k}{p}\right) \\ & - \frac{p^{j(n+1)} - 1}{p^j - 1} \sum_{k=1}^{N-1} \sigma_i\left(\frac{k}{p}\right) \sigma_j(N - k) - \frac{p^j - p^{j(n+1)}}{p^j - 1} \sum_{k=1}^{N-1} \sigma_i\left(\frac{k}{p}\right) \sigma_j\left(\frac{N - k}{p}\right) \end{aligned}$$

completes the proof. □

**Theorem 3.2.** *Let  $1 \leq k, l \leq n$  be integers. Then we obtain*

$$(11) \quad \left( \prod_{m=1}^k \frac{\mathcal{P}_{i,j}(p, m, n; N)}{\mathcal{P}_{i,j'}(p, m, n; N)} \right) \left( \prod_{m'=1}^l \frac{\mathcal{P}_{i,j}(p, m', n; N)}{\mathcal{P}_{i,j'}(p, m', n; N)} \right) = \prod_{m=1}^{k+l} \frac{\mathcal{P}_{i,j}(p, m, n; N)}{\mathcal{P}_{i,j'}(p, m, n; N)}.$$

*Proof.* From Theorem 3.1, we can write

$$\mathcal{P}_{i,j}(p, m, n; N) = \frac{p^{i(m+1)} - p^i}{p^i - 1} \eta \quad \text{and} \quad \mathcal{P}_{i,j'}(p, m, n; N) = \frac{p^{i(m+1)} - p^i}{p^i - 1} \eta'.$$

Therefore, we have

$$\begin{aligned} & \left( \prod_{m=1}^k \frac{\mathcal{P}_{i,j}(p, m, n; N)}{\mathcal{P}_{i,j'}(p, m, n; N)} \right) \left( \prod_{m'=1}^l \frac{\mathcal{P}_{i,j}(p, m', n; N)}{\mathcal{P}_{i,j'}(p, m', n; N)} \right) \\ = & \left( \prod_{m=1}^k \frac{p^{i(m+1)} - p^i}{p^i - 1} \right) \left( \frac{\eta}{\eta'} \right)^k \left( \prod_{m'=1}^l \frac{p^{i(m'+1)} - p^i}{p^i - 1} \right) \left( \frac{\eta}{\eta'} \right)^l = \left( \frac{\eta}{\eta'} \right)^{k+l} \\ = & \left( \prod_{m=1}^{k+l} \frac{p^{i(m+1)} - p^i}{p^i - 1} \right) \left( \frac{\eta}{\eta'} \right)^{k+l} = \prod_{m=1}^{k+l} \frac{\mathcal{P}_{i,j}(p, m, n; N)}{\mathcal{P}_{i,j'}(p, m, n; N)}. \end{aligned}$$

□

**Example 3.3.** We present here an example for Theorem 3.2. Let  $i = 1, j = 3, j' = 5, n = 2, k = 2$ , and  $l = 3$  in (11). Using Table 4, we get

TABLE 4.  $\mathcal{P}_{1,3}(2, m, 2; 6)$  and  $\mathcal{P}_{1,5}(2, m, 2; 6)$

$m$	1	2	3	4	5
$\mathcal{P}_{1,3}(2, m, 2; 6)$	7316	21948	51212	109740	226796
$\mathcal{P}_{1,5}(2, m, 2; 6)$	414884	1244652	2904188	6223260	12861404

$$(12) \quad \prod_{m=1}^2 \frac{\mathcal{P}_{1,3}(2, m, 2; 6)}{\mathcal{P}_{1,5}(2, m, 2; 6)} = \frac{3345241}{10758045841},$$

$$\prod_{m'=1}^3 \frac{\mathcal{P}_{1,3}(2, m', 2; 6)}{\mathcal{P}_{1,5}(2, m', 2; 6)} = \frac{6118445789}{1115835272674361},$$

and

$$(13) \quad \prod_{m=1}^5 \frac{\mathcal{P}_{1,3}(2, m, 2; 6)}{\mathcal{P}_{1,5}(2, m, 2; 6)} = \frac{20467675709640149}{12004207014435510303382601}.$$

From (12) and (13), it is easy to see that

$$\left( \prod_{m=1}^2 \frac{\mathcal{P}_{1,3}(2, m, 2; 6)}{\mathcal{P}_{1,5}(2, m, 2; 6)} \right) \left( \prod_{m'=1}^3 \frac{\mathcal{P}_{1,3}(2, m', 2; 6)}{\mathcal{P}_{1,5}(2, m', 2; 6)} \right) = \prod_{m=1}^5 \frac{\mathcal{P}_{1,3}(2, m, 2; 6)}{\mathcal{P}_{1,5}(2, m, 2; 6)}.$$

*Remark 3.4.* From Table 3, we observe that we can write in the matrix form

$$\begin{pmatrix} \mathcal{A}_1(m, n; p) \\ \mathcal{A}_2(m, n; p) \\ \mathcal{A}_3(m, n; p) \\ \mathcal{A}_4(m, n; p) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} S_1(q) \\ S_2(q) \\ S_3(q) \\ S_4(q) \end{pmatrix} := A \begin{pmatrix} S_1(q) \\ S_2(q) \\ S_3(q) \\ S_4(q) \end{pmatrix},$$

where

$$A = \frac{8^{m+1} - 8}{21} \begin{pmatrix} 20(3 \cdot 2^n - 1) & -24 \cdot 2^n & 15 \cdot 2^{n+1} - 7 & -6 \cdot 2^n \\ 20(3 \cdot 2^n - 2) & -24 \cdot 2^n & 2(15 \cdot 2^n - 7) & -6 \cdot 2^n \\ 60(2^n - 1) & -24 \cdot 2^n & 3(15 \cdot 2^{n+1} - 7) & -6 \cdot 2^n \\ 20 & 0 & 7 & 0 \end{pmatrix}.$$

Note that  $\det(A) = 0$ .

**Corollary 3.5.** *Let  $p = 2q + 1$  be an odd prime number and let*

$$L = \{\mathcal{A}_i(m, n; p), \mathcal{B}_j(m, n; p) \text{ with } 1 \leq i \leq 4 \text{ and } 1 \leq j \leq 3\}.$$

- (a) *If  $f \in L$ , then there exist  $a, b, c, d \in \mathbb{Q}$  satisfying  $f = aS_1(q) + bS_2(q) + cS_3(q) + dS_4(q)$  and  $4a = c$ .*
- (b) *If  $b = \alpha \cdot 8^n + \beta$  and  $d = \gamma \cdot 8^n + \delta$ , then  $\gamma = 2\alpha$ .*

*Proof.* Follows by Table 3. □

#### 4. Examples on Faulhaber sums and convolution sums

Let the convolution sums in Table 2 be

$$\sum = aS_4(q) + bS_3(q) + cS_2(q) + dS_1(q).$$

Then we find the coefficients  $a, b, c, d$  as listed in Table 5-1 and Table 5-2.



TABLE 5-1. Faulhaber sums  $(\sigma_{1,i}, \sigma_{3,j})$  with  $p = 2q + 1$ .

Convolution sum	$a$	$b$	$c$	$d$
$\sum_{k=1}^{p-1} \sigma_3(k)\sigma_1(2^n(p-k))$	$\frac{2}{3}(2^{n+5} - 11)$	$-8 \cdot 2^n$	$\frac{8}{3}(2^{n+2} - 1)$	$-2^{n+1}$
$\sum_{k=1}^{p-1} \sigma_1(k)\sigma_3(2^n(p-k))$	$\frac{2}{21}(5 \cdot 2^{3n+5} - 13)$	$-\frac{8}{7}(2^{3n+3} - 1)$	$\frac{8}{21}(23 \cdot 2^{3n} - 2)$	$-\frac{2}{7}(2^{3n+3} - 1)$
$\sum_{k=1}^{p-1} \sigma_3(k)\sigma_{1,0}(2^n(p-k); 2)$	$\frac{4}{3}(2^{n+4} - 11)$	$-2^{n+3}$	$\frac{16}{3}(2^{n+1} - 1)$	$-2^{n+1}$
$\sum_{k=1}^{p-1} \sigma_{1,0}(k; 2)\sigma_3(2^n(p-k))$	$\frac{20}{21}(8^{n+1} - 1)$	$-\frac{8}{7}(8^{n+1} - 1)$	$\frac{16}{21}(8^{n+1} - 1)$	$-\frac{2}{7}(8^{n+1} - 1)$
$\sum_{k=1}^{p-1} \tilde{\sigma}_3(k)\tilde{\sigma}_1(2^n(p-k))$	$-10$	$2^{n+3}$	$-8$	$2^{n+1}$
$\sum_{k=1}^{p-1} \tilde{\sigma}_1(k)\tilde{\sigma}_3(2^n(p-k))$	$-10$	$-\frac{8}{7}(2^{3n+3} - 15)$	$\frac{8}{7}(3 \cdot 2^n - 10)$	$-\frac{2}{7}(2^{3n+3} - 15)$
$\sum_{k=1}^{p-1} \sigma_3(k)\sigma_{1,1}(2^n(p-k); 2)$	$\frac{22}{3}$	$0$	$\frac{8}{3}$	$0$

TABLE 5-2. Faulhaber sums  $(\sigma_{1,i}, \sigma_{1,j})$  with  $p = 2q + 1$ .

Convolution sum	$a$	$b$	$c$	$d$
$\sum_{k=1}^{p-1} \sigma_1(k)\sigma_1(2^n(p-k))$	$0$	$0$	$2(2^{n+3} - 3)$	$-2(3 \cdot 2^n - 1)$
$\sum_{k=1}^{p-1} \sigma_{1,0}(k; 2)\sigma_1(2^n(p-k))$	$0$	$0$	$4(2^{n+1} - 1)$	$-2(2^{n+1} - 1)$
$\sum_{k=1}^{p-1} \tilde{\sigma}_1(k)\sigma_1(2^n(p-k))$	$0$	$0$	$2$	$2(2^n - 1)$
$\sum_{k=1}^{p-1} \sigma_1(k)\tilde{\sigma}_1(2^n(p-k))$	$0$	$0$	$-2(2^{n+3} - 9)$	$6(2^n - 1)$
$\sum_{k=1}^{p-1} \sigma_{1,0}(k; 2)\tilde{\sigma}_1(2^n(p-k))$	$0$	$0$	$-4(2^{n+1} - 3)$	$2(2^{n+1} - 3)$
$\sum_{k=1}^{p-1} \tilde{\sigma}_1(k)\tilde{\sigma}_1(2^n(p-k))$	$0$	$0$	$-6$	$-2(2^n - 3)$
$\sum_{k=1}^{p-1} \sigma_{1,0}(k; 2)\sigma_{1,0}(2^n(p-k); 2)$	$0$	$0$	$8(2^n - 1)$	$-4(2^n - 1)$
$\sum_{k=1}^{p-1} \sigma_1(k; 2)\sigma_{1,0}(2^n(p-k); 2)$	$0$	$0$	$4(2^{n+2} - 3)$	$-2(3 \cdot 2^n - 2)$
$\sum_{k=1}^{p-1} \tilde{\sigma}_1(k; 2)\sigma_{1,0}(2^n(p-k); 2)$	$0$	$0$	$4$	$2(2^n - 2)$
$\sum_{k=1}^{p-1} \sigma_{1,0}(k; 2)\sigma_{1,1}(2^n(p-k); 2)$	$0$	$0$	$4$	$-2$
$\sum_{k=1}^{p-1} \sigma_1(k)\sigma_{1,1}(2^n(p-k); 2)$	$0$	$0$	$6$	$-2$
$\sum_{k=1}^{p-1} \tilde{\sigma}_1(k)\sigma_{1,1}(2^n(p-k); 2)$	$0$	$0$	$-2$	$2$

Remark 4.1. This remark pertains to the last two quantities appearing in Table 5-2. More precisely, we can write them as

$$(14) \quad \sum_{k=1}^{2q} \sigma_1(k)\sigma_{1,1}(2^n(2q+1-k); 2) = 6S_2(q) - 2S_1(q)$$

and

$$(15) \quad \sum_{k=1}^{2q} \tilde{\sigma}_1(k)\sigma_{1,1}(2^n(2q+1-k); 2) = -2S_2(q) + 2S_1(q).$$

Adding (14) and (15), we get

$$\sum_{k=1}^{2q} (\sigma_1(k) + \tilde{\sigma}_1(k))\sigma_{1,1}(2^n(2q+1-k); 2) = 4S_2(q).$$

Since  $\sigma_1(k) = \sigma_{1,1}(k; 2) + \sigma_{1,0}(k; 2)$ ,  $\tilde{\sigma}_1(k) = \sigma_{1,1}(k; 2) - \sigma_{1,0}(k; 2)$  and  $\sigma_{1,0}(k; 2) = 2\sigma_1(\frac{k}{2})$ , we derive that

$$\sum_{k=1}^{2q} \sigma_{1,1}(k; 2)\sigma_{1,1}(2^n(2q+1-k); 2) = 2S_2(q),$$

and thus

$$(16) \quad \sum_{k=1}^q \sigma_{1,1}(k; 2)\sigma_{1,1}(2q + 1 - k; 2) = S_2(q) = \sum_{k=1}^q k^2$$

since  $p$  is an odd prime number and  $\sigma_{1,1}(k; 2)\sigma_{1,1}(p-k; 2) = \sigma_{1,1}(p-k; 2)\sigma_{1,1}(p-(p-k); 2)$ , where  $n = 0$ . Similarly, from (14) and (15), we also have

$$(17) \quad \sum_{k=1}^{2q} \left( \sigma_1(k) - 3\sigma_1\left(\frac{k}{2}\right) \right) \sigma_{1,1}(2q + 1 - k; 2) = S_1(q) = \sum_{k=1}^q k.$$

Let

$$h_1(X) := \sum_{k=1}^{2X} \left( \sigma_1(k) - 3\sigma_1\left(\frac{k}{2}\right) \right) \sigma_{1,1}(2X + 1 - k; 2),$$

$$h_2(X) := \sum_{k=1}^X \sigma_{1,1}(k; 2)\sigma_{1,1}(2X + 1 - k; 2),$$

$$S_1(X) := \sum_{k=1}^X k \quad \text{and} \quad S_2(X) := \sum_{k=1}^X k^2.$$

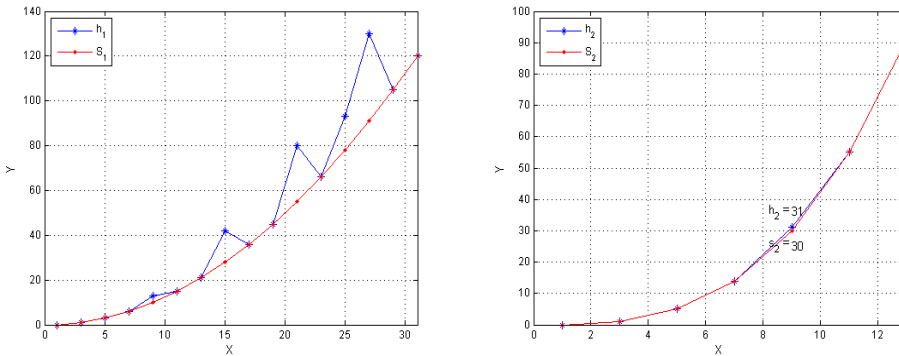


FIGURE 1.  $h_i(X)$  and  $S_i(X)(i = 1, 2)$

In Figure 1, the graphs of the functions  $(h_1(X), S_1(X))$  and  $(h_2(X), S_2(X))$  are drawn and it is easily seen that  $h_1(X) = S_1(X)$  and  $h_2(X) = S_2(X)$  when  $2X + 1$  is a prime number. Similar result on consecutive integers is in [6, Remark 3.3].

**5. Applications to Weierstrass  $\wp$ -function and some linear combinations of Eisenstein Series**

Let  $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$  ( $\tau \in \mathcal{H}$ , the complex upper-half plane) be a lattice and  $z \in \mathbb{C}$ . The Weierstrass  $\wp$ -function relative to  $\Lambda_\tau$  is defined by the series

$$\wp(z; \Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \left\{ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right\},$$

and the Eisenstein series of weight  $2k$  for  $\Lambda_\tau$  with  $k > 1$  is defined by

$$G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \omega^{-2k}.$$

We use the notations  $\wp(z)$  and  $G_{2k}$  instead of  $\wp(z; \Lambda_\tau)$  and  $G_{2k}(\Lambda_\tau)$ , respectively, when the lattice  $\Lambda_\tau$  has been fixed. Now, Laurent series for  $\wp(z)$  about  $z = 0$  is given by

$$\wp(z) := \wp(z; \Lambda_\tau) = z^{-2} + \sum_{k=1}^{\infty} (2k + 1)G_{2k+2}z^{2k}.$$

As is customary, by setting

$$g_2(\tau) = g_2(\Lambda_\tau) = 60G_4 \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140G_6,$$

the algebraic relation between  $\wp(z)$  and  $\wp'(z)$  is given by the equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau).$$

With  $q = e^{2\pi i\tau}$ , the normalized Eisenstein series  $E_{2k}(\tau)$  is given by (see [13, p. 58])

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{N=1}^{\infty} \sigma_{2k-1}(N)q^N,$$

where  $B_{2k}$  denotes the  $2k$ -th Bernoulli number. We define

$$E_{2k}^*(\tau) := E_{2k}(\tau) - E_{2k}(2\tau).$$

The aim of this section is to establish the following:

**Theorem 5.1.**

$$E_6^*(\tau) = \frac{189}{4\pi^6} \wp\left(\frac{\tau}{2}; \Lambda_\tau\right) \wp''\left(\frac{\tau}{2}; \Lambda_\tau\right).$$

To prove this theorem, we need the following lemmas.

**Lemma 5.2.** *Let  $p$  be a prime number. Then we have*

$$\sigma_s^*(pN; p) = p^s \sigma_s^*(N; p).$$

*Proof.* Let  $rl = pN$  and  $l \not\equiv 0 \pmod{p}$  for some  $r, l \in \mathbb{N}$  and prime  $p$ . Since  $l$  is not divisible by  $p$ , we can write  $r = pd$  for some  $d \in \mathbb{N}$ . Therefore, we have

$$\begin{aligned}
 \sigma_s^*(pN; p) &= \sum_{\substack{r|pN, \\ \frac{pN}{r} \not\equiv 0 \pmod{p}}} r^s = \sum_{\substack{d|N, \\ \frac{N}{d} \not\equiv 0 \pmod{p}}} (pd)^s \\
 &= p^s \sum_{\substack{d|N, \\ \frac{N}{d} \not\equiv 0 \pmod{p}}} d^s = p^s \sigma_s^*(N; p).
 \end{aligned}
 \tag{18}$$

□

**Lemma 5.3.** *Let  $N \geq 2$  be an integer and  $p$  be a prime. Then we have*

$$\sum_{k=1}^{N-1} \sigma_3^*(k) \sigma_{1,1}(N-k; 2) = \frac{1}{24} \left( \sigma_5^*(N) - \sigma_3^*(N) \right).$$

*Proof.* In Lemma 2.3(g), we consider the case when  $m = 1, n = 0$ . Then

$$\begin{aligned}
 A_4(1, 0; N) &= \sum_{k=1}^{N-1} \sigma_3^*(2k) \sigma_{1,1}(N-k; 2) \\
 &= 8 \sum_{k=1}^{N-1} \sigma_3^*(k) \sigma_{1,1}(N-k; 2) \\
 &= \frac{1}{3} \left( \sigma_5^*(N) - \sigma_3^*(N) \right)
 \end{aligned}
 \tag{19}$$

by Lemma 5.2. □

Let

$$(a; t)_\infty := \prod_{n=0}^{\infty} (1 - at^n)$$

and

$$(q)_\infty := (q; q)_\infty = \prod_{n=1}^{\infty} (1 - q^n).$$

From [7, (5), (28)], we see that

$$\begin{aligned}
 \wp\left(\frac{\tau}{2}; \Lambda_\tau\right) &= -\frac{\pi^2}{3} \left[ \frac{(q^2; q^2)_\infty^{20}}{(q)_\infty^8 (q^4; q^4)_\infty^8} + 16 \frac{q(q^4; q^4)_\infty^8}{(q^2; q^2)_\infty^4} \right] \\
 &= -\frac{\pi^2}{3} \left[ 1 + 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) q^N \right]
 \end{aligned}
 \tag{20}$$

and

$$\begin{aligned}
 \wp''\left(\frac{\tau}{2}; \Lambda_\tau\right) &= 32\pi^4 q \frac{(q^2; q^2)_\infty^{16}}{(q)_\infty^8} \\
 (21) \qquad &= 32\pi^4 \left[ \sum_{N=1}^\infty \sigma_3(N)q^N - \sum_{N=1}^\infty \sigma_3(N)q^{2N} \right] \\
 &= 32\pi^4 \sum_{N=0}^\infty \sigma_3^*(N)q^N
 \end{aligned}$$

since  $\sigma_3^*(0) = 0$ .

*Proof of Theorem 5.1.* From (20), we note that  $\sigma_{1,1}(0; 2) = \frac{1}{24}$ . Therefore, we see that

$$(22) \qquad -\frac{1}{8\pi^2} \wp\left(\frac{\tau}{2}; \Lambda_\tau\right) = \sum_{k=0}^\infty \sigma_{1,1}(k; 2)q^k.$$

From (19), we observe that

$$\sum_{k=1}^{N-1} \sigma_3^*(k)\sigma_{1,1}(N-k; 2) = \frac{1}{24} \left( \sigma_5^*(N) - \sigma_3^*(N) \right),$$

i.e.,

$$\begin{aligned}
 &\sum_{k=0}^N \sigma_3^*(k)\sigma_{1,1}(N-k; 2) \\
 &= \frac{1}{24} \sigma_5^*(N) - \frac{1}{24} \sigma_3^*(N) + \sigma_3^*(0)\sigma_{1,1}(N; 2) + \sigma_3^*(N)\sigma_{1,1}(0; 2) \\
 &= \frac{1}{24} \sigma_5^*(N).
 \end{aligned}$$

Since  $\sigma_{1,1}(0; 2) = \frac{1}{24}$  and  $\sigma_3^*(0) = 0$ , we have

$$(23) \qquad \sum_{k=0}^N \sigma_3^*(k)\sigma_{1,1}(N-k; 2) = \frac{1}{24} \sigma_5^*(N).$$

Therefore, (with  $B_6 = \frac{1}{42}$ ) we have

$$\begin{aligned}
 E_6^*(\tau) &= E_6(\tau) - E_6(2\tau) \\
 &= -\frac{12}{B_6} \sum_{N=1}^\infty \sigma_5^*(N)q^N \\
 &= -\frac{12}{B_6} \sum_{N=1}^\infty \left( 24 \sum_{k=0}^N \sigma_3^*(N)\sigma_{1,1}(N-k; 2) \right) q^N \\
 &= -\frac{12}{B_6} \cdot 24 \left( -\frac{1}{8\pi^2} \wp\left(\frac{\tau}{2}; \Lambda_\tau\right) \right) \left( \frac{1}{32\pi^4} \wp''\left(\frac{\tau}{2}; \Lambda_\tau\right) \right)
 \end{aligned}$$

$$= \frac{189}{4\pi^6} \wp\left(\frac{\tau}{2}; \Lambda_\tau\right) \wp''\left(\frac{\tau}{2}; \Lambda_\tau\right)$$

by (21) and (22). This completes the proof of the theorem. □

**Corollary 5.4.** *We have*

$$E_6^*(\tau) = -504 \left( q \frac{(q^2; q^2)_\infty^{36}}{(q)_\infty^{16} (q^4; q^4)_\infty^8} + 16q^2 \frac{(q^2; q^2)_\infty^{12} (q^4; q^4)_\infty^8}{(q)_\infty^8} \right).$$

Corollary follows from the  $q$ -expansion of  $\wp(\frac{\tau}{2}; \Lambda_\tau)$ ,  $\wp''(\frac{\tau}{2}; \Lambda_\tau)$  along with the theorem.

*Remark 5.5.* This product formula for  $E_6^*(\tau)$  may be compared with the formula  $g_3(\tau) = (2\pi)^6 \frac{1}{23 \cdot 3^3} E_6(\tau)$  of [8, (1.5)].

**Theorem 5.6.** *Let  $q = e^{2\pi i\tau}$  and write  $-\frac{\pi^2}{3} \cdot \frac{1}{\wp(\frac{\tau}{2})} := \sum_{N=0}^\infty \beta_N q^N$ . Then the coefficients  $\beta_p$  satisfy the congruence relation*

$$\beta_p \equiv 0 \pmod{3^4}$$

for any prime number  $p \equiv -1 \pmod{9}$ . Other cases are given in Table 5.

TABLE 5. Values  $\beta_p \pmod{3^4}$

$p$ prime number	$\beta_p \pmod{3^4}$	$p$ prime number	$\beta_p \pmod{3^4}$
$p \equiv 1 \pmod{9}$	33	$p \equiv 4 \pmod{9}$	6
$p \equiv 2 \pmod{9}$	54	$p \equiv 5 \pmod{9}$	27
$p = 3$	30	$p \equiv 7 \pmod{9}$	60

*Proof.* From [9, Theorem 2.3] or [7, (5), (28)], we see that the Weierstrass  $\wp$ -function satisfies the equation (with  $q = e^{2\pi i\tau}$ )

$$-\frac{\pi^2}{3} \wp\left(\frac{\tau}{2}\right) = 1 + 24 \sum_{n=1}^\infty \sigma_{1,1}(n; 2) q^n.$$

Therefore, for sufficiently small  $|q|$ , writing the geometric series expansion for  $-\frac{\pi^2}{3} \cdot \frac{1}{\wp(\frac{\tau}{2})}$ , we obtain

$$\begin{aligned} & -\frac{\pi^2}{3} \cdot \frac{1}{\wp\left(\frac{\tau}{2}\right)} \\ &= \frac{1}{1 + 24 \sum_{N=1}^\infty \sigma_{1,1}(N; 2) q^N} \\ &= 1 - 24 \sum_{N=1}^\infty \sigma_{1,1}(N; 2) q^N + 24^2 \sum_{N=2}^\infty \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_{1,1}(N - k; 2) q^N \end{aligned}$$

$$- 24^3 \sum_{N=3}^{\infty} \sum_{k=2}^{N-1} \sum_{k_1=1}^{k-1} \sigma_{1,1}(k_1; 2)\sigma_{1,1}(k - k_1; 2)\sigma_{1,1}(N - k; 2)q^N + \dots$$

Hence

$$\begin{aligned} \sum_{N=0}^{\infty} \beta_N q^N &\equiv 1 - 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2)q^N \\ &\quad + 24^2 \sum_{N=2}^{\infty} \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_{1,1}(N - k; 2)q^N \\ &\quad - 24^3 \sum_{N=3}^{\infty} \sum_{k=2}^{N-1} \sum_{k_1=1}^{k-1} \sigma_{1,1}(k_1; 2)\sigma_{1,1}(k - k_1; 2)\sigma_{1,1}(N - k; 2)q^N \\ &\pmod{3^4}. \end{aligned}$$

Comparing the coefficients, we get

$$\begin{aligned} \beta_N &\equiv -24\sigma_{1,1}(N; 2) + 24^2 \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_{1,1}(N - k; 2) \\ &\quad - 24^3 \sum_{k=2}^{N-1} \sum_{k_1=1}^{k-1} \sigma_{1,1}(k_1; 2)\sigma_{1,1}(k - k_1; 2)\sigma_{1,1}(N - k; 2) \pmod{3^4} \end{aligned}$$

with  $N \geq 3$ . From [6, (11), Theorem 3.7], we observe that

$$\sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_{1,1}(N - k; 2) = \frac{1}{24} [11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N; 2)]$$

and

$$\begin{aligned} &\sum_{\substack{k+l+h=N \\ k,l,h>0}} \sigma_{1,1}(k; 2)\sigma_{1,1}(l; 2)\sigma_{1,1}(h; 2) \\ &= -\frac{29}{768}\sigma_5(N) + \frac{1}{768}\sigma_5(2N) - \frac{11}{192}\sigma_3(N) + \frac{1}{192}\sigma_3(2N) + \frac{1}{192}\sigma_{1,1}(N; 2). \end{aligned}$$

Therefore, when  $N$  is an odd prime  $p$ , we have

$$\beta_p \equiv -24(3p^5 - 8p^3 + 6p + 1) \pmod{3^4}.$$

It is easy to check and conclude Theorem 5.6 for each prime  $p \equiv i \pmod{9}$  ( $i = 1, \dots, 9$ ) with  $(i, 9) = 1$ . □

*Remark 5.7.* For comparison, in [2, Theorem 4.1], we see that

$$\alpha_n \equiv 0 \pmod{3^4} \quad \text{for } n \equiv 2 \pmod{3}.$$

However, in our case, if the prime  $p \equiv 8 \pmod{9}$ , then  $\alpha_p \equiv \beta_p \pmod{3^4}$  and if the prime  $p \equiv 2, 5 \pmod{9}$ , then  $\alpha_p \not\equiv \beta_p \pmod{3^4}$ .

**Acknowledgements.** The author Ayyadurai Sankaranarayanan wishes to thank the National Institute for Mathematical Sciences (NIMS), Daejeon, Republic of Korea for its warm hospitality and generous support. This research was supported by the National Institute for Mathematical Science (NIMS) grant funded by the Korean government (B21303).

### References

- [1] B. C. Berndt, *Ramanujan's Notebooks. Part II*, Springer-Verlag, New York, 1989.
- [2] B. C. Berndt and A. J. Yee, *Congruences for the coefficients of quotients of Eisenstein series*, Acta Arith. **104** (2002), no. 3, 297–308.
- [3] J. W. L. Glaisher, *On the square of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **14** (1884), 156–163.
- [4] H. Hahn, *Convolution sums of some functions on divisors*, Rocky Mountain J. Math. **37** (2007), no. 5, 1593–1622.
- [5] J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams, *Elementary evaluation of certain convolution sums involving divisor functions*, Number theory for the millennium, II (Urbana, IL, 2000), 229–274, A K Peters, Natick, MA, 2002.
- [6] D. Kim, A. Kim, and Y. Li, *Convolution sums arising from the divisor functions*, J. Korean Math. Soc. **50** (2013), no. 2, 331–360.
- [7] D. Kim, A. Kim, and H. Park, *Congruences of the Weierstrass  $\wp(x)$  and  $\wp''(x)$  ( $x = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}$ ) - functions on divisors*, Bull. Korean Math. Soc. **50** (2013), no. 1, 241–261.
- [8] D. Kim and J. K. Koo, *Algebraic integers as values of elliptic functions*, Acta Arith. **100** (2001), no. 2, 105–116.
- [9] D. Kim and M. Kim, *Divisor functions and Weierstrass functions arising from  $q$ -series*, Bull. Korean Math. Soc. **49** (2012), no. 4, 693–704.
- [10] G. Melfi, *On some modular identities*, Number theory (Eger, 1996), 371–382, de Gruyter, Berlin, 1998.
- [11] S. Ramanujan, *Collected Papers of Srinivasa Ramanujan*, AMS Chelsea, Providence, 2000.
- [12] S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), no. 9, 159–184.
- [13] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer-Verlag, 1994.
- [14] K. S. Williams, *Number Theory in the Spirit of Liouville*, London Mathematical Society, Student Texts 76, Cambridge, 2011.

DAEYEOL KIM  
 NATIONAL INSTITUTE FOR MATHEMATICAL SCIENCES  
 DAEJEON 305-811, KOREA  
*E-mail address:* daeyeoul@nims.re.kr

AERAN KIM  
 DEPARTMENT OF MATHEMATICS AND INSTITUTE OF PURE AND APPLIED MATHEMATICS  
 CHONBUK NATIONAL UNIVERSITY  
 CHONBUK 561-756, KOREA  
*E-mail address:* ae\_ran\_kim@hotmail.com



AYYADURAI SANKARANARAYANAN  
SCHOOL OF MATHEMATICS  
TATA INSTITUTE OF FUNDAMENTAL RESEARCH  
HOMI BHABHA ROAD  
MUMBAI 400005, INDIA  
*E-mail address:* `sank@nims.re.kr`; `sank@math.tifr.res.in`