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# QUASICONFORMAL EXTENSIONS OF STARLIKE HARMONIC MAPPINGS IN THE UNIT DISC

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Abstract. Let $f$ be a harmonic mapping on the unit disc $\Delta$ in $\mathbb{C}$. We give some condition for $f$ to be a quasiconformal homeomorphism on $\Delta$ and to have a quasiconformal extension to the whole plane $\overline{\mathbb{C}}$. We also obtain quasiconformal extension results for starlike harmonic mappings of order $\alpha \in (0, 1)$.

## 1. Introduction

Let $f$ be a complex-valued function of class $C^1$ on $\Delta = \{z \in \mathbb{C}; |z| < 1\}$. The Jacobian of $f$ is given by $J_f(z) = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \overline{z}}\right|^2 = |f_z|^2 - |f_{\overline{z}}|^2$. Lewy [15] proved that if a harmonic mapping $f$ on $\Delta$ is locally univalent, then $J_f(z) \neq 0$ in $\Delta$. Thus a locally univalent harmonic mapping is either sense-preserving (if $J_f(z) > 0$ in $\Delta$) or sense-reversing (if $J_f(z) < 0$ in $\Delta$). A harmonic mapping of $\Delta$ has the unique representation $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\Delta$ and $g(0) = 0$. Note that $f$ is sense-preserving if and only if $|g'(z)| < |h'(z)|$ for all $z \in \Delta$ (For univalent harmonic mappings, see [5]).

Let $f = h + \overline{g}$ be a harmonic mapping of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

Recently, many mathematicians have studied about holomorphic or harmonic mappings of the above form by certain coefficient conditions. When $f$ is holomorphic, Fait, Krzyż and Zygmunt [6] gave a sufficient coefficient condition for $f$ to be a quasiconformal homeomorphism on $\Delta$ and to have a quasiconformal extension to the extended plane $\overline{\mathbb{C}}$ (see also Brodskiĭ [3], Curt, Kohr and Kohr [4], Graham, Hamada and Kohr [9], Hamada and Kohr [10], [11],

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[12]). When $f$ is harmonic, Avcı and Złotkiewicz [2], Silverman [17] gave a sufficient coefficient condition for $f$ to be univalent, sense-preserving and starlike when $b_1 = 0$. Jahangiri [13] generalized the result to the case that $b_1$ is not necessarily 0. He gave a sufficient coefficient condition for $f$ to be univalent, sense-preserving and starlike of order $\alpha \in [0, 1)$ when $b_1$ is not necessarily 0 (Theorem 2.1). He also showed that the condition is also necessary when $h$ has negative and $g$ has positive coefficients (Theorem 2.2). Ganczar [8] gave a sufficient coefficient condition for $f$ to be a quasiconformal homeomorphism on $\Delta$ and to have a quasiconformal extension to $\overline{\mathbb{C}}$ when $b_1 = 0$. Then the following natural questions arise:

**Question 1.1.** Can we give a sufficient coefficient condition for a harmonic mapping $f$ to be a quasiconformal homeomorphism on $\Delta$ and to have a quasiconformal extension to $\overline{\mathbb{C}}$ when $b_1$ is not necessarily 0?

**Question 1.2.** Can we give a sufficient coefficient condition for a starlike harmonic mapping $f$ of order $\alpha \in [0, 1)$ to be a quasiconformal homeomorphism on $\Delta$ and to have a quasiconformal extension to $\overline{\mathbb{C}}$ when $b_1$ is not necessarily 0?

In the present paper, we will give affirmative answers to the above questions. Namely, we consider the condition for a harmonic mapping $f = h + \overline{g}$ of the form (1.1) to be a quasiconformal homeomorphism on $\Delta$ and to have a quasiconformal extension $F$ to $\overline{\mathbb{C}}$ when $|b_1| < 1$. When $b_1 = 0$, our result also gives an improvement of the estimate of the complex dilatation $\mu_F$ given by Ganczar [8]. We also obtain quasiconformal extension results for starlike harmonic mappings of order $\alpha \in (0, 1)$ and give a counterexample when $\alpha = 0$.

First, we give an estimate of the complex dilatation by the coefficients of the harmonic mapping $f$. Next, we show that $f$ has a homeomorphic extension $\hat{f}$ on $\overline{\Delta}$ such that the curve $\hat{f}(\mathbb{T})$ is a quasicircle. Finally, we give an explicit mapping $F$ which is a quasiconformal extension of $f$ onto $\overline{\mathbb{C}}$. As a corollary, we obtain quasiconformal extension results for starlike harmonic mappings of order $\alpha \in (0, 1)$. We also give a counterexample when $\alpha = 0$.

## 2. Notation and preliminaries

First, we give the analytic definition of quasiconformality (cf. [14]).

**Definition.** Let $f : G \to \overline{\mathbb{C}}$ be a sense-preserving homeomorphism of the domain $G$ in $\overline{\mathbb{C}}$. We say that $f$ is a *$K$-quasiconformal mapping* of $G$ if $f$ satisfies the following two conditions:

1. $f$ is absolutely continuous on lines in $G$.
2. The dilatation condition

$$\max_{\alpha} |\partial_\alpha f(z)| \leq K \min_{\alpha} |\partial_\alpha f(z)| \tag{2.1}$$

holds almost everywhere in $G$, where $K \geq 1$ and $\partial_\alpha f(z) = f_z(z) + e^{-2i\alpha} f_{\overline{z}}(z)$.

When the above conditions are satisfied for some $K \geq 1$, we say that $f$ is *quasiconformal*.

Let $f$ be a sense-preserving homeomorphism $f$ on $G$ which is absolutely continuous on lines. Then there exists a null set $N$ in $G$ such that $f$ is differentiable at $z \in G \setminus N$. We set

$$\mu_f(z) = \frac{f_{\overline{z}}(z)}{f_z(z)}, \quad z \in G \setminus N.$$

It is called *the complex dilatation* of $f$. Since the condition (2.1) is equivalent to the condition

$$|\mu_f(z)| \leq \frac{K-1}{K+1},$$

a sense-preserving homeomorphism $f$ which is absolutely continuous on lines is quasiconformal if and only if

$$\sup_{z \in G \setminus N} |\mu_f(z)| < 1.$$

Let $f$ be a sense-preserving harmonic mapping $f$ on $\Delta$. The function

$$\omega_f = \frac{g'}{h'} = \frac{\overline{f_{\overline{z}}}}{f_z}$$

is analytic and satisfies $|\omega_f(z)| = |\mu_f(z)| < 1$ on $\Delta$. We call $\omega_f$ *the second complex dilatation of* $f$ and set

$$\|\omega_f\|_\infty = \sup_{z \in \Delta} |\omega_f(z)| = \sup_{z \in \Delta} \left| \frac{g'(z)}{h'(z)} \right|.$$

A sense-preserving harmonic homeomorphism $f$ on $\Delta$ is quasiconformal if and only if $\|\omega_f\|_\infty < 1$.

Recall that, a Jordan curve $\Gamma$ in $\overline{\mathbb{C}}$ is called a *quasicircle* if it is the image line of a circle under a quasiconformal automorphism of $\overline{\mathbb{C}}$. According to Ahlfors [1], the Jordan curve $\gamma$ on $\overline{\mathbb{C}}$ is a quasicircle if and only if

$$K(\gamma) = \sup \frac{|w_1 - w_2| \cdot |w_3 - w_4| + |w_1 - w_4| \cdot |w_2 - w_3|}{|w_1 - w_3| \cdot |w_2 - w_4|} \tag{2.2}$$

is finite, where supremum is taken over the set of all ordered quadruples $\{w_1, w_2, w_3, w_4\}$ of points on $\gamma$. A sufficient condition for a quasiconformal mapping $f$ on $\Delta$ to have a quasiconformal extension to $\overline{\mathbb{C}}$ is that the boundary curve of the image $f(\Delta)$ is a quasicircle (see [14, Theorem 8.3]).

For $0 \leq \alpha < 1$, a harmonic mapping $f$ of the form (1.1) is said to be starlike of order $\alpha$, if

$$\frac{\partial}{\partial \theta}(\arg f(re^{i\theta})) \geq \alpha, \quad |z| = r < 1.$$

Jahangiri [13, Theorems 1 and 2] gave the following criterions for harmonic mappings to be starlike of order $\alpha \in [0, 1)$.

**Theorem 2.1.** *Let $f = h+\overline{g}$ be a harmonic mapping of the form* (1.1)*. Assume that*

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 2, \tag{2.3}$$

*where $a_1 = 1$ and $0 \leq \alpha < 1$. Then $f$ is harmonic univalnt in $\Delta$ and starlike of order $\alpha$.*

**Theorem 2.2.** *Let $f = h + \overline{g}$ be a harmonic mapping of the form*

$$h(z) = z - \sum_{n=2}^{\infty} |a_n|z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n|z^n. \tag{2.4}$$

*Then $f$ is starlike of order $\alpha \in [0, 1)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 2,$$

*where $a_1 = 1$.*

## 3. Quasiconformal extension

In this section, we obtain generalizations of results on homeomorphic extension and quasiconformal extension of harmonic mappings to the case that $b_1$ is not necessarily 0.

For a sequence $\{\psi_n\}_{n=2,3,...}$ of positive real numbers $\psi_n$, we denote by $H(\psi_n)$ the set of harmonic mappings $f = h + \overline{g}$ of the form (1.1) that satisfy the conditions $|b_1| < 1$ and

$$|b_1| + \sum_{n=2}^{\infty} \psi_n(|a_n| + |b_n|) \leq 1. \tag{3.1}$$

When $\psi_2 < 2$, $f \in H(\psi_n)$ need not be univalent on $\Delta$ as the following example shows (cf. [7, Theorem 3]).

**Example 3.1.** Let

$$f(z) = z + b_1\overline{z} + i\frac{1-b_1}{2^p}\overline{z}^2,$$

where $0 < b_1 < 1$ and $0 < p < 1$. Then $f \in H(\psi_n)$ with $\psi_2 = 2^p < 2$. Since

$$f(ix) = i(1-b_1)x - i\frac{1-b_1}{2^p}x^2,$$

we have

$$f(ix) - f(iy) = i(1-b_1)(x-y)\left(1 - \frac{x+y}{2^p}\right).$$

This implies that $f(ix) = f(iy)$ for $x, y$ with $x + y = 2^p < 2$. Thus, $f$ is not univalent on $\Delta$.

So, we will restrict our attention to the case $\psi_2 \geq 2$. In this case, if

$$\frac{\psi_n}{n} \geq \frac{\psi_2}{2}$$

for $n \geq 3$, we have $H(\psi_n) \subset H(n)$. Thus, we obtain the following result from Theorem 2.1.

**Proposition 3.2.** *Let $\{\psi_n\}_{n=2,3,\dots}$ be a sequence of positive real numbers with the condition*

$$\frac{\psi_n}{n} \geq \frac{\psi_2}{2}$$

*for $n \geq 3$ and let $f = h + \overline{g} \in H(\psi_n)$. If $\psi_2 \geq 2$, then $f$ is a sense-preserving, univalent and harmonic mapping onto a starlike domain.*

The following lemma is a generalization of [8, Lemma 3] to the case that $b_1$ is not necessarily 0. We remark that the estimate of the second complex dilatation $\omega_f$ given in the following lemma is an improvement of that given in [8, Lemma 3]. Also, our proof is more simple than that in [8, Lemma 3].

**Lemma 3.3.** *Let $\{\psi_n\}_{n=2,3,\dots}$ be a sequence of positive real numbers with the condition*

$$\frac{\psi_n}{n} \geq \frac{\psi_2}{2}$$

*for $n \geq 3$ and let $f = h + \overline{g} \in H(\psi_n)$. If $\psi_2 > 2$, then*

$$\|\omega_f\|_\infty \leq |b_1| + \frac{2}{\psi_2}(1 - |b_1|) < 1. \tag{3.2}$$

*Proof.* From (1.1), we have

$$|\omega_f(z)| = \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{\sum_{n=1}^{\infty} n\,|b_n|}{1 - \sum_{n=2}^{\infty} n\,|a_n|}$$

for $z \in \Delta$. We set

$$t = \sum_{n=2}^{\infty} n\,|a_n| \quad \text{and} \quad k = \sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n|\,.$$

Then, from the condition (3.1), we obtain

$$k = |b_1| + \frac{2}{\psi_2}\sum_{n=2}^{\infty} \frac{\psi_2}{2} n\,(|a_n| + |b_n|) \leq |b_1| + \frac{2}{\psi_2}(1 - |b_1|) < 1, \tag{3.3}$$

where the last inequality follows from the assumption that $\psi_2 > 2$. Also, we have

$$\sum_{n=1}^{\infty} n\,|b_n| = k - t \geq 0.$$

It follows from this equality and the fact that the function

$$1-\frac{1-k}{1-t}$$

is monotone decreasing for $t \in [0,1)$ that

$$|\omega_f(z)| \leq \frac{k-t}{1-t} = 1-\frac{1-k}{1-t} \leq 1-\frac{1-k}{1-0} = k \tag{3.4}$$

for $z \in \Delta$. From (3.3) and (3.4), we obtain (3.2). □

**Corollary 3.4.** *Let $\{\psi_n\}_{n=2,3,\dots}$ be a sequence of positive real numbers with the condition*

$$\frac{\psi_n}{n} \geq \frac{\psi_2}{2}$$

*for $n \geq 3$ and let $f = h + \overline{g} \in H(\psi_n)$. If $\psi_2 > 2$, then $f$ is a quasiconformal homeomorphism on $\Delta$ such that the complex dilatation $\mu_f$ satisfies*

$$|\mu_f(z)| \leq k = \sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n| < 1.$$

*Proof.* Since $k < 1$, $f$ is a sense-preserving homeomorphism on $\Delta$ by Proposition 3.2. Thus $f$ is quasiconformal and $|\mu_f(z)| = |\omega_f(z)| \leq k$ from Lemma 3.3. □

**Theorem 3.5.** *Let $\{\psi_n\}_{n=2,3,\dots}$ be a sequence of positive real numbers with the condition*

$$\frac{\psi_n}{n} \geq \frac{\psi_2}{2}$$

*for $n \geq 3$ and let $f = h + \overline{g} \in H(\psi_n)$. Let $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$. If $\psi_2 > 2$, then $f$ has a homeomorphic extension $\hat{f}$ on $\overline{\Delta}$ such that the curve $\hat{f}(\mathbb{T})$ is a quasicircle.*

*Proof.* By the proof of Lemma 3.3, we have

$$k = \sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n| < 1.$$

Thus, for $z_1, z_2 \in \Delta$ with $z_1 \neq z_2$, we obtain

$$\begin{aligned}
|f(z_1) - f(z_2)| &= \left| z_1 - z_2 + \sum_{n=2}^{\infty} a_n(z_1^n - z_2^n) + \overline{\sum_{n=1}^{\infty} b_n(z_1^n - z_2^n)} \right| \\
&\leq |z_1 - z_2| \left(1 + \sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n|\right) \\
&\leq (1+k)\,|z_1 - z_2|
\end{aligned}$$

and

$$|f(z_1) - f(z_2)| \quad \geq \quad (1-k)\,|z_1 - z_2| > 0.$$

It follows from these inequalities that $f$ has a homeomorphic extension $\hat{f}$ to $\overline{\Delta}$ such that

$$(1-k)\,|z_1-z_2| \le \left|\hat{f}(z_1)-\hat{f}(z_2)\right| \le (1+k)\,|z_1-z_2| \tag{3.5}$$

for $z_1, z_2 \in \overline{\Delta}$. Therefore the image $\hat{f}(\mathbb{T})$ is a Jordan curve.

From (2.2) and (3.5), we have

$$\begin{aligned} K(\hat{f}(\mathbb{T})) &= \sup\left\{ \frac{\left|\hat{f}(w_1)-\hat{f}(w_2)\right| \cdot \left|\hat{f}(w_3)-\hat{f}(w_4)\right|}{\left|\hat{f}(w_1)-\hat{f}(w_3)\right| \cdot \left|\hat{f}(w_2)-\hat{f}(w_4)\right|} \right. \\ &\qquad \left. + \frac{\left|\hat{f}(w_1)-\hat{f}(w_4)\right| \cdot \left|\hat{f}(w_2)-\hat{f}(w_3)\right|}{\left|\hat{f}(w_1)-\hat{f}(w_3)\right| \cdot \left|\hat{f}(w_2)-\hat{f}(w_4)\right|} \right\} \\ &\le \left(\frac{1+k}{1-k}\right)^2 K(\mathbb{T}). \end{aligned}$$

Since $\mathbb{T}$ is a quasicircle, $K(\mathbb{T}) < \infty$, that is, $K(\hat{f}(\mathbb{T})) < \infty$. Thus $\hat{f}(\mathbb{T})$ is a quasicircle. □

**Theorem 3.6.** *Let $\{\psi_n\}_{n=2,3,\ldots}$ be a sequence of positive real numbers with the condition*

$$\frac{\psi_n}{n} \ge \frac{\psi_2}{2}$$

*for $n \ge 3$ and let $f = h + \overline{g} \in H(\psi_n)$. If $\psi_2 > 2$, then the mapping*

$$F(z) = \begin{cases} f(z) & \text{for } |z|<1, \\ z + \displaystyle\sum_{n=2}^{\infty} a_n \overline{z^{-n}} + \sum_{n=1}^{\infty} \overline{b_n} z^{-n} & \text{for } |z| \ge 1, \end{cases} \tag{3.6}$$

*is a quasiconformal extension of $f$ onto $\overline{\mathbb{C}}$ such that the complex dilatation $\mu_F$ satisfies*

$$|\mu_F(z)| \le k = \sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n| < 1.$$

*Proof.* By Corollary 3.4, $f$ is quasiconformal on $\Delta$ and $|\mu_F(z)| = |\mu_f(z)| \le k < 1$ for $z \in \Delta$.

First, we will show the inequalities

$$(1-k)|z_1-z_2| \le |F(z_1)-F(z_2)| \le (1+k)|z_1-z_2| \text{ for } z_1, z_2 \in \mathbb{C}. \tag{3.7}$$

If $z_1, z_2 \in \overline{\Delta}$, we obtain (3.7) from (3.5). For $z_1, z_2 \in \mathbb{C} \setminus \Delta$ with $z_1 \ne z_2$, we have

$$|F(z_1)-F(z_2)| \le |z_1-z_2| + \left|z_1^{-1}-z_2^{-1}\right| \left( \sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n| \right)$$

$$\leq |z_1 - z_2| \left(1 + \sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n|\right)$$
$$= (1+k)\,|z_1 - z_2|$$

and

$$|F(z_1) - F(z_2)| \geq |z_1 - z_2| \left(1 - \frac{\displaystyle\sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n|}{|z_1 z_2|}\right)$$
$$\geq (1-k)\,|z_1 - z_2| > 0.$$

Let $G(z) = F(z) - z$. By the above argument, we have

$$|G(w_1) - G(w_2)| \leq k|w_1 - w_2|,$$

if $w_1, w_2 \in \overline{\Delta}$ or $w_1, w_2 \in \mathbb{C} \setminus \Delta$. We consider the case $z_1 \in \Delta$ and $z_2 \in \mathbb{C} \setminus \overline{\Delta}$. Let $z_3 \in \mathbb{T} \cap [z_1, z_2]$, where $[z_1, z_2]$ is the line segment from $z_1$ to $z_2$. Then we have

$$\begin{aligned}|F(z_1) - F(z_2)| &\leq |z_1 - z_2| + |G(z_1) - G(z_2)| \\ &\leq |z_1 - z_2| + |G(z_1) - G(z_3)| + |G(z_3) - G(z_2)| \\ &\leq |z_1 - z_2| + k(|z_1 - z_3| + |z_3 - z_2|) \\ &= (1+k)|z_1 - z_2|\end{aligned}$$

and

$$\begin{aligned}|F(z_1) - F(z_2)| &\geq |z_1 - z_2| - |G(z_1) - G(z_2)| \\ &\geq |z_1 - z_2| - (|G(z_1) - G(z_3)| + |G(z_3) - G(z_2)|) \\ &\geq |z_1 - z_2| - k(|z_1 - z_3| + |z_3 - z_2|) \\ &= (1-k)|z_1 - z_2|.\end{aligned}$$

Thus, we obtain (3.7) for all $z_1, z_2 \in \mathbb{C}$. This implies that $F$ is Lipschitz continuous and univalent on $\mathbb{C}$ and $\lim_{z\to\infty} F(z) = \infty$. By [16, Theorem 2.7.2] and (3.7), $F$ is a sense-preserving homeomorphism of $\overline{\mathbb{C}}$ onto itself.

For $z \in \mathbb{C} \setminus \overline{\Delta}$, we obtain

$$\begin{aligned}|F_z| &= \left|1 - \sum_{n=1}^{\infty} n\overline{b_n} z^{-n-1}\right| \\ &\geq 1 - \sum_{n=1}^{\infty} n\,|b_n| \\ &\geq 1 - \left(\sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n|\right) \\ &= 1 - k > 0\end{aligned}$$

and

$$|\mu_F(z)| = \left|\frac{F_{\overline{z}}}{F_z}\right| \le \left|\frac{\displaystyle\sum_{n=2}^{\infty} na_n \overline{z^{-n-1}}}{1 - \displaystyle\sum_{n=1}^{\infty} n\overline{b_n} z^{-n-1}}\right|$$

$$\le \frac{\displaystyle\sum_{n=2}^{\infty} n\,|a_n|}{1 - \displaystyle\sum_{n=1}^{\infty} n\,|b_n|} \le k < 1.$$

Therefore $|\mu_F(z)| \le k < 1$ on $\mathbb{C}\backslash\mathbb{T}$. Since $\mathbb{T}$ is a null set, $F$ is quasiconformal on $\overline{\mathbb{C}}$. □

## 4. Harmonic starlike mappings of order $\alpha$

In this section, we show quasiconformal extension results for starlike harmonic mappings of order $\alpha \in (0, 1)$ and give a counterexample when $\alpha = 0$ (cf. Jahangiri [13, Theorems 1 and 2]).

**Theorem 4.1.** *Let $f = h+\overline{g}$ be a harmonic starlike mapping of order $\alpha \in (0, 1)$ of the form* (1.1) *which satisfies the condition*

$$\sum_{n=1}^{\infty} \left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \le 2,$$

*where $a_1 = 1$. Then $f$ is quasiconformal in $\Delta$ and extends to a quasiconormal homeomorphism of $\overline{\mathbb{C}}$.*

*Proof.* Let

$$\psi_n = \frac{n-\alpha}{1-\alpha} \qquad \text{for } n \ge 2,$$

where $0 \le \alpha < 1$. If $\alpha \in (0, 1)$, then the inequalities

$$\psi_2 = \frac{2-\alpha}{1-\alpha} = 2 + \frac{\alpha}{1-\alpha} > 2$$

and

$$\frac{\psi_n}{n} = \frac{1-\frac{\alpha}{n}}{1-\alpha} \ge \frac{1-\frac{\alpha}{2}}{1-\alpha} = \frac{\psi_2}{2}$$

for $n \ge 3$ hold. By Theorem 3.6, we obtain this theorem. □

Next, we obtain the following quasiconformal extension result for harmonic starlike mappings of order $\alpha \in (0, 1)$ by Theorems 2.2 and 4.1.

**Theorem 4.2.** *Let $f = h + \overline{g}$ be a harmonic starlike mapping of order $\alpha \in (0,1)$ of the form*

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n.$$

*Then $f$ is quasiconformal in $\Delta$ and extends to a quasiconormal homeomorphism of $\overline{\mathbb{C}}$.*

We will give an example such that the above theorems do not hold when $\alpha = 0$.

**Example 4.3.** Let

$$f(z) = z - \frac{1-b_1}{2} z^2 + b_1 \overline{z},$$

where $0 < b_1 < 1$. Then $f$ is univalent and starlike in $\Delta$ by Theorem 2.1. Since

$$\omega_f(z) = \frac{b_1}{1-(1-b_1)z} \to 1, \quad \text{as } z \to 1,$$

$f$ is not quasiconformal on $\Delta$. So, Theorems 4.1 and 4.2 do not hold when $\alpha = 0$.

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