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EINSTEIN LIGHTLIKE HYPERSURFACES OF A LORENTZ SPACE FORM WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. We study Einstein lightlike hypersurfaces M of a Lorentzian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection subject to the conditions; (1) M is screen conformal and (2) the structure vector field ζ of $\widetilde{M}$ belongs to the screen distribution $S(TM)$. The main result is a characterization theorem for such a lightlike hypersurface.

1. Introduction

The theory of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds produce models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons) [7, 11]. Lightlike submanifolds are also studied in the theory of electromagnetism [3]. Thus, large number of applications but limited information available, motivated us to do research on this subject matter. As for any semi-Riemannian manifold there is a natural existence of lightlike subspaces, Duggal and Bejancu published their work [3] on the general theory of lightlike submanifolds to fill a gap in the study of submanifolds. Since then there has been very active study on lightlike geometry of submanifolds (see up-to date results in two books [4, 6]). Although now we have lightlike version of a large variety of Riemannian submanifolds, the theory of lightlike submanifolds of semi-Riemannian manifolds equipped with semi-symmetric non-metric connections has not been introduced until quite recently.

Ageshe and Chafle [1] introduced the notion of a semi-symmetric non-metric connection on a Riemannian manifold. Yasar, Cöken and Yücesan [12] and Jin [9] studied lightlike hypersurfaces in a semi-Riemannian manifold admitting a semi-symmetric non-metric connection. Recently Jin [8] studied lightlike submanifolds of a semi-Riemannian manifold with a semi-symmetric non-metric

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connection subject to the conditions; (a) the structure vector field of $\widetilde{M}$ belongs to the screen distribution $S(TM)$ and (b) $S(TM)$ is totally umbilical in M.

The objective of this paper is to study Einstein lightlike hypersurfaces M of a Lorentzian space form $\widetilde{M}(c)$ with a semi-symmetric non-metric connection subject to the conditions; (1) M is screen conformal and (2) the structure vector field ζ of $\widetilde{M}(c)$ belongs to $S(TM)$. The reason for this geometric restriction on M is due to the fact that such a class admits a canonical integrable screen distribution and a symmetric induced Ricci tensor of M [5]. We prove a characterization theorem for such a lightlike hypersurface M.

2. Semi-symmetric non-metric connection

Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannian manifold. A connection $\widetilde{\nabla}$ on $\widetilde{M}$ is called a *semi-symmetric non-metric connection* [1] if $\widetilde{\nabla}$ and its torsion tensor $\widetilde{T}$ satisfy

$$(\widetilde{\nabla}_X \widetilde{g})(Y, Z) = -\pi(Y)\widetilde{g}(X, Z) - \pi(Z)\widetilde{g}(X, Y), \tag{2.1}$$

$$\widetilde{T}(X, Y) = \pi(Y)X - \pi(X)Y, \tag{2.2}$$

for any vector fields X, Y and Z on $\widetilde{M}$, where π is a 1-form associated with a non-vanishing vector field ζ, which called the *structure vector field* of $\widetilde{M}$, by

$$\pi(X) = \widetilde{g}(X, \zeta). \tag{2.3}$$

In the entire discussion of this article we shall assume that the structure vector field ζ to be unit spacelike unless otherwise specified.

Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$. Then the normal bundle $TM^\perp$ of M is a vector subbundle of TM of rank 1 and coincides the radical distribution $Rad(TM) = TM \cap TM^\perp$ of M. Therefore there exist a complementary non-degenerate vector bundle $S(TM)$ of $Rad(TM)$ in TM, which called a *screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \tag{2.4}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. It is well-known [3] that, for any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^\perp$ satisfying

$$\widetilde{g}(\xi, N) = 1, \quad \widetilde{g}(N, N) = \widetilde{g}(N, X) = 0, \quad \forall\, X \in \Gamma\left(S(TM)|_{\mathcal{U}}\right).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to $S(TM)$ respectively. Then $T\widetilde{M}$ is given by

$$T\widetilde{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{2.5}$$

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulas for M and $S(TM)$ are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.6}$$

$$\widetilde{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.7}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.8}$$

$$\nabla_X \xi = -A^*_\xi X - \sigma(X)\xi, \quad \forall\, X,\, Y \in \Gamma(TM), \tag{2.9}$$

where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$, respectively, B and C are the local second fundamental forms on TM and $S(TM)$, respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$, respectively and τ is a 1-form on TM defined by $\widetilde{g}(\widetilde{\nabla}_X N, \xi)$.

From (2.1), (2.2) and (2.6), we have the following equation

$$\begin{aligned}(\nabla_X g)(Y,Z) = &-\pi(Y)g(X,Z) - \pi(Z)g(X,Y)\\ &+ B(X,Y)\eta(Z) + B(X,Z)\eta(Y),\end{aligned} \tag{2.10}$$

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{2.11}$$

for all X, Y, $Z \in \Gamma(TM)$ and B is symmetric on TM, where T is the torsion tensor with respect to the connection ∇ and η is a 1-form on TM such that

$$\eta(X) = \widetilde{g}(X,N), \quad \forall\, X \in \Gamma(TM).$$

From the fact $B(X,Y) = \widetilde{g}(\widetilde{\nabla}_X Y, \xi)$, we know that B is independent of the choice of a screen distribution. Taking $Y = \xi$ to this and using (2.1), we get

$$B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.12}$$

Let a and b be the smooth functions defined by $a = \pi(N)$ and $b = \pi(\xi)$. Then the above second fundamental forms are related to their shape operators by

$$g(A^*_\xi X, Y) = B(X,Y) - bg(X,Y), \qquad \widetilde{g}(A^*_\xi X, N) = 0, \tag{2.13}$$

$$\begin{aligned}&g(A_N X, PY) = C(X,PY) - ag(X,PY) - \eta(X)\pi(PY),\\ &\widetilde{g}(A_N X, N) = -a\eta(X), \qquad \sigma(X) = \tau(X) - b\eta(X),\end{aligned} \tag{2.14}$$

for all X, $Y \in \Gamma(TM)$. By (2.13), we show that A^*_ξ is $S(TM)$-valued self-adjoint shape operator related to B and satisfies

$$A^*_\xi \xi = 0. \tag{2.15}$$

Theorem 2.1 ([9]). *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric metric connection. Then the following assertions are equivalent*:

(1) *The screen distribution $S(TM)$ is an integrable distribution.*

(2) *C is symmetric, i.e., $C(X,Y) = C(Y,X)$ for all X, $Y \in \Gamma(S(TM))$.*

(3) *The shape operator A_N is self-adjoint with respect to g, i.e.,*

$$g(A_N X, Y) = g(X, A_N Y), \quad \forall\, X,\, Y \in \Gamma(S(TM)).$$

Just as in the well-known case of locally product Riemannian or semi-Riemannian manifolds [3, 4, 5, 11], if $S(TM)$ is an integrable distribution, then M is locally a product manifold $\mathcal{C}_1 \times M^*$ where $\mathcal{C}_1$ is a null curve tangent to $Rad(TM)$ and M^* is a leaf of the integrable distribution $S(TM)$.

3. Induced Ricci curvature tensors

Denote by $\widetilde{R}$, R and R^* the curvature tensors of the semi-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$, respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\tilde{g}(\widetilde{R}(X,Y)Z,\, PW) = g(R(X,Y)Z,\, PW) + B(X,Z)g(A_{_N}Y, PW) - B(Y,Z)g(A_{_N}X, PW), \tag{3.1}$$

$$\begin{aligned}\tilde{g}(\widetilde{R}(X,Y)Z,\, \xi) = {} & (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ & + B(Y,Z)\{\tau(X) - \pi(X)\} \\ & - B(X,Z)\{\tau(Y) - \pi(Y)\},\end{aligned} \tag{3.2}$$

$$\tilde{g}(\widetilde{R}(X,Y)Z,\, N) = \tilde{g}(R(X,Y)Z,\, N) + a\{B(Y,Z)\eta(X) - B(X,Z)\eta(Y)\}, \tag{3.3}$$

$$\begin{aligned}\tilde{g}(\widetilde{R}(X,Y)\xi,\, N) &= B(X, A_{_N}Y) - B(Y, A_{_N}X) - 2d\tau(X,Y) \\ &= C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\sigma(X,Y),\end{aligned} \tag{3.4}$$

$$\begin{aligned}g(R(X,Y)PZ,\, PW) = {} & g(R^*(X,Y)PZ,\, PW) \\ & + C(X,PZ)g(A^*_\xi Y, PW) \\ & - C(Y,PZ)g(A^*_\xi X, PW)\end{aligned} \tag{3.5}$$

$$\begin{aligned}\tilde{g}(R(X,Y)PZ,\, N) = {} & (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ & + C(X,PZ)\{\sigma(Y) + \pi(Y)\} \\ & - C(Y,PZ)\{\sigma(X) + \pi(X)\},\end{aligned} \tag{3.6}$$

for any $X,\, Y,\, Z,\, W \in \Gamma(TM)$. Let $\widetilde{Ric}$ be the Ricci curvature tensor of $\widetilde{M}$ and $R^{(0,2)}$ the induced Ricci type tensor on M given respectively by

$$\begin{aligned}\widetilde{Ric}(X,Y) &= trace\{Z \to \widetilde{R}(Z,X)Y\}, \quad \forall\, X,\, Y \in \Gamma(T\widetilde{M}), \\ R^{(0,2)}(X,Y) &= trace\{Z \to R(Z,X)Y\}, \quad \forall\, X,\, Y \in \Gamma(TM).\end{aligned}$$

Consider a quasi-orthonormal frame field $\{\xi; W_a\}$ on M, where $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$ and let $E = \{\xi, N, W_a\}$ be the corresponding frame field on $\widetilde{M}$. Using this quasi-orthonormal frame field, we obtain

$$\begin{aligned}R^{(0,2)}(X,Y) = {} & \widetilde{Ric}(X,Y) + B(X,Y)trA_{_N} - g(A_{_N}X, A^*_\xi Y) \\ & - bg(A_{_N}X, Y) - \tilde{g}(\widetilde{R}(\xi,Y)X,\, N), \quad \forall\, X,\, Y \in \Gamma(TM).\end{aligned}$$

This shows that $R^{(0,2)}$ is not symmetric. The tensor field $R^{(0,2)}$ is called its *induced Ricci tensor* [4, 5], denoted by Ric, of M if it is symmetric. M is called *Ricci flat* if its induced Ricci tensor vanishes on M. For a lightlike hypersurface M of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection, it is known [9] that $R^{(0,2)}$ is an induced Ricci tensor of M if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.

Remark 3.1. If $R^{(0,2)}$ is symmetric, then there exists a null pair $\{\xi, N\}$ such that the corresponding 1-form τ satisfies $\tau = 0$ [3], which called a *canonical null pair* of M. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/Rad(TM)$ [10]. This implies that all screen distribution are mutually isomorphic. For this reason, in case $d\tau = 0$ we consider only lightlike hypersurfaces M endow with the canonical null pair.

We say that M is an *Einstein manifold* if the Ricci tensor of M satisfies

$$Ric = \kappa g. \tag{3.7}$$

It is well-known that if dim $M > 2$, then κ is a constant. For dim $M = 2$, any manifold M is Einstein but κ is not necessarily constant.

A complete simply connected semi-Riemannian manifold $\widetilde{M}$ of constant curvature c is called a *semi-Riemannian space form* and denote it by $\widetilde{M}(c)$. In this case, the curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is given by

$$\widetilde{R}(X,Y)Z = c\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\} \tag{3.8}$$

for all $X,\, Y,\, Z \in \Gamma(T\widetilde{M})$. In this case, $R^{(0,2)}$ is given by

$$\begin{aligned} R^{(0,2)}(X,Y) = {} & mc\, g(X,Y) + B(X,Y) tr A_{_N} - g(A_{_N}X, A^*_\xi Y) \\ & - bg(A_{_N}X, Y), \quad \forall\, X,\, Y \in \Gamma(TM). \end{aligned} \tag{3.9}$$

4. Main theorem

In this section, we study lightlike hypersurface M of a semi-Riemannian manifold admitting a semi-symmetric non-metric connection such that ζ is tangent to M. Then we show that $b = g(\zeta, \xi) = 0$ and $\tau = \sigma$ by $(2.14)_3$.

Theorem 4.1. *Let M be a lightlike hypersurface of a semi-Riemannian manifold admitting a semi-symmetric non-metric connection such that ζ is tangent to M. Then ζ is conjugate to any vector field on M, i.e., ζ satisfies*

$$B(X, \zeta) = \pi(A^*_\xi X) = 0, \quad \forall\, X \in \Gamma(TM). \tag{4.1}$$

Proof. As $\tau = \sigma$, from the two equations of (3.4), we obtain

$$B(X, A_{_N}Y) - B(Y, A_{_N}X) = C(Y, A^*_\xi X) - C(X, A^*_\xi Y)$$

for all $X, Y \in \Gamma(TM)$. Using (2.13), (2.14) and A^*_ξ is self-adjoint, we have

$$\pi(A^*_\xi X)\eta(Y) = \pi(A^*_\xi Y)\eta(X), \ \forall\, X, Y \in \Gamma(TM).$$

Replacing Y by ξ to this equation and using (2.15), we have (4.1). □

Definition. A lightlike hypersurface M of a semi-Riemannian manifold $\widetilde{M}$ is called *screen conformal* [4, 5] if the second fundamental forms B and C satisfy

$$C(X, PY) = \varphi B(X, Y), \quad \forall\, X, Y \in \Gamma(TM), \tag{4.2}$$

where φ is a non-vanishing function on a coordinate neighborhood $\mathcal{U}$ in M.

Remark 4.2. It follows from (4.2) that C is symmetric on $S(TM)$. Thus, by Theorem 2.1, $S(TM)$ is integrable and M is locally a product manifold $\mathcal{C}_1 \times M^*$ where $\mathcal{C}_1$ is a null curve tangent to $Rad(TM)$ and M^* is a leaf of $S(TM)$.

Theorem 4.3. *Let M be a lightlike hypersurface of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection such that ζ is tangent to M. If M is screen conformal, then we have $c = 0$.*

Proof. Substituting (3.8) into (3.2) such that $\tau = 0$, we have

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \pi(X)B(Y, Z) - \pi(Y)B(X, Z) \tag{4.3}$$

for any X, Y, $Z \in \Gamma(TM)$. Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = X[\varphi]B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this into (3.6) with $\sigma = 0$ and using (4.3), we have

$$\widetilde{g}(R(X, Y)PZ, N) = X[\varphi]\, B(Y, PZ) - Y[\varphi]\, B(X, PZ)$$

for all X, Y, $Z \in \Gamma(TM)$. Substituting this and (3.8) into (3.3), we get

$$\begin{aligned} &c\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ &= \{X[\varphi] + a\eta(X)\}B(Y, PZ) - \{Y[\varphi] + a\eta(Y)\}B(X, PZ). \end{aligned}$$

Taking $X = \xi$ and $Y = Z = \zeta$ to this and using (4.1), we have $c = 0$. □

Proposition 4.4. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. If ζ is tangent to M, then there exist a screen distribution $S(TM)$ which contains ζ.*

Proof. Assume that ζ belongs to $Rad(TM)$. Then ζ is decompose as $\zeta = a\xi$ and $a \neq 0$. Using this, we have $1 = \widetilde{g}(\zeta, \zeta) = a^2\widetilde{g}(\xi, \xi) = 0$. It is a contradiction. Thus ζ does not belong to $Rad(TM)$. This enables one to choose a screen distribution $S(TM)$ which contains ζ by (2.4). □

As all screen distributions are mutually isomorphic by Remark 3.1, in this section we consider only the screen distribution $S(TM)$ which contains ζ. This implies that if ζ is tangent to M, then it belongs to $S(TM)$. In the sequel, we shall assume that the structure vector field ζ belongs to $S(TM)$. In this case, we show that $a = 0$ and A_N is $S(TM)$-valued by $(2.14)_2$.

Theorem 4.5. *Let M be an Einstein screen conformal lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection. If ζ belongs to $S(TM)$, then $c = 0$ and M is Ricci flat. Moreover, if the mean curvature of M is constant, then M is locally a product manifold*

$M = \mathcal{C}_1 \times \mathcal{C}_2 \times M^\natural$, where $\mathcal{C}_1$ is a null curve tangent to $Rad(TM)$, $\mathcal{C}_2$ is a non-null curve and $M^\natural$ is an Euclidean space.

Proof. Using (2.13), (2.14), (4.2) and the facts $a = 0$ and $b = 0$, we show that M is screen conformal if and only if

$$A_{_N} X = \varphi A^*_\xi X - \eta(X)\zeta, \quad \forall\, X \in \Gamma(TM). \tag{4.4}$$

From (3.9) with $b = 0$, (4.1) and (4.4), we show that $R^{(0,\,2)}$ is the symmetric induced Ricci tensor of M and the 1-form τ satisfies $\tau = 0$ by Remark 3.1. As $g(A^*_\xi\zeta,\, X) = B(\zeta, X) = 0$ and $S(TM)$ is non-degenerate, we show that

$$A^*_\xi \zeta = 0. \tag{4.5}$$

Using (2.13), (3.7), (4.1) and (4.4), from (3.9) we have

$$g(A^*_\xi X, A^*_\xi Y) - \alpha g(A^*_\xi X, Y) + \varphi^{-1}\kappa g(X, Y) = 0, \tag{4.6}$$

for all $X,\, Y \in \Gamma(TM)$ due to $c = 0$, where $\alpha = tr\, A^*_\xi$. Taking $X = Y = \zeta$ to (4.6) and using (4.5), we have $\varphi^{-1}\kappa = 0$. Thus (4.6) reduce to

$$g(A^*_\xi X, A^*_\xi Y) - \alpha g(A^*_\xi X, Y) = 0, \qquad \kappa = 0. \tag{4.7}$$

From the second equation of (4.7), we show that M is Ricci flat.

As M is screen conformal and $\widetilde{M}$ is Lorentzian manifold, $S(TM)$ is a Riemannian vector bundle. Since ξ is an eigenvector field of A^*_ξ corresponding to the eigenvalue 0 due to (2.15) and A^*_ξ is $S(TM)$-valued real self-adjoint operator, A^*_ξ have m real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of A^*_ξ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$ and $A^*_\xi E_i = \lambda_i E_i$. Put $X = Y = E_i$ in (4.7), each eigenvalue λ_i is a solution of the equation

$$x^2 - \alpha x = 0.$$

As this equation has at most two distinct solutions 0 and α, there exists $p \in \{0, 1, \ldots, m\}$ such that $\lambda_1 = \cdots = \lambda_p = 0$ and $\lambda_{p+1} = \cdots = \lambda_m = \alpha$, by renumbering if necessary. As $tr\, A^*_\xi = 0p + (m - p)\alpha$, we have

$$\alpha = tr\, A^*_\xi = (m - p)\alpha.$$

So $p = m - 1$, i.e.,

$$A^*_\xi = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \alpha \end{pmatrix}.$$

Consider two distributions D_o and D_α on $S(TM)$ given by

$$D_o = \{X \in \Gamma(S(TM)) \mid A^*_\xi X = 0 \text{ and } X \neq 0\},$$
$$D_\alpha = \{U \in \Gamma(S(TM)) \mid A^*_\xi U = \alpha U \text{ and } U \neq 0\}.$$

Clearly we show that $D_o \cap D_\alpha = \{0\}$ as $\alpha \neq 0$. In the sequel, we take $X, Y \in \Gamma(D_o)$, $U, V \in \Gamma(D_\alpha)$ and $Z, W \in \Gamma(S(TM))$. Since X and U are eigenvector fields of the real self-adjoint operator A^*_ξ corresponding to the different eigenvalues 0 and α, respectively, we have $g(X, U) = 0$. From this and the fact $B(X, U) = g(A^*_\xi X, U) = 0$, we show that $D_\alpha \perp_g D_o$ and $D_\alpha \perp_B D_o$ respectively. Since $\{E_i\}_{1 \leq i \leq m-1}$ and $\{E_m\}$ are vector fields of D_o and D_α respectively and D_o and D_α are mutually orthogonal, we show that D_o and D_α are non-degenerate distributions of rank $(m-1)$ and rank 1, respectively. Thus the screen distribution $S(TM)$ is decomposed as $S(TM) = D_\alpha \oplus_{orth} D_o$.

From (4.7), we get $A^*_\xi(A^*_\alpha - \alpha P) = 0$. Let $W \in Im\, A^*_\xi$. Then there exists $Z \in \Gamma(S(TM))$ such that $W = A^*_\xi Z$. Then $(A^*_\xi - \alpha P)W = 0$ and $W \in \Gamma(D_\alpha)$. Thus $Im\, A^*_\xi \subset \Gamma(D_\alpha)$. By duality, we have $Im(A^*_\xi - \alpha P) \subset \Gamma(D_o)$.

Applying ∇_X to $B(Y, U) = 0$ and using (2.13) and $A^*_\xi Y = 0$, we obtain

$$(\nabla_X B)(Y, U) = -g(A^*_\xi \nabla_X Y, U).$$

Substituting this into (4.3) and using (2.11) and $A^*_\xi X = A^*_\xi Y = 0$, we get

$$g(A^*_\xi [X, Y],\, U) = 0.$$

As $Im\, A^*_\xi \subset \Gamma(D_\alpha)$ and D_α is non-degenerate, we get $A^*_\xi [X, Y] = 0$. This implies $[X, Y] \in \Gamma(D_o)$. Thus D_o is an integrable distribution.

Applying ∇_U to $B(X, Y) = 0$ and ∇_X to $B(U, Y) = 0$, we have

$$(\nabla_U B)(X, Y) = 0, \quad (\nabla_X B)(U, Y) = -\alpha g(\nabla_X Y, U).$$

Substituting this two equations into (4.3), we have $\alpha g(\nabla_X Y,\, U) = 0$. As $g(A^*_\xi \nabla_X Y,\, U) = B(\nabla_X Y,\, U) = \alpha g(\nabla_X Y,\, U) = 0$ and $Im\, A^*_\xi \subset \Gamma(D_\alpha)$ and D_α is non-degenerate, we get $A^*_\xi \nabla_X Y = 0$. This implies $\nabla_X Y \in \Gamma(D_o)$. Thus D_o is an auto-parallel distribution on $S(TM)$.

As $A^*_\xi \zeta = 0$, ζ belongs to D_o. Thus $\pi(U) = 0$ for any $U \in \Gamma(D_\alpha)$. Applying ∇_X to $g(U, Y) = 0$ and using (2.10) and the fact D_o is auto-parallel, we get $g(\nabla_X U, Y) = 0$. This implies $\nabla_X U \in \Gamma(D_\alpha)$.

Applying ∇_U to $B(V, X) = 0$ and using $A^*_\xi X = 0$, we obtain

$$(\nabla_U B)(V, X) = -\alpha g(V, \nabla_U X).$$

Substituting this into (4.3) and using the fact $D_o \perp_B D_\alpha$, we get

$$g(V, \nabla_U X) = g(U, \nabla_V X).$$

Applying ∇_U to $g(V, X) = 0$ and using (2.10), we get

$$g(\nabla_U V, X) = \pi(X) g(U, V) - g(V, \nabla_V X).$$

Taking the skew-symmetric part of this and using (2.11), we obtain

$$g([U, V], X) = 0.$$

This implies $[U, V] \in \Gamma(D_\alpha)$ and D_α is an integrable distribution.

Applying ∇_X to $B(U, V) = \alpha g(U, V)$ and ∇_U to $B(X, V) = 0$, we have

$$(\nabla_X B)(U, V) = 0, \quad (\nabla_U B)(X, V) = -\alpha g(\nabla_U X, V),$$

due to the mean curvature of M is constant. Substituting this two equations into (4.3) and using $D_o \perp_B D_\alpha$, we have

$$g(\nabla_U X,\, V) = \pi(X)g(U, V).$$

Applying ∇_U to $g(X, V) = 0$ and using (2.10), we obtain

$$g(X, \nabla_U V) = \pi(X)g(U, V) - g(\nabla_U X, V) = 0.$$

Thus D_α is also an integrable and auto-parallel distribution.

Since the leaf M^* of $S(TM)$ is a Riemannian manifold and $S(TM) = D_\alpha \oplus_{orth} D_o$, where D_α and D_o are auto-parallel distributions of M^*, by the decomposition theorem of de Rham [2] we have $M^* = \mathcal{C}_2 \times M^\natural$, where $\mathcal{C}_2$ is a leaf of D_α and $M^\natural$ is a totally geodesic leaf of D_o. Consider the frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of A^*_ξ such that $\{E_i\}_i$ is an orthonormal frame field of $S(TM)$, then $B(E_i, E_j) = C(E_i, E_j) = 0$ for $1 \le i < j \le m$ and $B(E_i, E_i) = C(E_i, E_i) = 0$ for $1 \le i \le m-1$. From (3.1) and (3.5), we have $\widetilde{g}(\widetilde{R}(E_i, E_j)E_j, E_i) = g(R^*(E_i, E_j)E_j, E_i) = 0$. Thus the sectional curvature K of the leaf $M^\natural$ of D_o is given by

$$K(E_i, E_j) = \frac{g(R^*(E_i, E_j)E_j,\, E_i)}{g(E_i, E_i)g(E_j, E_j) - g^2(E_i, E_j)} = 0.$$

Thus M is a locally product $\mathcal{C}_1 \times \mathcal{C}_2 \times M^\natural$, where $\mathcal{C}_1$ is a null curve, $\mathcal{C}_2$ is a non-null curve and $M^\natural$ is an $(m-1)$-dimensional Euclidean space. □

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