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CLASSIFICATIONS OF HELICOIDAL SURFACES WITH L_1-POINTWISE 1-TYPE GAUSS MAP

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Abstract. In this paper, we study rotational and helicoidal surfaces in Euclidean 3-space in terms of their Gauss map. We obtain a complete classification of these type of surfaces whose Gauss maps G satisfy $L_1 G = f(G+C)$ for some constant vector $C \in \mathbb{E}^3$ and smooth function f, where L_1 denotes the Cheng-Yau operator.

1. Introduction

Let M be a hypersurface of the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$. A smooth mapping $\phi : M \to \mathbb{E}^N$ is said to be of k-type if it can be expressed as a sum of eigenvectors of Laplace operator Δ corresponding to k distinct eigenvalues of Δ ([6]). If ϕ is an immersion from M into $\mathbb{E}^{n+1}$ is of k-type, then the submanifold M is said to be of k-type ([3]). A good survey on finite type submanifolds is [4].

On the other hand, if the Gauss map G of M satisfies

$$\Delta G = \lambda(G + C) \tag{1.1}$$

for a constant $\lambda \in \mathbb{R}$ and a constant vector C, M is said to have 1-type Gauss map, [7]. However, the Gauss map of some important submanifolds such as a helicoid and a catenoid in $\mathbb{E}^3$ satisfies a very similar equation to (1.1), namely,

$$\Delta G = f(G + C) \tag{1.2}$$

for a smooth function $f \in C^\infty(M)$ and a constant vector C ([10]). These submanifolds whose Gauss maps G satisfying (1.2) are said to have pointwise 1-type Gauss map. Submanifolds with pointwise 1-type Gauss map have been studied in several papers (cf. [5, 10, 18, 20, 22]).

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In the recent years, the definition of L_k-finite type hypersurface has been given by changing the Laplace operator Δ in the definition of finite type hypersurfaces with the sequence of operators $L_0,\ L_1,\ L_2,\ \dots,\ L_{n-1}$ such that $L_0 = -\Delta$. Note that L_k is the linearized operator of the first variation of the $(k+1)$-th mean curvature arising from the normal variations of M for $k = 2, 3, \dots, n-1$. In particular, L_1 is called the Cheng-Yau operator introduced in [8]. L_k-finite type hypersurfaces have been studied in [1, 15].

On the other hand, the following definition was given by the authors in [17]:

Definition 1. A surface M of the Euclidean space $\mathbb{E}^3$ is said to have L_1-pointwise 1-type Gauss map if its Gauss map satisfies

$$L_1 G = f(G + C) \tag{1.3}$$

for a smooth function $f \in C^\infty(M)$ and a constant vector $C \in \mathbb{E}^3$. More precisely, a L_1-pointwise 1-type Gauss map is said to be of the first kind if (1.3) is satisfied for $C = 0$; otherwise, it is said to be of the second kind. Moreover, if (1.3) is satisfied for a constant function f, then we say that M has L_1-(global) 1-type Gauss map.

In the same paper, the authors proposed the following problem.

Open Problem. Classify surfaces in $\mathbb{E}^3$ with L_1-1-type Gauss map.

On the other hand, rotational and helicoidal surfaces of Euclidean or semi-Euclidean spaces with pointwise 1-type Gauss map with respect to the Laplace operator Δ have been studied in several papers. For example, in [5] and [16], the rotational surfaces of $\mathbb{E}^3$ and $\mathbb{E}^3_1$ with (Δ-) pointwise 1-type Gauss map respectively have been studied. Furthermore, several classification theorems on rotational surfaces in $\mathbb{E}^4$ and $\mathbb{E}^4_2$ satisfying (1.2) were given in [14, 19, 22, 23]. Recently, the classifications of helicoidal surfaces in $\mathbb{E}^3$ and $\mathbb{E}^3_1$ with (Δ-) pointwise 1-type Gauss map have been obtained in [9, 11, 12].

In this paper, we study rotational and helicoidal surfaces in $\mathbb{E}^3$ with L_1-pointwise 1-type Gauss map. In Section 2, we give some basic notations, facts and definitions on the theory of submanifolds of Euclidean spaces. In Section 3, we obtain the complete classification of rotational surfaces with L_1-pointwise 1-type Gauss map. Finally, in Section 4, we prove that a genuine helicoidal surface has pointwise 1-type Gauss map if and only if its Gaussian curvature is constant.

2. Preliminaries

Let M be a hypersurface in the Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$. We denote the Levi-Civita connections of $\mathbb{E}^{n+1}$ and M by $\widetilde{\nabla}$ and ∇, respectively and D stands for the normal connection of M. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal frame of the tangent bundle of M and N the unit normal vector field of M. Then, the Gauss and Weingarten formulas are

$$\widetilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + h_{ij} N, \tag{2.1}$$

$$\widetilde{\nabla}_{e_i} N = -\sum_{k=1}^{n} h_{ik} e_k, \quad i,j = 1,2,\ldots,n, \tag{2.2}$$

where h_{ij} are the components of the second fundamental form h of M, i.e., $h_{ij} = \langle h(e_i, e_j), N\rangle$. The covariant derivative of h is defined by

$$(\bar{\nabla}_{e_i} h)(e_j, e_k) = D_{e_i} h(e_j, e_k) - h(\nabla_{e_i} e_j, e_k) - h(e_j, \nabla_{e_i} e_k).$$

Then, the Codazzi equation is given by

$$(\bar{\nabla}_{e_i} h)(e_j, e_k) = (\bar{\nabla}_{e_j} h)(e_i, e_k), \quad i,j,k = 1,2,\ldots,n. \tag{2.3}$$

The shape operator S of M is defined by $S(X) = -\widetilde{\nabla}_X N$. Since S is a self-adjoint linear mapping, it has n eigenvalues up to multiplicities. Moreover, the eigenvectors of S corresponding to these eigenvalues span the tangent space T_pM of M at every point $p \in M$. The eigenvalues and eigenvectors of S are called the principal curvatures and principal directions of M, respectively.

Now, let M be a surface in Euclidean 3-space, $k_1,\ k_2$ the principal curvatures and $e_1,\ e_2$ the corresponding principal directions of M. The mean curvature H and the Gaussian curvature K of M are defined by

$$H = (k_1 + k_2)/2, \tag{2.4}$$

$$K = k_1 k_2 \tag{2.5}$$

and the Gauss equation is given by

$$R(e_1, e_2, e_2; e_1) = K, \tag{2.6}$$

where R is the curvature tensor associated with the induced connection ∇. Then, M is said to be minimal (resp., flat) if its mean curvature H (resp., Gaussian curvature K) vanishes identically.

We will use $C^\infty(M, \mathbb{E}^3)$ to denote the space of all real smooth functions from M into $\mathbb{E}^3$ and $C^\infty(M)$ stands for the space of all real smooth functions defined on M. If $X \in C^\infty(M, \mathbb{E}^3)$ is tangent to M, its divergence $\mathrm{div}X$ is defined as the map which sends every point p of M to the trace of the linear mapping $Y(p) \mapsto (\nabla_Y X)(p)$. On the other hand, the gradient of a function $f \in C^\infty(M)$ is defined by $\nabla f = e_1(f)e_1 + e_2(f)e_2$. If $\{y, t\}$ is a local coordinate system in M such that $\langle \partial_y, \partial_y\rangle = g_{11}$, $\langle \partial_y, \partial_t\rangle = g_{12}$ and $\langle \partial_t, \partial_t\rangle = g_{22}$, then the gradient of a smooth function f is

$$\nabla f = \frac{1}{g_{11}g_{22} - (g_{12})^2}\{(g_{22}f_y - g_{12}f_t)\,\partial_y + (-g_{12}f_y + g_{11}f_t)\,\partial_t\}, \tag{2.7}$$

where f_y and f_t are the partial derivatives of f with respect to y and t, respectively.

2.1. Surfaces with L_1-pointwise 1-type Gauss map

Let M be a hypersurface in $\mathbb{E}^{n+1}$. The mapping given by

$$\begin{aligned} G : M &\rightarrow \mathbb{S}^n \subset \mathbb{E}^{n+1} \\ p &\mapsto N(p) \end{aligned}$$

is called the Gauss map of M which sends every point $p \in M$ to the unit normal vector at p associated with the orientation of M, where $\mathbb{S}^n$ denotes the unit hypersphere centered at the origin of $\mathbb{E}^{n+1}$.

The functions s_k given by

$$s_k(p) = \sigma_k(k_1(p), k_2(p), \ldots, k_n(p)), \quad 1 \leq k \leq n$$

are called the algebraic invariants of the shape operator S of M, where σ_k is the k-th symmetric function in $\mathbb{R}^n$ given by

$$\sigma_k(t_1, t_2, \ldots, t_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} t_{i_1} t_{i_2} \cdots t_{i_k}.$$

The operator $L_k : C^\infty(M) \to C^\infty(M)$ is defined by

$$L_k(f) = \mathrm{tr}(P_k \circ \nabla^2 f),$$

where $\nabla^2 f$ is the Hessian of f and P_k are the operators defined by $P_k = s_k P_0 - S \circ P_{k-1}$ and $P_0 = I$ is the identity operator acting on the tangent bundle of M ([1]).

In particular, if M is a surface in $\mathbb{E}^3$, then the action of the Cheng-Yau operator $L_1 = L_1$ on vector fields defined on M is given by $L_1 : C^\infty(M, \mathbb{E}^3) \to C^\infty(M, \mathbb{E}^3)$,

(2.8)

$$L_1 = e_1(k_2)\widetilde{\nabla}_{e_1} + e_2(k_1)\widetilde{\nabla}_{e_2} + k_2\left(\widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_1} - \widetilde{\nabla}_{\nabla_{e_2}e_2}\right) + k_1\left(\widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_2} - \widetilde{\nabla}_{\nabla_{e_1}e_1}\right).$$

We will use the following lemma and theorems that we obtained in [17]:

Lemma 2.1 ([17])**.** *Let M be a surface in $\mathbb{E}^3$ with Gaussian curvature K and mean curvature H. Then, the Gauss map G of M satisfies*

$$L_1 G = -\nabla K - 2HKG. \tag{2.9}$$

Theorem 2.2 ([17])**.** *A surface M in $\mathbb{E}^3$ has L_1-harmonic Gauss map if and only if it is flat, i.e., its Gaussian curvature vanishes identically.*

Theorem 2.3 ([17])**.** *A surface M in $\mathbb{E}^3$ has L_1-pointwise 1-type Gauss map of the first kind if and only if it has constant Gaussian curvature.*

2.2. Lie point symmetries of ordinary differential equations

Consider the n-th order non-linear ordinary differential equation given by

$$F(s, y, y^{(1)}, y^{(2)}, \ldots, y^{(n)}) = 0. \tag{2.10}$$

Let $\mathcal{T}$ be a 1-parameter transformation group given by $s = s(\tilde{s}, \tilde{y}; \varepsilon)$, $y = y(\tilde{s}, \tilde{y}; \varepsilon)$ with group parameter ε and the vector field X as its infinitesimal generator, i.e., $X = \xi(s, y)\partial_s + \eta(s, y)\partial_y$ with $\xi(\tilde{s}, \tilde{y}) = s_\varepsilon(\tilde{s}, \tilde{y}, 0)$, $\eta(\tilde{s}, \tilde{y}) = y_\varepsilon(\tilde{s}, \tilde{y}, 0)$. The differential equation given by (2.10) is said to admit $\mathcal{T}$ as a Lie point symmetry if it is left invariant under all of the transformations in $\mathcal{T}$. In this case, a transformation by $Y = Y(s, y)$ and $\phi = \phi(s, y)$ satisfying $XY = 0$,

$X\phi = 1$ and $\frac{\partial(Y,\phi)}{\partial(s,y)} \neq 0$ transforms the ordinary differential equation given by (2.10) into $\tilde{F}(Y, \phi, \phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(n)}) = 0$ with $\frac{\partial \tilde{F}}{\partial \phi} = 0$. Thus, by putting $\psi = \phi'$, one can reduce the order of this equation to $n-1$ ([2]).

3. Rotational surfaces

Up to congruence, the position vector of a rotational surface M is given by

$$F(s,t) = (x(s), y(s)\cos t, y(s)\sin t), \tag{3.1}$$

where x and y are some smooth functions defined on an open interval I of $\mathbb{R}$. The curve α given by $\alpha(s) = (x(s), y(s), 0),\ s \in I$ is called the profile curve of the rotational surface M. By a simple calculation, one can check that F is an immersion if and only if $(x'^2 + y'^2)y \neq 0$. Therefore, α has to be regular and its coordinate function y is nowhere vanishing. Thus, without loss of generality, we may assume that $y > 0$ and $x'^2 + y'^2 = 1$, that is, α is parameterized by its arc-length. We should note that as α is arc-length parameterized, there exists a smooth function θ such that $x' = \sin\theta$ and $y' = \cos\theta$ which satisfy

$$\theta' = x''y' - x'y'', \tag{3.2}$$

$$\theta'^2 = x''^2 + y''^2, \tag{3.3}$$

$$\theta' x' = -y'', \tag{3.4}$$

$$\theta' y' = x''. \tag{3.5}$$

We choose local orthonormal frame fields for the tangent space of M as

$$e_1 = \partial_s, \tag{3.6}$$

$$e_2 = \frac{1}{y}\partial_t. \tag{3.7}$$

In addition, the Gauss map $G : M \to \mathbb{S}^2 \subset \mathbb{E}^3$ of M is given by

$$G(F(s,t)) = (y'(s), -x'(s)\cos t, -x'(s)\sin t), \tag{3.8}$$

where $'$ denotes the ordinary differentiation with respect to the variable s. By a direct computation, we obtain

$$\widetilde{\nabla}_{e_1} e_1 = \theta' G, \tag{3.9}$$

$$\widetilde{\nabla}_{e_1} e_2 = 0, \tag{3.10}$$

$$\widetilde{\nabla}_{e_2} e_2 = -\frac{y'}{y} e_1 + \frac{x'}{y} G. \tag{3.11}$$

Then, (3.10) implies that e_1 and e_2 are the principal directions of M. On the other hand, from (3.9) and (3.11) we obtain

$$k_1 = \theta', \quad k_2 = \frac{x'}{y}, \tag{3.12}$$

$$\omega_1 = 0, \quad \omega_2 = \frac{y'}{y}, \tag{3.13}$$

where k_1 and k_2 are the principal curvatures of M corresponding to e_1 and e_2, respectively and ω_1, ω_2 are the functions defined by $\omega_1 = \langle \nabla_{e_1} e_1, e_2 \rangle$ and $\omega_2 = -\langle \nabla_{e_2} e_2, e_1 \rangle$.

3.1. Rotational surfaces with L_1-harmonic Gauss map

From (3.4) and (3.12) we get

$$K = -\frac{y''}{y}. \tag{3.14}$$

Thus, a rotational surface given by (3.1) is flat if and only if $y'' = x'' \equiv 0$ on I because of $x'^2 + y'^2 = 1$ and (3.2)-(3.5). Therefore, Theorem 2.2 implies that a rotational surface in $\mathbb{E}^3$ has L_1-harmonic Gauss map if and only if it is congruent to a surface given by

$$F(s,t) = \left(as, (\sqrt{1-a^2}s + b)\cos t, (\sqrt{1-a^2}s + b)\sin t\right) \tag{3.15}$$

for some constants a and b. Note that the cases $a = 0$, $a = \pm 1$ and $a \notin \{-1, 0, 1\}$ imply M is an open part of a plane, a right circular cylinder and a right circular cone, respectively. Hence, we state:

Theorem 3.1. *Let M be a rotational surface in $\mathbb{E}^3$. Then, M has L_1-harmonic Gauss map if and only if it is an open part of a plane, a right circular cylinder or a right circular cone.*

3.2. Rotational surfaces with L_1-pointwise 1-type Gauss map of the first kind

Equation (3.14) implies that a rotational surface given by (3.1) has constant Gaussian curvature $K = -\kappa$ if and only if $y'' = \kappa y$. By multiplying both sides of this equation by y' and integrating the resulting equation, we obtain $y'^2 = \kappa y^2 + \lambda$ for a constant λ, which implies $x'^2 = -\kappa y^2 + 1 - \lambda$. Thus, we have

$$\frac{dx}{dy} = \pm\sqrt{\frac{-\kappa y^2 + 1 - \lambda}{\kappa y^2 + \lambda}}.$$

Hence, Theorem 2.3 implies:

Theorem 3.2. *Let M be a rotational surface in $\mathbb{E}^3$. Then, M has L_1-pointwise 1-type Gauss map of the first kind if and only if it is congruent to the surface given by*

$$F(y,t) = \left(\int_{x_0}^{y} \sqrt{\frac{-\kappa y^2 + 1 - \lambda}{\kappa y^2 + \lambda}}dy, y\cos t, y\sin t\right) \tag{3.16}$$

for some constants x_0, κ and λ.

Remark 1. The classification of rotational surfaces with constant Gaussian curvature is very well-known and can be found, for instance, in [21].

3.3. Rotational surfaces with L_1-pointwise 1-type Gauss map of the second kind

Let M be a rotational surface in $\mathbb{E}^3$ given by (3.1) with L_1-pointwise 1-type Gauss map of the second kind. Then, equation (1.3) is satisfied for a constant vector $C \neq 0$ and a smooth function f.

Now, consider the open subset $\mathcal{U} = \{p \mid K(p) \neq 0\}$ of M. From equations (2.1) and (3.12) we obtain

$$L_1 G = \left(\frac{y''}{y}\right)' e_1 + \frac{y''}{y}\left(\theta' + \frac{x'}{y}\right) G.$$

This equation and (1.3) imply

$$fC_1 = \left(\frac{y''}{y}\right)', \tag{3.17}$$

$$C_2 = 0, \tag{3.18}$$

$$f(C_3 + 1) = \frac{y''}{y}\left(\theta' + \frac{x'}{y}\right) \tag{3.19}$$

on $\mathcal{U}$, where $C_1 = \langle C, e_1\rangle$, $C_2 = \langle C, e_2\rangle$ and $C_3 = \langle C, G\rangle$. On the other hand, since C is a constant vector, (3.18) implies $e_2(C_2) = -C_1\omega_2 + k_2 C_3 = 0$. By using (3.12), (3.13) and this equation, we obtain

$$y'C_1 = x'C_3. \tag{3.20}$$

Moreover, as C is a constant vector, we have

$$\widetilde{\nabla}_{e_i} C = 0, \; i = 1, 2. \tag{3.21}$$

From equation (3.21) for $i = 2$ we obtain $e_2(C_1) = e_2(C_3) = 0$. Therefore, we have $C_1 = C_1(s)$, $C_3 = C_3(s)$, functions of s only. On the other hand, by using (3.12), (3.13) in equation (3.21) for $i = 1$, we get

$$C_1' = \theta' C_3, \tag{3.22}$$

$$C_3' = -\theta' C_1. \tag{3.23}$$

By multiplying both sides of (3.22) by x' and using (3.5) and (3.20), we obtain $x'C_1' = x''C_1$ which implies

$$C_1 = cx' \tag{3.24}$$

for a constant c. Thus, (3.20) implies

$$x'(C_3 - cy') = 0. \tag{3.25}$$

Note that if $x' = 0$ at a point p of $\mathcal{U}$, then (3.4) implies $y'' = 0$. Therefore, from (3.14) we have $K(p) = 0$ which is a contradiction. Therefore, from (3.25) we have

$$C_3 = cy' \tag{3.26}$$

on $\mathcal{U}$. From (3.6), (3.8), (3.24) and (3.26) we obtain $C = C_1 e_1 + C_3 G = (c, 0, 0)$. As $C \neq 0$, we have $c \neq 0$.

On the other hand, by using (3.24) in (3.17), we get

$$f = \frac{yy''' - y'y''}{cy^2 x'}. \tag{3.27}$$

Moreover, from (3.19), (3.26) and (3.27) we obtain

$$\frac{1 + cy'}{cx'}\left(\frac{y''}{y}\right)' = \frac{y''}{y}\left(\theta' + \frac{x'}{y}\right). \tag{3.28}$$

By a direct computation by using (3.2)-(3.5), (3.28) implies

$$(1 + cy')yy''' - (c + y')y'' + cyy''^2 = 0. \tag{3.29}$$

Conversely, suppose that y satisfies equation (3.29) for a constant c. Then, by a direct calculation, we see that the Gauss map of the rotational surface given by (3.1) satisfies (1.3) for $C = (c, 0, 0)$ and the function f given by (3.27).

Hence, we state the following lemma:

Lemma 3.3. *Let M be a rotational surface in $\mathbb{E}^3$ given by* (3.1). *Assume that the open subset $\mathcal{U} = \{p \mid K(p) \neq 0\}$ of M is not empty. Then, M has L_1-pointwise 1-type Gauss map of the second kind if and only if equation* (3.29) *is satisfied for a non-zero constant c. In that case, the Gauss map G of M satisfies* (1.3) *for the constant vector $C = (c, 0, 0)$ and the smooth function f given by* (3.27).

By integrating (3.29), we obtain

$$(1 + cy')yy'' - y'^2 - \frac{c}{3}y'^3 - cy' = A \tag{3.30}$$

on $\mathcal{U}$, where A is a constant. Note that the transformation by $y = \varepsilon\tilde{y}$, $s = \varepsilon\tilde{s}$ leaves equation (3.30) invariant for all $\varepsilon \in \mathbb{R}$. Therefore, (3.30) admits the Lie point symmetry whose infinitesimal generator is $X = s\partial_s + y\partial_y$. So, we use the transformation

$$y = e^Y, \quad s = \phi e^Y \tag{3.31}$$

in (3.30) to obtain

$$(\psi + c)\dot{\psi} + \psi^2 + \frac{c}{3}\psi + c\psi^3 = A\psi^4, \tag{3.32}$$

where $\dot{}$ denotes the ordinary differentiation with respect to Y and $\psi = \phi + \dot{\phi}$. Note that the transformation given by (3.31) implies

$$y' = \frac{1}{\psi}\ . \tag{3.33}$$

If ψ is a constant on a component $\mathcal{O}$ of $\mathcal{U}$ such that $\psi^2 + \frac{c}{3}\psi + c\psi^3 - A\psi^4 = 0$, then equation (3.33) implies $y'' = 0$. Therefore, from (3.14) we see that $\mathcal{O}$ is flat, which is a contradiction.

If ψ is not a constant function, then the interior of $\mathcal{U}_2 = \{p \mid \psi^2 + \frac{c}{3}\psi + c\psi^3 - A\psi^4 = 0 \text{ at } p\}$ is an empty set. Therefore, we may assume that $\mathcal{U}_2 = \emptyset$ locally. Thus, (3.32) implies

$$\frac{(\psi + c)d\psi}{A\psi^4 - (\psi^2 + \frac{c}{3}\psi + c\psi^3)} = dY.$$

The general solution of this equation is $\psi = \eta^{-1}(Y)$ where η is a function defined by

$$\eta(\psi) = \int_{\psi_0}^{\psi} \frac{\xi + c}{\xi(A\xi^3 - c\xi^2 - \xi - \frac{c}{3})} d\xi. \tag{3.34}$$

Thus, from (3.33) we have $y' = \frac{1}{\eta^{-1}(\ln y)}$ which implies

$$\frac{dx}{dy} = \sqrt{\frac{1 - y'^2}{y'^2}} = \sqrt{(\eta^{-1}(\ln y))^2 - 1}.$$

Hence, we have:

Theorem 3.4. *Let M be a rotational surface in $\mathbb{E}^3$. Then, M has L_1-pointwise 1-type Gauss map of the second kind if and only if M is an open part of one of the following four types of surfaces*:

(1) *a plane,*
(2) *a right circular cylinder,*
(3) *a right circular cone,*
(4) *a surface which is locally congruent to the surface given by*

$$F(y,t) = \left(\int_{y_0}^{y} \sqrt{(\eta^{-1}(\ln \xi))^2 - 1} d\xi, y\cos t, y \sin t \right) \tag{3.35}$$

for some constants y_0, A and $c \neq 0$ where η is the function defined by (3.34).

4. Helicoidal surfaces

Let M be a helicoidal surface in $\mathbb{E}^3$ given by

$$F(y,t) = (\phi + at, y\cos t, y\sin t)\,, \tag{4.1}$$

where $\phi = \phi(y)$ is a smooth function defined on an open interval of $\mathbb{R}$ and a is a constant. Note that if $a = 0$, then M becomes a rotational surface with the profile curve $\alpha(y) = (\phi(y), y, 0)$ (see Section 3). Therefore, we may assume $a \neq 0$, i.e., M is a genuine helicoidal surface ([9]).

By a simple calculation, we obtain

$$F_y = (\phi', \cos t, \sin t)\,, \quad F_t = (a, -y\sin t, y\cos t)\,, \tag{4.2}$$
$$F_{yy} = (\phi'', 0, 0)\,, \quad F_{yt} = (0, -\sin t, y\cos t)\,, \quad F_{tt} = (0, -y\cos t, -y\sin t)$$

and

$$g_{11} = \phi'^2 + 1, \quad g_{12} = a\phi', \quad g_{22} = a^2 + y^2, \tag{4.3}$$
$$h_{11} = \tfrac{y\phi''}{(a^2+y^2(\phi'^2+1))^{1/2}}, \quad h_{12} = \tfrac{-a}{(a^2+y^2(\phi'^2+1))^{1/2}}, \quad h_{22} = \tfrac{y^2\phi'}{(a^2+y^2(\phi'^2+1))^{1/2}},$$

where g and h are the first and second fundamental forms of M. By using these equations, we obtain the mean curvature H and the Gaussian curvature K of M as

$$\begin{aligned} K = K(y) &= \frac{y^3\phi'\phi'' - a^2}{(a^2 + y^2(\phi'^2 + 1))^2}, \\ H = H(y) &= \frac{y\phi''(a^2 + y^2) + 2a^2\phi' + (\phi'^2 + 1)y^2\phi'}{2(a^2 + y^2(\phi'^2 + 1))^{3/2}} \end{aligned}$$

and the Gauss map of M is obtained by

$$G(F(y,t)) = \frac{1}{(a^2 + y^2(\phi'^2 + 1))^{1/2}} \left(y, a\sin t - y\phi'\cos t, -a\cos t - y\phi'\sin t\right). \tag{4.4}$$

By using (2.7) and (4.3), we obtain

$$\nabla K = \frac{K'}{a^2 + y^2(\phi'^2 + 1)} \left(\left(a^2 + y^2\right)\partial_y - a\phi'\partial_t\right).$$

Then, equation (2.9) with the aid of this equation, (4.2) and (4.4) yields

$$(L_1G)(F(s,t)) = (C(y), A(y)\cos t + B(y)\sin t, A(y)\sin t - B(y)\cos t), \tag{4.5}$$

where

$$A(y) = -\frac{(a^2 + y^2)K'}{a^2 + y^2(\phi'^2 + 1)} + \frac{2yHK\phi'}{(a^2 + y^2(\phi'^2 + 1))^{1/2}}, \tag{4.6}$$

$$B(y) = -\frac{ay\phi'K'}{a^2 + y^2(\phi'^2 + 1)} - \frac{2aHK}{(a^2 + y^2(\phi'^2 + 1))^{1/2}}, \tag{4.7}$$

$$C(y) = -\frac{y^2\phi'K'}{a^2 + y^2(\phi'^2 + 1)} - \frac{2HKy}{(a^2 + y^2(\phi'^2 + 1))^{1/2}}. \tag{4.8}$$

Now, we suppose that M has L_1-pointwise 1-type Gauss map of the second kind. Then, there exists a smooth function $f \in C^\infty(M)$ and a constant vector $C = (c_1, c_2, c_3)$ in $\mathbb{E}^3$ such that (1.3) is satisfied. From (1.3), (4.4) and (4.5), we obtained

$$C(y) = \frac{fy}{(a^2 + y^2(\phi'^2 + 1))^{1/2}} + c_1, \tag{4.9}$$

$$A(y)\cos t + B(y)\sin t = \frac{f(a\sin t - y\phi'\cos t)}{(a^2 + y^2(\phi'^2 + 1))^{1/2}} + c_2, \tag{4.10}$$

$$A(y)\sin t - B(y)\cos t = -\frac{f(a\cos t + y\phi'\sin t)}{(a^2 + y^2(\phi'^2 + 1))^{1/2}} + c_3. \tag{4.11}$$

Since $\{1, \cos t, \sin t\}$ forms a set of linearly independent functions, (4.10) and (4.11) imply $c_2 = c_3 = 0$ and

$$A(y) = -\frac{fy\phi' \cos t}{(a^2 + y^2(\phi'^2 + 1))^{1/2}}, \tag{4.12}$$

$$B(y) = \frac{fa}{(a^2 + y^2(\phi'^2 + 1))^{1/2}}, \tag{4.13}$$

from which, we obtain $aA + y\phi' B = 0$. This equation, (4.6) and (4.7) imply $aK'(a^2 + y^2(\phi'^2 + 1)) = 0$. Hence, we obtain

Theorem 4.1. *A genuine helicoidal surface in $\mathbb{E}^3$ has L_1-pointwise 1-type Gauss map of the second kind if and only if it is flat.*

By combining Theorem 2.3 and Theorem 4.1, we give the following characterization theorem:

Theorem 4.2. *Let M be a genuine helicoidal surface in $\mathbb{E}^3$ given by* (4.1)*. Then M has L_1-pointwise 1-type Gauss map if and only if its Gaussian curvature is constant.*

Remark 2. Helicoidal surfaces in $\mathbb{E}^3$ with constant Gaussian curvature were obtained in [13].

Consequently, by summing up Theorem 3.4 and Theorem 4.1, we obtain following theorem:

Theorem 4.3. *Let M be a helicoidal surface in $\mathbb{E}^3$. Then, M has L_1-pointwise 1-type Gauss map of the second kind if and only if M is an open part of one of the following four types of surfaces*:

(1) *a plane,*
(2) *a right circular cylinder,*
(3) *a right circular cone,*
(4) *a surface which is locally congruent to the rotational surface given by* (3.34) *and* (3.35).

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