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SOME TRANSLATION SURFACES IN THE 3-DIMENSIONAL HEISENBERG GROUP

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Abstract. In this paper, we define translation surfaces in the 3-dimensional Heisenberg group $\mathcal{H}_3$ obtained as a product of two planar curves lying in planes, which are not orthogonal, and study minimal translation surfaces in $\mathcal{H}_3$.

1. Introduction

Minimal surfaces are one of main objects which have drawn geometers' interest for a very long time. In 1744, L. Euler found that the only minimal surfaces of revolution are the planes and the catenoids, and in 1842 E. Catalan proved that the planes and the helicoids are the only minimal ruled surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. Also, H. F. Scherk in 1835 studied translation surfaces in $\mathbb{E}^3$ defined as graph of the function $z(x,y) = f(x) + g(y)$ and he proved that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a}\log\left|\frac{\cos(ax)}{\cos(ay)}\right| = \frac{1}{a}\log|\cos(ax)| - \frac{1}{a}\log|\cos(ay)|, \tag{1.1}$$

where $f(x)$ and $g(y)$ are smooth functions on some interval of $\mathbb{R}$ and a is a non-zero constant. These surfaces are now referred as Scherk's minimal surfaces. The study of minimal surfaces of revolution, ruled surfaces and translation surfaces in the Euclidean space was extended to the Lorentz version by O. Kobayashi [4] and I. V. de Woestijne [8]. R. López [5] studied translation surfaces in the 3-dimensional hyperbolic space $\mathbb{H}^3$ and classified minimal translation surfaces.

Translation surfaces can be defined in any 3-dimensional Lie group equipped with left invariant Riemannian metric. A translation surface in the 3-dimensional Lie group equipped with a left invariant metric is a surface in the group

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parametrized as a product of two curves (cf. [3]). In [3] J. Inoguchi, R. López and M. I. Munteanu defined translation surfaces in the 3-dimensional Heisenberg group $\mathcal{H}_3$ in terms of a pair of two planar curves lying in orthogonal planes and they classified minimal translation surfaces in $\mathcal{H}_3$. Also, R. López and M. I. Munteanu [6] constructed translation surfaces in Sol_3 and investigated properties of minimal one. The space Sol_3 is a simply connected homogeneous the 3-dimensional manifold whose isometry group has dimension 3 and it is one of the eight models of geometry of W. Thurston [7].

On the other hand, Scherk's minimal surface given by (1.1) can be also parametrized by

$$x(s,t) = \left(a, 0, \frac{1}{a}\log|\cos(as)|\right) + \left(0, t, \frac{-1}{a}\log|\cos(at)|\right), \tag{1.2}$$

and it is defined as the sum of two planar curves lying in orthogonal planes. In [1] authors considered translation surfaces generated as the sum of planar curves lying in planes, which are not orthogonal, and they classified such minimal one.

In this paper, we classify translation surfaces in the 3-dimensional Heisenberg group $\mathcal{H}_3$ obtained as a product of two planar curves lying in any two planes.

2. Preliminaries

The 3-dimensional Heisenberg group $\mathcal{H}_3$ is a matrix group which is given by

$$\mathcal{H}_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}.$$

The Heisenberg group $\mathcal{H}_3$ is represented as the Cartesian 3-space $\mathbb{R}^3(x, y, z)$ with the group operation (cf. [2]):

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = \left(x + \bar{x}, y + \bar{y}, z + \bar{z} + \frac{1}{2}x\bar{y} - \frac{1}{2}y\bar{x}\right). \tag{2.1}$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is $(-x, -y, -z)$. It is simply connected and connected 2-step nilpotent Lie group.

On the other hand, the orthonormal basis of the tangent space at the identity are

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the left invariant metric $\tilde{g}$ in $\mathcal{H}_3$ is given by

$$\tilde{g} = dx^2 + dy^2 + \left(dz + \frac{1}{2}(ydx - xdy)\right)^2. \tag{2.2}$$

And the left invariant orthonormal frame on $\mathcal{H}_3$ are given by

$$e_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},$$

for which we have the Lie brackets

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0.$$

Also, the Livi-Civita connection $\tilde{\nabla}$ of $\mathcal{H}_3$ is expressed as

$$\begin{aligned}
&\tilde{\nabla}_{e_1} e_1 = 0, && \tilde{\nabla}_{e_1} e_2 = \frac{1}{2} e_3, && \tilde{\nabla}_{e_1} e_3 = -\frac{1}{2} e_2, \\
&\tilde{\nabla}_{e_2} e_1 = -\frac{1}{2} e_3, && \tilde{\nabla}_{e_2} e_2 = 0, && \tilde{\nabla}_{e_2} e_3 = \frac{1}{2} e_1, \\
&\tilde{\nabla}_{e_3} e_1 = -\frac{1}{2} e_2, && \tilde{\nabla}_{e_3} e_2 = \frac{1}{2} e_1, && \tilde{\nabla}_{e_3} e_3 = 0.
\end{aligned}$$

A translation surface $\Sigma(\alpha, \beta)$ in the 3-dimensional Heisenberg group $\mathcal{H}_3$ is a surface parametrized by

$$x : \Sigma \to \mathcal{H}_3, \quad x(s,t) = \alpha(s) * \beta(t),$$

where α and β are any generating curves in $\mathbb{R}^3$. Since the group operation $*$ is not commutative, we have two translation surfaces, namely $\Sigma(\alpha, \beta)$ and $\Sigma(\beta, \alpha)$, which are different.

Now, we define translation surfaces in $\mathcal{H}_3$ generated by two planar curves α and β lying in not orthogonal planes. According to planar curves α and β, we distinguish six types as follows:

First, we assume that $\alpha(s)$ lies in the xz-plane of $\mathbb{R}^3$ and $\beta(t)$ in the plane with equation $x \cos\theta - y \sin\theta = 0$. This means that

$$\begin{aligned}
\alpha(s) &= (s, 0, f(s)), \\
\beta(t) &= (t \sin\theta, t \cos\theta, g(t)).
\end{aligned}$$

In this case, we have two translation surfaces $\Sigma_1(\alpha, \beta)$ and $\Sigma_4(\alpha, \beta)$ parametrized by, respectively

$$\begin{aligned}
x(s,t) &= \alpha(s) * \beta(t) \\
&= \left(s + t \sin\theta, t \cos\theta, f(s) + g(t) + \frac{st \cos\theta}{2} \right)
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
x(s,t) &= \beta(t) * \alpha(s) \\
&= \left(s + t \sin\theta, t \cos\theta, f(s) + g(t) - \frac{st \cos\theta}{2} \right),
\end{aligned} \tag{2.4}$$

which are called the translation surfaces of type 1 and 4.

Second, if a curve $\alpha(s)$ lies in the xz-plane of $\mathbb{R}^3$ and $\beta(t)$ in the plane with equation $z \cos\theta - y \sin\theta = 0$, that is,

$$\begin{aligned}
\alpha(s) &= (s, 0, f(s)), \\
\beta(t) &= (g(t), \cos\theta t, \sin\theta t),
\end{aligned}$$

then the parametrization of $\Sigma_2(\alpha, \beta)$ is given by

$$\begin{aligned} x(s,t) &= \alpha(s) * \beta(t) \\ &= \left(s + g(t), \cos\theta t, f(s) + \sin\theta t + \frac{st\cos\theta}{2}\right) \end{aligned} \tag{2.5}$$

and the parametrization of $\Sigma_5(\beta, \alpha)$ is given by

$$\begin{aligned} x(s,t) &= \beta(t) * \alpha(s) \\ &= \left(s + g(t), \cos\theta t, f(s) + \sin\theta t - \frac{st\cos\theta}{2}\right). \end{aligned} \tag{2.6}$$

The surfaces Σ_2 and Σ_5 are said to be of type 2 and type 5, respectively.

Third, we consider two curves $\alpha(s)$ and $\beta(t)$ lying the yz-plane of $\mathbb{R}^3$ and the plane with equation $\sin\theta x - \cos\theta z = 0$, respectively. In this case, $\alpha(s)$ and $\beta(t)$ are given by

$$\begin{aligned} \alpha(s) &= (0, s, f(s)), \\ \beta(t) &= (\cos\theta t, g(t), \sin\theta t). \end{aligned}$$

Thus, the translation surface $\Sigma_3(\alpha, \beta)$ is parametrized by

$$\begin{aligned} x(s,t) &= \alpha(s) * \beta(t) \\ &= \left(\cos\theta t, s + g(t), f(s) + \sin\theta t - \frac{st\cos\theta}{2}\right) \end{aligned} \tag{2.7}$$

and the translation surface $\Sigma_6(\beta, \alpha)$ is parametrized by

$$\begin{aligned} x(s,t) &= \beta(t) * \alpha(s) \\ &= \left(\cos\theta t, s + g(t), f(s) + \sin\theta t + \frac{st\cos\theta}{2}\right), \end{aligned} \tag{2.8}$$

which are called the translation surfaces of type 3 and type 6, respectively.

Remark 2.1. If $\theta = k\pi, k \in \mathbb{N}$ in all types, the surfaces defined in [3] appear. So, translation surfaces of type 1-6 is generalization of surfaces given in [3].

3. Minimal translation surfaces of type 1 and type 4

Let Σ_1 be a translation surface of type 1 in the 3-dimensional Heisenberg space $\mathcal{H}_3$. Then, Σ_1 is parametrized by

$$x(s,t) = \left(s + t\sin\theta, t\cos\theta, f(s) + g(t) + \frac{st\cos\theta}{2}\right). \tag{3.1}$$

We have the natural frame $\{x_s, x_t\}$ given by

$$\begin{aligned} \frac{\partial x}{\partial s} &:= x_s = e_1 + (f'(s) + t\cos\theta)e_3, \\ \frac{\partial x}{\partial t} &:= x_t = \sin\theta e_1 + \cos\theta e_2 + g'(t)e_3. \end{aligned}$$

From this, the unit normal vector field U of Σ_1 is given by

$$U = w^{-\frac{1}{2}}\left(-\cos\theta(f'(s)+t\cos\theta)e_1 + (\sin\theta(f'(s)+t\cos\theta) - g'(t))e_2 + \cos\theta e_3\right),$$

where $w = ||x_s \times x_t||^2$.

The coefficients of the first fundamental form of Σ_1 are given by

$$\begin{aligned} E &= \langle x_s, x_s\rangle = 1 + (f'(s) + t\cos\theta)^2, \\ F &= \langle x_s, x_t\rangle = \sin\theta + g'(t)(f'(s) + t\cos\theta), \\ G &= \langle x_t, x_t\rangle = 1 + g'(t)^2. \end{aligned}$$

To compute the second fundamental form of Σ_1, we have to calculate the following:

$$\begin{aligned} &\tilde{\nabla}_{x_s} x_s = -(f'(s) + t\cos\theta)e_2 + f''(s)e_3, \\ &\tilde{\nabla}_{x_s} x_t = \frac{1}{2}\cos\theta(f'(s)+t\cos\theta)e_1 - \frac{1}{2}(\sin\theta(f'(s)+t\cos\theta) + g'(t))e_2 + \frac{1}{2}\cos\theta e_3, \\ &\tilde{\nabla}_{x_t} x_t = \cos\theta g'(t)e_1 - \sin\theta g'(t)e_2 + g''(t)e_3, \end{aligned}$$

which imply the coefficients of the second fundamental form of Σ_1 are given by

$$\begin{aligned} L = \langle\tilde{\nabla}_{x_s} x_s, U\rangle &= -w^{-1/2}\left(\sin\theta f'(s)^2 + 2f'(s)\sin\theta\cos\theta t - f'(s)g'(t)\right. \\ &\quad \left. + \sin\theta\cos^2\theta t^2 - \cos\theta g'(t)t - \cos\theta f''(s)\right), \\ M = \langle\tilde{\nabla}_{x_s} x_t, U\rangle &= -\frac{1}{2}w^{-1/2}(f'(s)^2 + 2t\cos\theta f'(s) + t^2\cos^2\theta - g'(t)^2 - \cos\theta), \\ N = \langle\tilde{\nabla}_{x_t} x_t, U\rangle &= -w^{-1/2}(f'(s)g'(t) + t\cos\theta g'(t) - \sin\theta g'(t)^2 - \cos\theta g''(t)). \end{aligned}$$

Thus, the mean curvature of Σ_1 is given by

(3.2)

$$\begin{aligned} H = \frac{1}{2}w^{-3/2}\cos\theta(&-g''(t) - f''(s) + t\cos^2\theta g'(t) + \cos\theta f'(s)g'(t) - f'(s)^2 g''(t) \\ &- t^2\cos^2\theta g''(t) - f''(s)g'(t)^2 + \sin\theta\cos\theta - 2t\cos\theta f'(s)g''(t)). \end{aligned}$$

If $\cos\theta = 0$, then $H = 0$, that is, minimal. In this case, Σ_1 is an open part of a plane (trivial minimal surface).

We suppose that the translation surface Σ_1 of type 1 is non-trivial minimal. Then from (3.2) we obtain

$$\begin{aligned} &f''(s)(1 + g'(t)^2) - (\cos\theta f'(s) + t\cos^2\theta)g'(t) \\ &+ (1 + (f'(s) + t\cos\theta)^2)g''(t) - \sin\theta\cos\theta = 0. \end{aligned} \tag{3.3}$$

In order to solve the above ordinary differential equation, divide by $1 + g'(t)^2 \neq 0$. We obtain

$$\begin{aligned} &f''(s) - (\cos\theta f'(s) + t\cos^2\theta)\frac{g'(t)}{1 + g'(t)^2} \\ &+ \frac{g''(t)}{1 + g'(t)^2}(1 + (f'(s) + t\cos\theta)^2) - \frac{\sin\theta\cos\theta}{1 + g'(t)^2} = 0. \end{aligned}$$

Taking the derivative with respect to s, we have

$$f'''(s) - \cos\theta f''(s)\frac{g'(t)}{1+g'(t)^2} + 2\frac{g''(t)}{1+g'(t)^2}(f'(s)+t\cos\theta)f''(s) = 0. \tag{3.4}$$

The case $f''(s) = 0$ will be treated separately. First of all, let us suppose that $f''(s) \neq 0$ on an open interval. Then equation (3.4) becomes

$$\frac{f'''(s)}{f''(s)} - \cos\theta\frac{g'(t)}{1+g'(t)^2} + 2\frac{g''(t)}{1+g'(t)^2}(f'(s)+t\cos\theta) = 0. \tag{3.5}$$

Differentiating (3.5) with respect to t, we have

$$\cos\theta\frac{d}{dt}\left(\frac{g'(t)}{1+g'(t)^2}\right) - 2(f'(s)+t\cos\theta)\frac{d}{dt}\left(\frac{g''(t)}{1+g'(t)^2}\right) - 2\frac{g''(t)}{1+g'(t)^2}\cos\theta = 0. \tag{3.6}$$

If $\frac{d}{dt}\left(\frac{g''(t)}{1+g'(t)^2}\right) \neq 0$, then by (3.6) $f'(s)+t\cos\theta$ depends only on t. So, we get $f''(s) = 0$, it is a contradiction. Thus, $\frac{g''(t)}{1+g'(t)^2} = A$ (=constant). It follows that equation (3.6) rewritten as the form:

$$\cos\theta A\left(1+\frac{2g'(t)^2}{1+g'(t)^2}\right) = 0.$$

From this, we have $A = 0$ because $\cos\theta \neq 0$. Thus, $g''(t) = 0$. We put $g(t) = ct + d$ $(c, d \in \mathbb{R})$. Substituting it in (3.3), we get

$$f''(s)(1+c^2) - (\cos\theta f'(s) + t\cos^2\theta)c - \sin\theta\cos\theta = 0,$$

it follows that $c = 0$ and $f''(s) = \cos\theta\sin\theta$. We put $f'(s) = \sin\theta\cos\theta s + a$ $(a \in \mathbb{R})$. Substituting it in (3.3), one obtain

$$\sin\theta\cos\theta(1+c^2) - (\sin\theta\cos^2\theta + a\cos\theta + t\cos^2\theta)c = 0,$$

which implies $c = 0$ and $\sin\theta\cos\theta = 0$. It is a contradiction. Consequently, for any minimal translation surface of type 1, we have $f''(s) = 0$.

Take $f(s) = as + b$ $(a, b \in \mathbb{R})$. From this, equation (3.3) becomes

$$g''(t) - \frac{a\cos\theta + t\cos^2\theta}{1+(a+t\cos\theta)^2}g'(t) - \frac{\sin\theta\cos\theta}{1+(a+t\cos\theta)^2} = 0.$$

We can easily find a general solution of the above ODE and its solution is given by

$$\begin{aligned} g(t) &= \frac{c_1}{\cos\theta}\left[(a+t\cos\theta)\sqrt{1+(a+t\cos\theta)^2} + \ln(a+t\cos\theta+\sqrt{1+(a+t\cos\theta)^2})\right] \\ &\quad + \sin\theta(at + \frac{1}{2}\cos\theta t^2 + c_2), \end{aligned} \tag{3.7}$$

where c_1 and c_2 are constants of integration.

Thus, we have the following:

Theorem 3.1. *A translation surface of type 1 in the 3-dimensional Heisenberg space $\mathcal{H}_3$ is a non-trivial minimal surface if and only if the surface can be parametrized as*

$$x(s,t) = \left(s + t\sin\theta, t\cos\theta, f(s) + g(t) + \frac{st\cos\theta}{2}\right),$$

where $f(s) = as + b$ $(a, b \in \mathbb{R})$ and $g(t)$ is given by (3.7).

Let Σ_4 be a translation surface of type 4 in the 3-dimensional Heisenberg space $\mathcal{H}_3$. Then, Σ_4 is parametrized by

$$x(s,t) = \left(s + t\sin\theta, t\cos\theta, f(s) + g(t) - \frac{st\cos\theta}{2}\right). \tag{3.8}$$

By a long computation, we can obtain the mean curvature H as

$$\begin{aligned} H = \frac{1}{2}w^{-3/2}\Big(&g''(t) + f''(s) - s\cos^2\theta f'(s) + \cos\theta f'(s)g'(t) + f''(s)g'(t)^2 \\ &+s^2\cos^2\theta f''(s) + f'(s)^2 g''(t) + \sin\theta\cos\theta - 2s\cos\theta f''(s)g'(t)\Big). \end{aligned}$$

Suppose that the translation surface Σ_4 of type 4 is non-trivial minimal. Then, by using similar method of the translation surface of type 1, we have the following result:

Theorem 3.2. *A translation surface of type 4 in the 3-dimensional Heisenberg space $\mathcal{H}_3$ is a non-trivial minimal surface if and only if the surface can be parametrized as*

$$x(s,t) = \left(s + t\sin\theta, t\cos\theta, f(s) + g(t) - \frac{st\cos\theta}{2}\right),$$

where

$$\begin{aligned} &f(s) \\ =& \frac{c}{\cos\theta}\left[(a + s\cos\theta)\sqrt{1 + (a+s\cos\theta)^2} + \ln(a + s\cos\theta + \sqrt{1 + (a + s\cos\theta)^2})\right] \\ &+ \sin\theta(as + \frac{1}{2}\cos\theta s^2 + d) \end{aligned}$$

and $g(t) = -at + b$ with $a, b, c, d \in \mathbb{R}$.

4. Minimal translation surfaces of type 2 and type 5

Let Σ_2 be a translation surface of type 2 in the 3-dimensional Heisenberg space $\mathcal{H}_3$. Then, Σ_2 is parametrized by

$$x(s,t) = \left(s + g(t), \cos\theta t, f(s) + \sin\theta t + \frac{st\cos\theta}{2}\right). \tag{4.1}$$

It follows that we have

$$x_s = e_1 + (f'(s) + t\cos\theta)e_3,$$
$$x_t = g'(t)e_1 + \cos\theta e_2 + \left(\frac{1}{2}\cos\theta g'(t)t - \frac{1}{2}\cos\theta g(t) + \sin\theta\right)e_3$$

and the unit normal vector U is

$$U = w^{-1/2}\Big(-\cos\theta(f'(s) + t\cos\theta)e_1 + \left[g'(t)(f'(s) + \cos\theta t) - \frac{1}{2}\cos\theta g'(t)t + \frac{1}{2}\cos\theta g(t) - \sin\theta\right]e_2 + \cos\theta e_3\Big).$$

On the other hand, the coefficients of the first fundamental form of Σ_2 are given by

$$E = 1 + (f'(s) + t\cos\theta)^2,$$
$$F = g'(t) + (f'(s) + \cos\theta t)\left(\frac{1}{2}\cos\theta g'(t)t - \frac{1}{2}\cos\theta g(t) + \sin\theta\right),$$
$$G = g'(t)^2 + 1 + \frac{\cos^2\theta}{4}(g'(t)t - g(t))^2 + \sin\theta\cos\theta(g'(t)t - g(t)).$$

By a straightforward calculation, we obtain

$$\tilde{\nabla}_{x_s}x_s = -(f'(s) + t\cos\theta)e_2 + f''(s)e_3,$$
$$\tilde{\nabla}_{x_s}x_t = \frac{1}{2}\cos\theta(f'(s) + t\cos\theta)e_1 - \frac{1}{2}\Big(g'(t)(f'(s) + \cos\theta t) + \frac{1}{2}\cos\theta g'(t)t - \frac{1}{2}\cos\theta g(t) + \sin\theta\Big)e_2 + \frac{1}{2}\cos\theta e_3,$$
$$\tilde{\nabla}_{x_t}x_t = \left(g''(t) + \cos\theta\left[\frac{1}{2}\cos\theta g'(t)t - \frac{1}{2}\cos\theta g(t) + \sin\theta\right]\right)e_1 - g'(t)\left(\frac{1}{2}\cos\theta g'(t)t - \frac{1}{2}\cos\theta g(t) + \sin\theta\right)e_2 + \frac{1}{2}\cos\theta g''(t)te_3.$$

Using the data described above, we can calculate the coefficients of the second fundamental form of Σ_2 and obtain also the mean curvature H as follows:

$$H = \frac{1}{8}w^{-3/2}\cos\theta\left(T_0 f''(s) + T_1 f'(s) + T_2 f'(s)^2 + T_3 f'(s)^3 + T_4\right), \tag{4.2}$$

where

$$T_0 = 4g'(t)^2 + 4 + \cos^2\theta(g'(t)t - g(t))^2 + 4\sin\theta\cos\theta(g'(t)t - g(t)) = 4G,$$
$$T_1 = -4g''(t) - 4\sin\theta\cos\theta + 2\cos^2\theta g(t) - 2\cos^2\theta g'(t)t - 8\cos^2\theta t^2 g''(t),$$
$$T_2 = -10\cos\theta g''(t)t,$$
$$T_3 = -4g''(t),$$
$$T_4 = -4\sin\theta\cos^2\theta t + 2\cos^3\theta g(t)t - 2\cos^3\theta g'(t)t^2 - 4\cos\theta g'(t) - 2\cos\theta g''(t)t - 2\cos^3\theta t^3 g''(t).$$

If $\cos\theta = 0$, then $H = 0$, that is, Σ_2 is an open part of a plane (trivial minimal surface).

Suppose that the translation surface Σ_2 of type 2 is non-trivial minimal. Then from (4.2) we have

$$T_0 f''(s) + T_1 f'(s) + T_2 f'(s)^2 + T_3 f'(s)^3 + T_4 = 0. \tag{4.3}$$

Since $T_0 = \cos^2\theta(g'(t)t - g(t) + 2\tan\theta)^2 + 4(g'(t)^2 + \cos^2\theta) \neq 0$, we divide (4.3) by T_0 and take the derivatives with respect to s and t, respectively. We obtain

$$\frac{d}{dt}\left(\frac{T_1}{T_0}\right) f''(s) + 2\frac{d}{dt}\left(\frac{T_2}{T_0}\right) f'(s) f''(s) + 3\frac{d}{dt}\left(\frac{T_3}{T_0}\right) f'(s)^2 f''(s) = 0.$$

The case $f''(s) = 0$ will be treated separately. From now on, let us suppose that $f''(s) \neq 0$ on an open interval. Dividing by $f''(s)$ and differentiating the above equation with respect to s twice, we can obtain

$$\frac{d}{dt}\left(\frac{T_1}{T_0}\right) = 0, \quad \frac{d}{dt}\left(\frac{T_2}{T_0}\right) = 0, \quad \frac{d}{dt}\left(\frac{T_3}{T_0}\right) = 0.$$

By using the previous equation and the relation of T_2 and T_3, we show that $g''(t) = 0$, that is, $g(t) = at + b$ $(a, b \in \mathbb{R})$. Substituting it in (4.3), one obtain

$$f(s) = \frac{a\cos\theta}{2(a^2 + 1 + 2\sin^2\theta)} s^2 + cs + d,$$

where $c, d \in \mathbb{R}$.

Now, we consider $f''(s) = 0$. Then, $f(s) = as + b$ $(a, b \in \mathbb{R})$. It follows that equation (4.3) becomes

$$\begin{aligned}&\left[(2a + \cos\theta t)(1 + (\cos\theta t + a)^2)\right] g''(t) + \left[2\cos\theta + t(a\cos^2\theta + \cos^3\theta t)\right] g'(t)\\ &- (a\cos^2\theta + \cos^3\theta t) g(t) + 2\sin\theta\cos\theta + 2\sin\theta\cos^2\theta t = 0.\end{aligned}$$

Denote $p(t) = (2a + \cos\theta t) g(t)$. The above equation can be rewritten as the form:

$$p''(t) - \frac{\cos\theta(a + \cos\theta t)}{1 + (a + \cos\theta)^2} p'(t) = \frac{-2\sin\theta\cos\theta - 22\sin\theta\cos^2\theta t}{1 + (a + \cos\theta)^2},$$

which has the solution

$$\begin{aligned}&p(t)\\ &= \frac{c_1}{2}\left((a + \cos\theta t)\sqrt{1 + (a + \cos\theta t)^2} + \ln(a + \cos\theta t + \sqrt{1 + (a + \cos\theta t)^2})\right)\\ &\quad + 2\sin\theta t + c_2,\end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$. Thus, a function $g(t)$ is given by

$$\begin{aligned} g(t) = & \frac{c_1}{2(2a+\cos\theta t)}\Big((a+\cos\theta t)\sqrt{1+(a+\cos\theta t)^2} \\ & +\ln(a+\cos\theta t+\sqrt{1+(a+\cos\theta t)^2})\Big) \\ & +\frac{2\sin\theta t}{2a+\cos\theta t}+\frac{c_2}{2a+\cos\theta t}, \end{aligned} \tag{4.4}$$

Consequently, we have the following:

Theorem 4.1. *A translation surface of type 2 in the 3-dimensional Heisenberg space $\mathcal{H}_3$ is a non-trivial minimal surface if and only if the surface can be parametrized as*

$$x(s,t) = \left(s+g(t), \cos\theta t, f(s)+\sin\theta t+\frac{st\cos\theta}{2}\right),$$

where
(i) *either* $f(s) = \frac{a\cos\theta}{2(a^2+1+2\sin^2\theta)}s^2+cs+d$ *and* $g(t) = at+b$ *with* $a, b, c, d \in \mathbb{R}$,
(ii) *or* $f(s) = as+b$ *and* $g(t)$ *is given by* (4.4) *with* $a, b \in \mathbb{R}$.

Let Σ_5 be a translation surface of type 5 in the 3-dimensional Heisenberg space $\mathcal{H}_3$. Then, Σ_5 is parametrized by

$$x(s,t) = \left(s+g(t), \cos\theta t, f(s)+\sin\theta t-\frac{st\cos\theta}{2}\right). \tag{4.5}$$

In this case, by straightforward computation the mean curvature is given by

$$H = \frac{1}{8}w^{-3/2}\cos\theta\left(P_0 f''(s)+P_1 f'(s)+P_2 f'(s)^2+P_3 f'(s)^3+P_4\right),$$

where

$$\begin{aligned} P_0 &= 4g'(t)^2+4+\cos^2\theta(tg'(t)-g(t))^2+4\sin\theta\cos\theta(tg'(t)-g(t)) \\ &\quad -4\cos^2\theta s(tg'(t)-g(t))-8\sin\theta\cos\theta s+4\cos^2\theta s^2, \\ P_1 &= -4g''(t)+2\cos^2\theta(tg'(t)-g(t))-4\cos^2\theta s+4\sin\theta\cos\theta, \\ P_2 &= 2\cos\theta g''(t)t, \\ P_3 &= -4g''(t), \\ P_4 &= 2\cos\theta(g''(t)t+2g'(t)). \end{aligned}$$

If Σ_5 is a non-trivial minimal translation surface, then it satisfies the equation

$$P_0 f''(s)+P_1 f'(s)+P_2 f'(s)^2+P_3 f'(s)^3+P_4 = 0. \tag{4.6}$$

Differentiating (4.6) with respect to s, we have

$$P_0 f'''(s)+((P_0)_s+P_1)f''(s)+(P_1)_s f'(s)+2P_2 f'(s)f''(s)+3P_3 f'(s)^2 f''(s) = 0. \tag{4.7}$$

With respect to the surface Σ_5 of type 5, equation (4.7) is expressed as a very complicated ordinary differential equation. So, we give examples of non-trivial minimal translation surfaces by distinguishing some special cases:

1. If f is a constant function, then equation (4.6) leads to $g''(t)t+2g'(t)=0$. It follows that $g(t)=b$ or $g(t)=\frac{c}{t}+d$ $(b,c,d\in\mathbb{R})$.

2. Assume $f''(s)=0$. Then, from (4.7) $(P_1)_s f'(s)=0$, which implies $f'(s)=0$. This case is contained in the previous one.

3. If g is a constant function, that is, $g(t)=a$ $(a\in\mathbb{R})$, then, equation (4.6) writes as

$$\begin{aligned}&\left(\cos^2\theta(2s+a)^2-4\sin\theta\cos\theta(2s+a)+4\right)f''(s)\\&+(4\sin\theta\cos\theta-2\cos^2\theta a-4\cos^2\theta s)f'(s)=0.\end{aligned}$$

In this case, we have $f(s)=$ constant or

$$\begin{aligned}f(s)=\frac{b\cos\theta}{4}&\Big((2s+a-2\tan\theta)\sqrt{(2s+a-2\tan\theta)^2+4}\\&+4\ln(2s+a-2\tan\theta+\sqrt{(2s+a-2\tan\theta)^2+4})+c\Big),\end{aligned}\tag{4.8}$$

where b and c are constants of integration.

4. Suppose that $g''(t)=0$, that is, $g(t)=at+b$ $(a,b\in\mathbb{R})$. In this case, from (4.6) we obtain the following differential equation

$$\begin{aligned}&\left(4a^2+4+\cos^2\theta(2s+b)^2-4\sin\theta\cos\theta(2s+b)\right)f''(s)\\&+(-2\cos^2\theta b-4\cos^2\theta s+4\sin\theta\cos\theta)f'(s)+4a\cos\theta=0\end{aligned}$$

with the general solution

$$\begin{aligned}f(s)=&-\frac{a\cos\theta}{a^2+\cos^2\theta}s^2-\frac{a}{a^2+\cos^2\theta}(b\cos\theta+2\sin\theta)s+c_1\\&+\frac{c_2(\cos\theta(2s+b)-2\sin\theta)}{2\cos\theta}\sqrt{[\cos\theta(2s+b)-2\sin\theta]^2+4(a^2+\cos^2\theta)}\\&+\frac{2c_2(a^2+\cos^2\theta)}{\cos\theta}\ln\big(\cos\theta(2s+b)-2\sin\theta\\&\quad+\sqrt{[\cos\theta(2s+b)-2\sin\theta]^2+4(a^2+\cos^2\theta)}\big),\end{aligned}\tag{4.9}$$

where c_1 and c_2 are constants of integration.

Proposition 4.2. *Examples of non-trivial minimal translation surfaces of type 5 in the 3-dimensional Heisenberg group are the surfaces whose parametrization is*

$$x(s,t)=\left(s+g(t),\cos\theta t,f(s)+\sin\theta t-\frac{st\cos\theta}{2}\right)$$

given by

(1) $f(s)=a$ *and* $g(t)=b$.
(2) $f(s)=a$ *and* $g(t)=\frac{b}{t}+c$.
(3) $g(t)=a$ *and* $f(s)$ *is given by* (4.8).

(4) $g(t) = at + b$ *and* $f(s)$ *is given by* (4.9),
where $a, b, c \in \mathbb{R}$.

5. Minimal translation surfaces of type 3 and type 6

Let Σ_3 be a translation surface of type 3 in the 3-dimensional Heisenberg space $\mathcal{H}_3$. Then, Σ_3 is parametrized by

$$x(s,t) = \left(\cos\theta t, s + g(t), f(s) + \sin\theta t - \frac{st\cos\theta}{2}\right). \tag{5.1}$$

It follows that we have

$$\begin{aligned} x_s &= e_2 + (f'(s) - t\cos\theta)e_3, \\ x_t &= \cos\theta e_1 + g'(t)e_2 + \left(-\frac{1}{2}\cos\theta g'(t)t + \frac{1}{2}\cos\theta g(t) + \sin\theta\right)e_3, \end{aligned}$$

which imply the coefficients of the first fundamental form of Σ_3 are given by

$$\begin{aligned} E &= 1 + (f'(s) - t\cos\theta)^2, \\ F &= g'(t) + (f'(s) - \cos\theta t)\left(-\frac{1}{2}\cos\theta g'(t)t + \frac{1}{2}\cos\theta g(t) + \sin\theta\right), \\ G &= g'(t)^2 + 1 + \frac{\cos^2\theta}{4}(g'(t)t - g(t))^2 + \sin\theta\cos\theta(g(t) - g'(t)t), \end{aligned}$$

and the unit normal vector is given by

$$\begin{aligned} U = \frac{1}{2}w^{-1/2}\big(&[-2g'(t)(f'(s) - \cos\theta t) - \cos\theta g'(t)t + \cos\theta g(t) + 2\sin\theta]e_1 \\ &+2\cos\theta(f'(s) - t\cos\theta)e_2 - 2\cos\theta e_3\big). \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} \tilde{\nabla}_{x_s}x_s &= (f'(s) - t\cos\theta)e_1 + f''(s)e_3, \\ \tilde{\nabla}_{x_s}x_t &= \frac{1}{2}\left(g'(t)(f'(s) - \cos\theta t) - \frac{1}{2}\cos\theta g'(t)t + \frac{1}{2}\cos\theta g(t) + \sin\theta\right)e_1 \\ &\quad - \frac{1}{2}\cos\theta(f'(s) - t\cos\theta)e_2 - \frac{1}{2}\cos\theta e_3, \\ \tilde{\nabla}_{x_t}x_t &= g'(t)\left(-\frac{1}{2}\cos\theta g'(t)t + \frac{1}{2}\cos\theta g(t) + \sin\theta\right)e_1 \\ &\quad + \left(g''(t) - \cos\theta\left(-\frac{1}{2}\cos\theta g'(t)t + \frac{1}{2}\cos\theta g(t) + \sin\theta\right)\right)e_2 \\ &\quad - \frac{1}{2}\cos\theta g''(t)te_3. \end{aligned}$$

Using the data described above, we get the mean curvature H as

$$H = \frac{1}{8}w^{-3/2}\cos\theta\left[Q_0 f''(s) + Q_1 f'(s) + Q_2 f'(s)^2 + Q_3 f'(s)^3 + Q_4\right], \tag{5.2}$$

where

$$\begin{aligned}
Q_0 &= -[4g'(t)^2 + 4 + \cos^2\theta(g'(t)t - g(t))^2 - 4\sin\theta\cos\theta(g'(t)t - g(t))] \\
&= -4G, \\
Q_1 &= 4g''(t) - 4\sin\theta\cos\theta - 2\cos^2\theta g(t) + 2\cos^2\theta g'(t)t + 8\cos^2\theta t^2 g''(t) \\
Q_2 &= -10\cos\theta g''(t)t, \\
Q_3 &= 4g''(t), \\
Q_4 &= -4\sin\theta\cos^2\theta t - 2\cos^3\theta g(t)t + 2\cos^3\theta g'(t)t^2 + 4\cos\theta g'(t) \\
&\quad + 2\cos\theta g''(t)t + 2\cos^3\theta t^3 g''(t).
\end{aligned}$$

If $\cos\theta = 0$, then $H = 0$, that is, Σ is an open part of a plane (trivial minimal surface).

Suppose that the translation surface Σ of type 3 is non-trivial minimal. Then from (5.2) we have

$$Q_0 f''(s) + Q_1 f'(s) + Q_2 f'(s)^2 + Q_3 f'(s)^3 + Q_4 = 0.$$

Using the same algebraic techniques as in the case of surfaces of type 2, we get:

Theorem 5.1. *A translation surface of type 3 in the 3-dimensional Heisenberg space $\mathcal{H}_3$ is a non-trivial minimal surface if and only if the surface can be parametrized as*

$$x(s,t) = \left(\cos\theta t, s + g(t), f(s) + \sin\theta t - \frac{st\cos\theta}{2}\right),$$

where

(i) *either* $f(s) = \frac{a\cos\theta}{2(a^2+1+2\sin^2\theta)}s^2 + cs + d$ *and* $g(t) = -at + b$ *with* $a, b, c, d \in \mathbb{R}$,

(ii) *or* $f(s) = as + b$ *and*

$$\begin{aligned}
g(t) = {} & \frac{c}{2(2a - \cos\theta t)}\Big[(a - \cos\theta t)\sqrt{1 + (a - \cos\theta t)^2} \\
& + \ln(a - \cos\theta t + \sqrt{1 + (a - \cos\theta t)^2})\Big] \\
& + \frac{2\sin\theta t}{2a - \cos\theta t} + \frac{d}{2a - \cos\theta t},
\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

Let Σ_6 be a translation surface of type 6 in the 3-dimensional Heisenberg space $\mathcal{H}_3$. Then, Σ_6 is parametrized by

$$x(s,t) = \left(\cos\theta t, s + g(t), f(s) + \sin\theta t + \frac{st\cos\theta}{2}\right). \tag{5.3}$$

If Σ_6 is a non-trivial minimal translation surface, then it satisfies the equation

$$R_0 f''(s) + R_1 f'(s) + P_2 f'(s)^2 - P_3 f'(s)^3 + P_4 = 0, \tag{5.4}$$

where

$$\begin{aligned} R_0 &= -4g'(t)^2 - 4 - \cos^2\theta(tg'(t) - g(t))^2 + 4\sin\theta\cos\theta(tg'(t) - g(t)) \\ &\quad + 4\cos^2\theta s(tg'(t) - g(t)) - 8\sin\theta\cos\theta s - 4\cos^2\theta s^2, \\ R_1 &= 4g''(t) - 2\cos^2\theta(tg'(t) - g(t)) + 4\cos^2\theta s + 4\sin\theta\cos\theta. \end{aligned}$$

Applying the same method as in the case of surfaces of type 5, we can obtain the following:

Proposition 5.2. *Examples of non-trivial minimal translation surfaces of type 6 in the 3-dimensional Heisenberg group are the surfaces whose parametrization is*

$$x(s,t) = \left(\cos\theta t, s + g(t), f(s) + \sin\theta t + \frac{st\cos\theta}{2}\right)$$

given by

(1) $f(s) = a$ *and* $g(t) = b$.

(2) $f(s) = a$ *and* $g(t) = \frac{b}{t} + c$.

(3) $g(t) = a$ *and*

$$\begin{aligned} f(s) = \frac{b\cos\theta}{4}\Big((2s + a - 2\tan\theta)\sqrt{(2s + a - 2\tan\theta)^2 + 4} \\ +4\ln(2s + a - 2\tan\theta + \sqrt{(2s + a - 2\tan\theta)^2 + 4}) + c\Big). \end{aligned}$$

(4) $g(t) = at + b$ *and*

$$\begin{aligned} f(s) = &\frac{a\cos\theta}{a^2 + \cos^2\theta}s^2 - \frac{a}{a^2 + \cos^2\theta}(b\cos\theta + 2\sin\theta)s + c_1 \\ &+ \frac{c_2(\cos\theta(2s + b) - 2\sin\theta)}{2\cos\theta}\sqrt{[\cos\theta(2s + b) - 2\sin\theta]^2 + 4(a^2 + \cos^2\theta)} \\ &+ \frac{2c_2(a^2 + \cos^2\theta)}{\cos\theta}\ln\Big(\cos\theta(2s + b) - 2\sin\theta \\ &\quad + \sqrt{[\cos\theta(2s + b) - 2\sin\theta]^2 + 4(a^2 + \cos^2\theta)}\Big), \end{aligned}$$

where $a, b, c, c_1, c_2 \in \mathbb{R}$.

Remark 5.3. There are infinite numbers of minimal surfaces for every $\theta \in \mathbb{R}$.

References

[1] F. Dillen, I. V. de Woestijne, L. Verstraelen, and J. Walrave, *The surface of Scherk in* $\mathbb{E}^3$ *: a special case in the class of minimal surfaces defined as the sum of two curves*, Bull. Inst. Math. Acad. Sinica **26** (1998), no. 4, 257–267.

[2] J. Inoguchi, T. Kumamoto, N. Ohsugi, and Y. Suyama, *Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces. II*, Fukuoka Univ. Sci. Rep. **30** (2000), no. 1, 17–47.

[3] J. Inoguchi, R. López, and M. I. Munteanu, *Minimal translation surfaces in the Heisenberg group* Nil_3, Geom. Dedicata **161** (2012), 221–231.

[4] O. Kobayashi, *Maximal surfaces in the 3-dimensional Minkowski space* L^3, Tokyo J. Math. **6** (1983), no. 2, 297–309.

[5] R. López, *Minimal translation surfaces in hyperbolic space*, Beitr. Algebra Geom. **52** (2011), no. 1, 105–112.

[6] R. López and M. I. Munteanu, *Minimal translation surfaces in* Sol_3, J. Math. Soc. Japan **64** (2012), no. 3, 985–1003.

[7] W. Thurston, *Three Dimensional Geometry and Topology. Vol. 1*, Princeton Math. Ser. 35, Princeton University Press, Princeton. NJ, 1997.

[8] I. V. de Woestijne, *Minimal surfaces in the 3-dimensional Minkowski space*, Geomety and Topology of submanifolds, II, 344–369, World Scientific, Singapore, 1990.

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