Bull. Korean Math. Soc. **50** (2013), No. 4, pp. 1297–1302
http://dx.doi.org/10.4134/BKMS.2013.50.4.1297

FINITENESS OF MAPPING CLASS GROUPS

Sungbok Hong and Jin Yang

Abstract. We prove that the mapping class group of a non-Haken orientable irreducible 3-manifold is finite and we show that the quotient group of the mapping class group by the rotation group is virtually torsion-free if the manifold does not have 2-sphere boundary components.

1. Introduction

A compact irreducible 3-manifold M is called Haken if it contains a two sided incompressible surface. If M is a compact 3-manifold and either $H_1(M)$ is infinite or $\partial M \neq \emptyset$, then M contains a properly embedded 2-sided, incompressible surface. Hence M is Haken (see [9, Lemma 6.6] or [10, Theorem III.10] for more details). It is known that every closed manifold obtained by Dehn surgery from figure eight knot complement is non-Haken and $\pi_1(M)$ is infinite.

Thurston's geometrization theorem asserts that if each boundary component of a compact atoroidal Haken 3-manifold has zero Euler characteristic, then the interior of the manifold admits a complete hyperbolic metric of finite volume. Perelman among other things proved Thurson's geometrization conjecture for non-Haken manifolds which states that if M is an orientable non-Haken 3-manifold such that $\pi_1(M)$ is infinite, then M is either a Seifert fibered space or has a hyperbolic structure. This result gives a motivation to prove finiteness of mapping class groups for non-Haken 3-manifolds.

The mapping class group of a manifold M, denoted by $\mathcal{H}(M)$ is the group of isotopy classes of homeomorphisms of M. That is, $\mathcal{H}(M)$ is

$$\mathrm{Homeo}(M)/\mathrm{Homeo}_0(M),$$

where $\mathrm{Homeo}_0(M)$ is the normal subgroup of homeomorphism which are isotopic to the identity. We denote by $\mathcal{H}_+(M)$ the subgroup of $\mathcal{H}(M)$ consisting of elements represented by orientation preserving homeomorphisms. It is known that if F_g is a closed orientable surface of genus $g(\geq 2)$, then the mapping class group of F_g is generated by $2g+1$ Dehn twist homeomorphisms.

Received September 25, 2012; Revised February 15, 2013.
2010 *Mathematics Subject Classification.* Primary 57M99.
Key words and phrases. non-Haken manifold, mapping class group, 3-manifold.

Let M be an irreducible 3-manifold with infinite fundamental group then M is a $K(\pi, 1)$ space since it does not have any essential 2-sphere, its universal cover is noncompact and by use of the Hurewicz theorem (see the proof of Theorem in p. 26 of [14] for more details).

Consider a map $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ which sends an isotopy class $\langle f \rangle$ to the corresponding induced homomorphism f_* of $\pi_1(M)$. It is well known that if M is aspherical and $f : M \to M$ is a homeomorphism such that $f_* =$ identity, then f is homotopic to the identity. This implies that $\mathrm{Ker}(\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M)))$ consists of mapping classes whose representatives are homotopic to the identity.

We remark that Waldhausen [17, 18] showed that if M is a closed orientable sufficiently large 3-manifold, then the map $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ is an isomorphism.

Let $\{F_i\}, 1 \le i \le m$ be a collection of closed connected 2-manifolds, none of which is simply connected. Form a connected irreducible 3-manifold V from $\bigcup_{i=1}^m F_i \times I$ by attaching k 1-handles to $\bigcup_{i=1}^m F_i \times \{1\}$. V is called a compression body. $\pi_1(V)$ is isomorphic to $\pi_1(F_1) * \cdots * \pi_1(F_m) * H$ where H is a free group of rank $k - (m-1)$.

Let D be a properly embedded 2-disk in a 3-manifold M and let $D \times I \subset M$ be a product region such that $D \times I \cap \partial M = \partial D \times I$. Define a *twist homeomorphism* $t_D : M \to M$ by $t_D(x) = x$ if $x \notin D \times I$ and $t_D(re^{i\theta}, s) = (re^{i(\theta + 2\pi s)}, s)$. On the boundary of M, it is a Dehn twist about ∂D. Note that $t_{D*} = id$ and t_D is homotopic to the identity. This is because $(M - (D \times I)) \cup (\{0\} \times I)$ is a deformation retract of M and t_D restricts to the identity map on this subspace. When ∂D is not contractible in ∂M, then t_D can not be isotopic to the identity because an isotopy from t_D to the identity would restrict on ∂M to an isotopy from the Dehn twist about ∂D to the identity. But this Dehn twist induces a nontrivial outer automorphism on $\pi_1(\partial M)$.

McCullough and Miller [15] showed that if V is a compression body, then $\mathrm{Ker}(\mathcal{H}_+(V) \to \mathrm{Out}(\pi_1(V)))$ is the subgroup generated by twist homeomorphisms. Actually they showed that if M is a compact orientable irreducible 3-manifold with nonempty boundary, then the group $\mathcal{T}$ of twist homeomorphisms is $\mathrm{Ker}(\mathcal{H}_+(V) \to \mathrm{Out}(\pi_1(V)))$.

For finite index property of the image, they showed that if V is an orientable compression body, then the image of $\mathcal{H}(V)$ in $\mathrm{Out}(\pi_1(V))$ has finite index.

Generalizing the construction of Dehn twist homeomorphisms of 2-manifolds, define a *Dehn homeomorphism* h of M as follows: Let $(F^2 \times I, \partial F^2 \times I) \subset (M^3, \partial M^3)$, where F is a connected surface and $F \times I \cap \partial M = \partial F \times I$. Let ϕ_t be an element of $\pi_1(\mathrm{Homeo}(F), 1_F)$, that is, for $0 \le t \le 1$, ϕ_t is a continuous family of homeomorphisms of F such that $\phi_0 = \phi_1 = 1_F$. Define $h \in \pi_0(\mathrm{Homeo}(M)) = \mathcal{H}(M)$ by

$$h(x,t) = \begin{cases} (\phi_t(x), t) & \text{if } (x,t) \in F \times I, \\ h(m) = m & \text{if } m \notin F \times I. \end{cases}$$

Note that when $\pi_1(\mathrm{Homeo}(F))$ is trivial, a Dehn homeomorphism must be isotopic to the identity. Define the *Dehn subgroup* $\mathcal{D}(M)$ of $\mathcal{H}(M)$ to be the subgroup generated by the isotopy classes of Dehn homeomorphisms.

The following table lists $\pi_1(\mathrm{Homeo}(F))$ for connected 2-manifolds, and the names of the corresponding Dehn homeomorphisms of 3-manifolds.

F	$\pi_1(\mathrm{Homeo}(F))$	Dehn homeomorphism
$S^1 \times S^1$	$\mathbb{Z} \times \mathbb{Z}$	*Dehn twist about a torus*
$S^1 \times I$	$\mathbb{Z}$	*Dehn twist about an annulus*
D^2	$\mathbb{Z}$	*twist*
S^2	$\mathbb{Z}/2\mathbb{Z}$	*rotation about a sphere*
$\mathbb{RP}^2$	$\mathbb{Z}/2\mathbb{Z}$	*rotation about a projective plane*
Klein bottle	$\mathbb{Z}$	*Dehn twist about a Klein bottle*
Möbius band	$\mathbb{Z}$	*Dehn twist about a Möbius band*
$\chi(F) < 0$	$\{0\}$	

We thank to the referee for corrections and suggestions.

2. Mapping class groups of 3-manifolds

Now we state the theorem about the finiteness of mapping class groups of closed orientable, irreducible non-Haken 3-manifolds and the virtual freeness of the quotient group of mapping class groups by rotation groups if the 3-manifolds do not have any 2-sphere boundary components.

Theorem 2.1. *Let M be a closed orientable irreducible non-Haken 3-manifold. Then $\mathcal{H}(M)$ and $\mathrm{Out}(\pi_1(M))$ are finite, and if M is not S^3 or $\mathbb{RP}^3$, then $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ is injective.*

Proof. When $\pi_1(M)$ and hence $\mathrm{Out}(\pi_1(M))$ are finite, the Geometrization Theorem implies that M is the quotient of S^3 by a finite group of isometries. For these manifolds, as detailed in the proof of Theorem 3.1 in [13], the work of many authors shows that, apart from S^3 and $\mathbb{RP}^3$, $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ is injective.

When $\pi_1(M)$ is infinite, we appeal to the Geometrization Theorem again to deduce that every non-Haken irreducible orientable 3-manifold with infinite fundamental group is either a Seifert-fibered space or a hyperbolic manifold.

Gabai, Meyerhoff, and N. Thurston [7] have proven that $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ is an isomorphism when M is a closed hyperbolic 3-manifold. For any closed hyperbolic n-manifold with $n \geq 3$, $\mathrm{Out}(\pi_1(M))$ is finite by Mostow Rigidity theorem (see R. Benedetti and C. Petronio [1, Theorem C.5.6]).

The non-Haken Seifert manifolds with infinite fundamental group fiber over S^2 with exactly three exceptional fibers. For these manifolds, $\mathrm{Out}(\pi_1(M))$ is also finite by McCullough [12, p. 21]. Scott [16] and Boileau and Otal [2, 3] showed that $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ is injective in all such cases. □

Denote by $\mathcal{D}_{>0}(M)$ the subgroup of $\mathcal{D}(M)$ generated by Dehn homeomorphisms using D^2, S^2, and $\mathbb{RP}^2$ (the surfaces of positive Euler characteristic).

Define the *rotation subgroup* $\mathcal{R}(M)$ to be the subgroup generated by rotations about 2-spheres and 2-sided projective planes in M. It is a finite normal abelian subgroup of $\mathcal{H}(M)$.

Using the results of Theorem 1.5 in [11], Theorem 2.1 and the Geometrization Theorem, one can show that $\mathcal{D}_{>0}(M)$ is actually full kernel of $\mathcal{H}_+(M) \to \mathrm{Out}(\pi_1(M))$.

Proposition 2.2. *If M is a compact orientable 3-manifold which has no 2-sphere boundary components, then $\mathcal{D}_{>0}(M)$ equals the kernel of $\mathcal{H}_+(M) \to \mathrm{Out}(\pi_1(M))$.*

Here the assumption that M has no 2-sphere boundary components is necessary because otherwise *slide homeomorphisms* move a D^3 summand around an arc in M. Note that a D^3 summand is just a neighborhood of a 2-sphere boundary component.

A slide homeomorphism is defined as follow; Let S be an imbedded 2-sphere in M which bounds a 3 ball B and let α be an arc properly imbedded in $M - \mathrm{int}\, B$, both of whose endpoints lie in S. Take two regular neighborhoods N' and N'' ($N' \subset \mathrm{int}(N'')$) of $S \cup \alpha$ in M. Then $\mathrm{int}\,(N'' - N')$ has two components, one of which is homeomorphic to $S \times (0,1)$ and the other is homeomorphic to $T^2 \times (0,1)$ which we denote by $T(S,\alpha)$. Using a coordinate function $c : T(S,\alpha) \to T^2 \times (0,1)$ a slide homeomorphism is defined by

$$s(x) = \begin{cases} c^{-1}(\theta + 2\pi t, \phi, t) & \text{if } x = c^{-1}(\theta, \phi, t), \\ x & \text{otherwise.} \end{cases}$$

We note that a slide homeomorphism is isotopic to a Dehn twist about the torus $c^{-1}(T^2 \times \frac{1}{2})$. Several different kind of homeomorphisms including a slide homeomorphism and related results can be found in Section 1 of [11] and chapters 9, 10 of [5].

We need the following algebraic result of V. Guirardel and G. Levitt [8, Corollary 5.3] to show the virtual freeness of mapping class groups.

Proposition 2.3 (V. Guirardel and G. Levitt). *Let G be a free product $G_1 * \cdots * G_p * F_k$, with each G_i indecomposable and with F_k free. If each G_i has a subgroup H_i of finite index with H_i and $H_i/Z(H_i)$ torsion-free, and* $\mathrm{Out}(H_i)$ *virtually torsion-free, then* $\mathrm{Out}(G)$ *is virtually torsion-free.*

Theorem 2.4. *Let M be a compact orientable 3-manifold with no 2-sphere boundary components and incompressible boundary. Assume that each irreducible summand M_i of M has the property that $\mathcal{H}(M_i) \to \mathrm{Out}(M_i)$ has image of finite index. Then $\mathcal{H}(M)/\mathcal{R}(M)$ is virtually torsion-free.*

Proof. It suffices to prove that $\mathcal{H}_+(M)/\mathcal{R}(M)$ is virtually torsion-free. Since M has no essential compressing disks, $\mathcal{R}(M)$ is the kernel of $\mathcal{H}_+(M) \to$

$\mathrm{Out}(\pi_1(M))$ by the previous proposition and it is a finite group, so it suffices to show that $\mathrm{Out}(\pi_1(M))$ is virtually torsion-free. We will apply the theorem of Guirardel and Levitt, with $G_i = \pi_1(M_i)$.

$G = \pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_n)$, where each M_i is a prime summand of M, and we take each $G_i = \pi_1(M_i)$. We need to verify the hypotheses of the Guirardel-Levitt theorem.

Case 1. M_i is closed.

If $\pi_1(M_i)$ is finite, then we may take H_i trivial, and if it is infinite cyclic, we take $H_i = \pi_1(M_i)$. So we may assume that M_i is aspherical.

Suppose first that M_i is Seifert-fibered. If M_i is the 3-torus, then we may take $H_i = \pi_1(M_i) = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ and $\mathrm{Out}(\pi_1(M_i)) = \mathrm{GL}(3,\mathbb{Z})$ is torsion free by a theorem of A. Borel and J.-P. Serre [4]. Otherwise, M_i admits a finite covering by a circle bundle $\widetilde{M_i}$ over a closed orientable 2-manifold F_i of genus at least 1. Take $H_i = \pi_1(\widetilde{M_i})$. Since $\widetilde{M_i}$ is a circle bundle but not the 3-torus, there is a central extension

$$1 \to \mathbb{Z} \to \pi_1(\widetilde{M_i}) \to \pi_1(F_i) \to 1$$

so $H_i/Z(H_i) = \pi_1(F_i)$ is torsion-free. Since $\widetilde{M_i}$ is Haken, we have $\mathrm{Out}(H_i) \cong \mathcal{H}(\widetilde{M_i})$ by F. Waldhausen's result [17], and $\mathcal{H}(\widetilde{M_i})$ is torsion free by a result of McCullough [12]. The remaining case is when M_i is aspherical and not Seifert-fibered. We take $H_i = \pi_1(M_i)$. If M_i is Haken, then again $\mathrm{Out}(\pi_1(M_i))$ is torsion free by a result of McCullough [12] and if M_i is non-Haken, then $\mathrm{Out}(\pi_1(M_i))$ is finite by Theorem 2.1.

Case 2. M_i is compact with nonempty boundary.

If M_i is not Seifert fibered and M_i is Haken, then the conclusion follows from the result of McCullough [12]. If it is non-Haken, then the conclusion follows from Mostow Rigidity theorem for compact hyperbolic 3-manifolds.

If M_i is Seifert-fibered, then it has a finite covering $\widetilde{M_i} \to M_i$, where now $\widetilde{M_i}$ is a product $F_i \times S^1$ with F_i an aspherical orientable surface, and we take $H_i = \pi_1(\widetilde{M_i})$, which is torsion-free.

Suppose first that F_i is an annulus. Then $\pi_1(\widetilde{M_i}) = \mathbb{Z}\times\mathbb{Z}$. So $H_i/Z(H_i)$ is trivial, $\mathrm{Out}(H_i) \cong \mathrm{GL}(2,\mathbb{Z})$ is virtually free.

If F_i is not an annulus, then $H_i/Z(H_i) = \pi_1(F_i)$ is free of rank at least 2, hence is torsion-free. For $\mathrm{Out}(H_i)$, we regard H_i as a direct product $F\times\mathbb{Z}$ where F is free of rank at least 2. The $\mathbb{Z}$-factor is characteristic, and is fixed by an index-2 subgroup of $\mathrm{Out}(F\times\mathbb{Z})$. There is a surjection from this subgroup to $\mathrm{Out}(F)$, and $\mathrm{Out}(F)$ is virtually torsion-free by work of M. Culler and K. Vogtmann [6]. Since the kernel $\mathrm{Hom}(F,\mathbb{Z}) \cong H^1(F)$ of this surjection is torsion-free, $\mathrm{Out}(F\times\mathbb{Z})$ is also virtually torsion-free. □

References

[1] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Spriner Verlag, Berlin, 1992.

[2] M. Boileau and J.-P. Otal, *Groupe des difféotopies de certaines variétés de Seifert*, C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), no. 1, 19–22.
[3] ______, *Scindements de Heegaard et groupe des homéotopies des petites variétés de Seifert*, Invent. Math. **106** (1991), no. 1, 85–107.
[4] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491.
[5] R. Canary and D. McCullough, *Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups*, Mem. Amer. Math. Soc. **172** (2004) no. 812, 1–218.
[6] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. Math. **84** (1986), no. 1, 91–119.
[7] D. Gabai, G. R. Meyerhoff, and N. Thurston, *Homotopy hyperbolic 3-manifolds are hyperbolic*, Ann. of Math. (2) **157** (2003), no. 2, 335–431.
[8] V. Guirardel and G. Levitt, *The outer space of a free product*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 3, 695–714.
[9] J. Hempel, *3-manifolds*, American Mathematical Society, 2004.
[10] W. Jaco, *Lectures on Three Manifold Topology*, American Mathematical Society, 1980.
[11] D. McCullough, *Topological and algebraic automorphisms of 3-manifolds*, Groups of self-equivalences and related topics (Montreal, PQ, 1988), 102–113, Lecture Notes in Math., 1425, Springer, Berlin, 1990.
[12] ______, *Virtually geometrically finite mapping class groups of 3-manifolds*, J. Differential Geom. **33** (1991), no. 1, 1–65.
[13] ______, *Isometries of elliptic 3-manifolds*, J. London Math. Soc. (2) **65** (2002), no. 1, 167–182.
[14] ______, *Three manifolds and their mappings*, Lecture Notes Series, 1–82, no. 26, Seoul National University, 1995.
[15] D. McCullough and A. Miller, *Homeomorphisms of 3-manifolds with compressible boundary*, Mem. Amer. Math. Soc. **61** (1986) no. 344, 1–100.
[16] P. Scott, *There are no fake Seifert fibre spaces with infinite* π_1, Ann. of Math. (2) **117** (1983), no. 1, 35–70.
[17] F. Waldhausen, *On irreducible* 3*-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88.
[18] ______, *Recent results on sufficiently large 3-manifolds*, In Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, pp. 21–38, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978.

Sungbok Hong
Department of Mathematics
Korea University
Seoul 136-713, Korea
E-mail address: shong@korea.ac.kr

Jin Yang
Department of Mathematics
Korea University
Seoul 136-713, Korea
E-mail address: supjin@korea.ac.kr