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# SOME IDENTITIES ON THE BERNSTEIN AND $q$-GENOCCHI POLYNOMIALS

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Abstract. Recently, T. Kim has introduced and analysed the $q$-Euler polynomials (see [3, 14, 35, 37]). By the same motivation, we will consider some interesting properties of the $q$-Genocchi polynomials. Further, we give some formulae on the Bernstein and $q$-Genocchi polynomials by using $p$-adic integral on $\mathbb{Z}_p$. From these relationships, we establish some interesting identities.

## 1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The $p$-adic norm is normally defined by $|p|_p = 1/p$. As an indeterminate, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ (see [1-43]). Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by T. Kim as follows:

$$\begin{aligned} I_{-1}(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) \\ &= \lim_{n\to\infty} \sum_{0\le x\le p^n-1} f(x)\mu_{-1}(x+p^n\mathbb{Z}_p) \\ &= \lim_{n\to\infty} \frac{1}{p^n} \sum_{0\le x\le p^n-1} f(x)(-1)^x, \quad \text{(see [1, 21, 22, 25])}. \end{aligned} \tag{1}$$

From (1), we can derive the following integral equation on $\mathbb{Z}_p$:

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \tag{2}$$

where $f_1(x) = f(x+1)$ (see [1, 21, 22, 25]) .

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As is well known, the Genocchi polynomials are defined by the generating function as follows:

$$\frac{2t}{e^t+1}e^{xt} = e^{G(x)t} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \tag{3}$$

with the usual convention about replacing $G^n(x)$ by $G_n(x)$ . Taking $x = 0$ into (3), we get $G_n(0) = G_n$ is called the $n$-th Genocchi number (see [1-4, 11, 12, 20, 24, 28, 33, 34]). From (3), we have the following recurrence relations of Genocchi numbers as follows:

$$G_0 = 0 \quad \text{and} \quad (G+1)^n + G_n = 2\delta_{1,n}, \tag{4}$$

where $\delta_{1,n}$ is the Kronecker symbol and $n \in \mathbb{N}^*$ (see [2, 28, 36]).

As is well known, the Frobenius-Euler polynomials, $H_n(u|x)$, are defined by the generating function as follows:

$$\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(u|x)\frac{t^n}{n!}, \ \ u \in \mathbb{C}_p \ \text{ with } \ u \neq 1 \ \text{ (see [6, 16, 25, 32, 39])}. \tag{5}$$

In the special case, $x = 0$, $H_n(u|0) = H_n(u)$ is called the $n$-th Frobenius-Euler number (see [6, 16, 25, 32, 39]). For $n, k \in \mathbb{N}^*$ with $n > k$ and $x \in \mathbb{Z}_p$, the Bernstein polynomials of degree $n$ is defined by

$$B_{k,n}(x) = \binom{n}{k} x^k(1-x)^{n-k} = \binom{n}{n-k}(1-x)^{n-k}x^k = B_{n-k,n}(1-x) \tag{6}$$

(see [19, 32, 33, 35, 37]).

In this paper, we investigate some identities for the $q$-Genocchi numbers and polynomials by using $p$-adic integral on $\mathbb{Z}_p$. From these relationships, we establish some interesting identities in the next section.

## 2. Some identities on the Bernstein and $q$-Genocchi polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. As is well known, the $q$-Genocchi polynomials are defined by the generating function as follows:

$$\frac{2t}{qe^t+1}e^{xt} = e^{G_q(x)t} = \sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!}, \tag{7}$$

with the usual convention about replacing $G_q^n(x)$ by $G_{n,q}(x)$. In the special case, $x = 0$, then we have $G_{n,q}(0) = G_{n,q}$ is called the $n$-th $q$-Genocchi number (see [1, 4, 11, 20, 24, 33, 34]). From (7), we have the following recurrence relations of $q$-Genocchi numbers as follows:

$$G_{0,q} = 0 \quad \text{and} \quad q(G_q+1)^n + G_{n,q} = 2\delta_{1,n}. \tag{8}$$

From (8), we easily see that

$$G_{1,q} = \frac{2}{[2]_q}, \quad \lim_{q\to 1} G_{1,q} = G_1, \quad \text{and} \quad G_{2,q} = -\frac{2^2 q}{{[2]_q}^2}, \tag{9}$$

where $[x]_q = \frac{1-q^x}{1-q}$ and $x \in \mathbb{Z}_p$ . By the definition of $q$-Genocchi numbers, we note that

$$G_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{l,q} x^{n-l}. \tag{10}$$

From (8), we get

$$q(G_q + 1)^n + G_{n,q} = qG_{n,q}(1) + G_{n,q} = 2\delta_{1,n}. \tag{11}$$

From (10) and (11), we have

$$\begin{aligned} qG_{n,q}(2) &= q(G_q + 2)^n = q(G_q + 1 + 1)^n \\ &= q\sum_{l=0}^{n} \binom{n}{l} (G_q + 1)^l = q\sum_{l=0}^{n} \binom{n}{l} G_{l,q}(1). \end{aligned} \tag{12}$$

By (11) and (12), we can derive the following equation:

$$\begin{aligned} q^2G_{n,q}(2) &= q^2(G_q + 2)^n = q^2(G_q + 1 + 1)^n \\ &= q\sum_{l=0}^{n} \binom{n}{l} q(G_q + 1)^l = q\sum_{l=1}^{n} \binom{n}{l} qG_{l,q}(1) \\ &= q\sum_{l=2}^{n} \binom{n}{l} qG_{l,q}(1) + q\left[\binom{n}{1} qG_{1,q}(1)\right] \\ &= -q\sum_{l=2}^{n} \binom{n}{l} G_{l,q} + nq(2 - G_{1,q}) \\ &= -q\sum_{l=0}^{n} \binom{n}{l} G_{l,q} + 2nq = -q(G_q + 1)^n + 2nq \\ &= -qG_{n,q}(1) + 2nq = -2\delta_{1,n} + G_{n,q} + 2nq. \end{aligned} \tag{13}$$

From (13), we have the following theorem.

**Theorem 1.** *For $n \in \mathbb{N}^*$, we have*

$$q^2G_{n,q}(2) = G_{n,q} + 2nq - 2\delta_{1,n}.$$

**Corollary 2.** *For $n \in \mathbb{N}$ with $n \geq 2$, we have*

$$q^2G_{n,q}(2) = G_{n,q} + 2nq.$$

By (7) and (8), we can derive the following equation:

$$\frac{2t}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} G_{n,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{G_{n+1,q}}{n+1} \frac{t^{n+1}}{n!}. \tag{14}$$

Also, we note that

$$\frac{2t}{qe^t + 1} e^{xt} = \left(\frac{2t}{1+q}\right)\left(\frac{1+q^{-1}}{e^t + q^{-1}}\right) e^{xt} = \frac{2}{[2]_q} \sum_{n=0}^{\infty} H_n\left(-q^{-1}\right) \frac{t^{n+1}}{n!}, \tag{15}$$

where $H_n\left(-q^{-1}\right)$ are the $n$-th Frobenius-Euler number.

Thus, by (14) and (15), we have

$$\frac{G_{n+1,q}}{n+1}=\frac{2}{[2]_q}H_n\left(-q^{-1}\right). \tag{16}$$

Therefore, by (16), we obtain the following proposition.

**Proposition 3.** *For $n\in\mathbb{N}^*$, we have*

$$\frac{G_{n+1,q}}{n+1}=\frac{2}{[2]_q}H_n\left(-q^{-1}\right),$$

*where $H_n(-q^{-1})$ are the $n$-th Frobenius-Euler number.*

Let us take $f(x)=q^xe^{xt}$. Then, by (2), we get

$$\int_{\mathbb{Z}_p}q^xe^{xt}d\mu_{-1}(x)=\sum_{n=0}^{\infty}\frac{G_{n+1,q}}{n+1}\frac{t^n}{n!}. \tag{17}$$

From Proposition 3 and (17), we have the following theorem.

**Theorem 4.** *For $n\in\mathbb{N}^*$, we have*

$$\int_{\mathbb{Z}_p}q^xx^nd\mu_{-1}(x)=\frac{G_{n+1,q}}{n+1}=\frac{2}{[2]_q}H_n\left(-q^{-1}\right).$$

By (2), (7), and (17), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}q^y(x+y)^nd\mu_{-1}(y)&=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}\int_{\mathbb{Z}_p}q^yy^ld\mu_{-1}(y)\\&=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}\frac{G_{l+1,q}}{l+1}\\&=\sum_{l=1}^{n+1}\binom{n}{l-1}x^{n+1-l}\frac{G_{l,q}}{l}\\&=\frac{1}{n+1}\sum_{l=1}^{n+1}\binom{n+1}{l}x^{n+1-l}G_{l,q}\\&=\frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}x^{n+1-l}G_{l,q}\\&=\frac{1}{n+1}G_{n+1,q}(x).\end{aligned} \tag{18}$$

From (18), we obtain the following theorem.

**Theorem 5.** *For $n\in\mathbb{N}^*$, we have*

$$\int_{\mathbb{Z}_p}q^y(x+y)^nd\mu_{-1}(y)=\frac{1}{n+1}G_{n+1,q}(x)=\frac{2}{[2]_q}H_n(-q^{-1}|x).$$

Now, we consider the symmetric property for the $q$-Genocchi polynomials as follows:

$$
\begin{aligned}
q\sum_{n=0}^{\infty} G_{n,q}(1-x)\frac{t^n}{n!} &= \frac{2qt}{qe^t+1}e^{(1-x)t} \\
&= -\frac{-2t}{1+q^{-1}e^{-t}}e^{-xt} \\
&= -\sum_{n=0}^{\infty} G_{n,q^{-1}}(x)\frac{(-t)^n}{n!} \\
&= \sum_{n=0}^{\infty} G_{n,q^{-1}}(x)(-1)^{n+1}\frac{t^n}{n!}.
\end{aligned} \tag{19}
$$

From (19), we get

$$
q\sum_{n=0}^{\infty} G_{n,q}(1-x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,q^{-1}}(x)(-1)^{n+1}\frac{t^n}{n!}.
$$

Therefore, we have the following theorem.

**Theorem 6.** *For $n \in \mathbb{N}^*$, we have*

$$
qG_{n,q}(1-x) = (-1)^{n+1}G_{n,q^{-1}}(x).
$$

For $n \in \mathbb{N}^*$ with $n \geq 2$, by Theorems 4, 5, 6, and Corollary 2, we have

$$
\begin{aligned}
\int_{\mathbb{Z}_p} q^{-x}(1-x)^{n-1}d\mu_{-1}(x) &= (-1)^{n-1}\int_{\mathbb{Z}_p} q^{-x}(x-1)^{n-1}d\mu_{-1}(x) \\
&= (-1)^{n-1}\frac{G_{n,q^{-1}}(-1)}{n} \\
&= q\frac{G_{n,q}(2)}{n} = \frac{1}{nq}(G_{n,q}+2nq). \\
&= \frac{1}{nq}(G_{n,q}+2nq) \\
&= \frac{1}{q}\frac{G_{n,q}}{n}+2 \\
&= \frac{1}{q}\int_{\mathbb{Z}_p} q^x x^{n-1}d\mu_{-1}(x)+2.
\end{aligned} \tag{20}
$$

Therefore, by (20), we have the following theorem.

**Theorem 7.** *For $n \in \mathbb{N}^*$ with $n \geq 2$, we have*

$$
\int_{\mathbb{Z}_p} q^{-x}(1-x)^{n-1}d\mu_{-1}(x) = \frac{1}{q}\int_{\mathbb{Z}_p} q^x x^{n-1}d\mu_{-1}(x)+2.
$$

Now, let $n, k \in \mathbb{N}^*$ with $n > k$. Then, by (6) and Theorem 5, we see that

$$\begin{aligned} I &= \int_{\mathbb{Z}_p} B_{k,n}(x) q^x d\mu_{-1}(x) \\ &= \int_{\mathbb{Z}_p} \binom{n}{k} x^k (1-x)^{n-k} q^x d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \int_{\mathbb{Z}_p} x^{l+k} q^x d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \frac{G_{l+k+1,q}}{l+k+1}. \end{aligned} \tag{21}$$

From the same method, we have

$$\begin{aligned} I &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x) q^x d\mu_{-1}(x) \\ &= \int_{\mathbb{Z}_p} \binom{n}{n-k} (1-x)^{n-k} x^k q^x d\mu_{-1}(x) \\ &= \binom{n}{n-k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} (1-x)^{n-l} q^x d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left[ q \int_{\mathbb{Z}_p} q^{-x} x^{n-l} d\mu_{-1}(x) \right] \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left[ 2 + q \frac{G_{n-l+1,q^{-1}}}{n-l+1} \right]. \end{aligned} \tag{22}$$

Thus, by (21) and (22), we obtain the following theorem.

**Theorem 8.** *For $n, k \in \mathbb{N}^*$ with $n > k$, we have*

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \frac{G_{l+k+1,q}}{l+k+1} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left[ 2 + q \frac{G_{n-l+1,q^{-1}}}{n-l+1} \right].$$

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