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LIPSCHITZ TYPE CHARACTERIZATIONS OF HARMONIC BERGMAN SPACES

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Abstract. Wulan and Zhu [16] have characterized the weighted Bergman space in the setting of the unit ball of $\mathbf{C}^n$ in terms of Lipschitz type conditions in three different metrics. In this paper, we study characterizations of the harmonic Bergman space on the upper half-space in $\mathbf{R}^n$. Furthermore, we extend harmonic analogues in the setting of the unit ball to the full range $0 < p < \infty$. In addition, we provide the application of characterizations to showing the boundedness of a mapping defined by a difference quotient of harmonic function.

1. Introduction

For a fixed positive integer $n \geq 2$, let $\mathbf{B}$ be the open unit ball in $\mathbf{R}^n$ and let $\mathbf{H} = \mathbf{R}^{n-1} \times \mathbf{R}_+$ be the upper half-space where $\mathbf{R}_+$ denotes the set of all positive real numbers. For a domain Ω in $\mathbf{R}^n$, we denote by $h(\Omega)$ the space of all complex-valued harmonic functions on Ω.

Given $\alpha > -1$ and $0 < p < \infty$, we denote by $b^p_\alpha(\Omega)$ the weighted harmonic Bergman space consisting of all $f \in h(\Omega)$ for which the norm

$$\|f\|_{b^p_\alpha(\Omega)} := \left\{ \int_\Omega |f|^p \, dV_\alpha \right\}^{1/p}$$

is finite. Here, we denote by V_α the weighted measure on Ω (see Sections 3 and 4).

Wulan and Zhu [16] obtain characterizations for weighted Bergman space in the setting of the unit ball of $\mathbf{C}^n$ in terms of Lipschitz type conditions with Euclidean, hyperbolic and pseudo-hyperbolic metrics.

In this paper, we study characterizations on harmonic Bergman space in the setting of the half-space. Let ρ be the pseudohyperbolic distance on $\mathbf{H}$. The following characterization is caused by the unboundedness of the half-space.

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Theorem 1.1. *Let $\alpha > -1$ and $0 < p < \infty$. Suppose $f \in b^p_\alpha(\mathbf{H})$. Then there exists a positive continuous function $g \in L^p_\alpha(\mathbf{H})$ such that*

$$|f(z) - f(w)| \leq \rho(z,w)[g(z) + g(w)]$$

for all $z, w \in \mathbf{H}$. Furthermore, if $1 \leq p < \infty$, then the function g associated with f can be chosen in such a way that $\|g\|_{L^p_\alpha(\mathbf{H})}$ is comparable to $\|f\|_{b^p_\alpha(\mathbf{H})}$.

Also, Theorem 1.2 implies the characterizations with Euclidean and hyperbolic metrics. See Theorem 3.5 and Theorem 3.6.

In Section 2 we collect several well-known results that we need later. In Section 3 we first give characterizations of harmonic Bergman space in the setting of the half-space. In addition, we prove the boundedness of a mapping defined by difference quotient of harmonic function for an application of our characterizations. In [11], it has been studied harmonic analogues in the setting of the unit ball, but with restriction $1 \leq p < \infty$. In Section 4, we extend characterizations to the full range $0 < p < \infty$.

Words on constants. In the rest of the paper we use the same letter C to denote various positive constants, depending only on allowed parameters, which may change at each occurrence. Also, for nonnegative quantities X and Y, we often write $X \lesssim Y$, if X is dominated by Y times some *inessential* positive constant. We write $X \approx Y$ if $X \lesssim Y \lesssim X$.

2. Preliminaries

In this section we introduce notations and collect some preliminary results that we need later.

Recall the hyperbolic metric β on $\mathbf{H}$ defined by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}, \qquad z, w \in \mathbf{H}$$

where

$$\rho(z,w) = \frac{|z-w|}{|z-\overline{w}|}.$$

The pseudohyperbolic distance ρ on $\mathbf{H}$ is horizontal translation and dilation invariant. Also, ρ is a distance function on $\mathbf{H}$. See Lemma 3.1 in [5].

We denote by $B_a(x)$ the Euclidean ball with radius $a > 0$ and center $x \in \mathbf{R}^n$. For $z \in \mathbf{H}$ and $r \in (0,1)$, let $E_r(z)$ denote the pseudohyperbolic ball with radius r and center z. A straightforward calculation shows that $E_r(z)$ is a Euclidean ball $B_a(x)$ where

$$x = \left(z', \frac{1+r^2}{1-r^2}z_n\right) \qquad \text{and} \qquad a = \frac{2r}{1-r^2}z_n. \tag{2.1}$$

Let $d(x,\Omega)$ be the Euclidean distance between a point x and a set Ω in $\mathbf{R}^n$. Then we have

$$d\big(z, \partial E_r(z)\big) = \frac{2r}{1+r}z_n. \tag{2.2}$$

The following lemma comes from Lemma 3.3 in [5].

Lemma 2.1. *The inequality*

$$\frac{1-\rho(v,z)}{1+\rho(v,z)} \le \frac{|v-\overline{w}|}{|z-\overline{w}|} \le \frac{1+\rho(v,z)}{1-\rho(v,z)}$$

holds for all $v, z, w \in \mathbf{H}$.

Given $0 < r < 1$, Lemma 2.1 implies

$$z_n \approx w_n, \qquad w \in E_r(z). \tag{2.3}$$

Whenever f is harmonic on a domain in $\mathbf{R}^n$ and $1 \le p < \infty$, $|f|^p$ satisfies the submean-value inequality. The following lemma, coming from Lemma 3.5 in [6] or Theorem (KLFS) in [14], shows that the submean-value inequality can be extended even to the case $0 < p < 1$.

Let dw be the Lebesgue volume measure on $\mathbf{R}^n$.

Lemma 2.2. *Let* $0 < p < \infty$. *If* $f \in h(\Omega)$, *then*

$$|f(z)|^p \le \frac{C}{r^n} \int_{B_r(z)} |f(w)|^p \, dw$$

whenever $B_r(z) \subset \Omega$, *where* C *is a positive constant depending only on* p.

For a given multi-index $m = (m_1, \dots, m_n)$ of nonnegative integers, we use notations $|m| = m_1 + \cdots + m_n$ and $\partial^m = \partial_1^{m_1} \cdots \partial_n^{m_n}$ where ∂_i denotes the differentiation with respect to the i-th component. Then Lemma 2.2 and Cauchy's estimate imply the following lemma. See Corollary 8.2 in [1] for a proof of the following lemma in the case $1 \le p < \infty$.

Lemma 2.3. *Given* $0 < p < \infty$ *and a multi-index* $m = (m_1, \dots, m_n)$, *there is a constant* $C = C(p, m) > 0$ *such that*

$$|\partial^m f(z)|^p \le \frac{C}{d(z, \partial\Omega)^{n+p|m|}} \int_\Omega |f(w)|^p \, dw, \qquad z \in \Omega$$

whenever f *is harmonic on a domain* Ω *in* $\mathbf{R}^n$.

3. Characterizations of harmonic Bergman space over the half-space

This section is devoted to the proof of our main results Theorems 3.4, 3.5 and 3.6. In addition, we provide the application of Theorem 3.4 to showing the boundedness of a mapping defined by the difference quotient of harmonic functions.

3.1. Lipschitz type characterizations of harmonic Bergman space

We write a point $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$. Also, given $z \in \mathbf{H}$, we let $\overline{z} = (z', -z_n)$. Given α real, we let the weighted measure $dV_\alpha(z) = z_n^\alpha dz$.

We recall the following integral estimate. See Lemma 4.2 in [3] for the proof.

Lemma 3.1. *Given $\alpha > -1$ and c real, the estimates*

$$\int_{\mathbf{H}} \frac{dV_\alpha(w)}{|z - \overline{w}|^{n+\alpha+c}} \approx \begin{cases} z_n^{-c} & \text{if} \quad c > 0 \\ \infty & \text{if} \quad c \leq 0 \end{cases}$$

hold for all $z \in \mathbf{H}$. The constants suppressed above are independent of z.

In the case $\alpha > -1$, $1 \leq p < \infty$ and $f \in b^p_\alpha(\mathbf{H})$, the following inequality (3.1) was proved by using the reproducing property of the nonorthogonal projection of $b^p_\alpha(\mathbf{H})$ onto $L^p_\alpha(\mathbf{H})$. See Lemma 5 in [12] or, for the unweighted case, Theorem 4.5 in [15]. The next proposition is the extension of this result to the case $0 < p < \infty$ for every harmonic functions on $\mathbf{H}$.

Proposition 3.2. *Given $\alpha > -1$ and $0 < p < \infty$, there is a positive constant $C = C(\alpha, p)$ such that*

$$\| z_n | \bigtriangledown f | \|_{L^p_\alpha(\mathbf{H})} \leq C \| f \|_{b^p_\alpha(\mathbf{H})} \tag{3.1}$$

for all $f \in h(\mathbf{H})$.

Proof. Fix $\alpha > -1$ and $0 < p < \infty$. Let $f \in h(\mathbf{H})$ and $z \in \mathbf{H}$. For each fixed $0 < r < 1$, Lemma 2.3 and (2.2) give us that

$$\begin{aligned} |\partial_i f(z)|^p &\lesssim z_n^{-(n+p)} \int_{E_r(z)} |f(w)|^p \, dw \\ &\approx z_n^{-(n+\alpha+p)} \int_{E_r(z)} |f(w)|^p \, dV_\alpha(w), \qquad i = 1, \ldots, n \end{aligned}$$

where we used (2.3) for the second equality. Integrating both sides of the above against the measure $dV_\alpha(z)$, we obtain

$$\begin{aligned} \| z_n | \bigtriangledown f | \|^p_{L^p_\alpha(\mathbf{H})} &\lesssim \int_{\mathbf{H}} z_n^{-(n+\alpha)} \int_{E_r(z)} |f(w)|^p \, dV_\alpha(w) \, dV_\alpha(z) \\ &= \int_{\mathbf{H}} |f(w)|^p \int_{E_r(w)} z_n^{-n} dz \, dV_\alpha(w) \end{aligned}$$

by Fubini's theorem. Note that by (2.3) and (2.1), the inner integral of the double integral of the above is comparable to

$$w_n^{-n} V(E_r(w)) \approx 1. \tag{3.2}$$

In other words, we have

$$\| z_n | \bigtriangledown f | \|_{L^p_\alpha(\mathbf{H})} \lesssim \| f \|_{b^p_\alpha(\mathbf{H})}.$$

The constants suppressed above all are independent of f. This completes the proof of the proposition. $\square$

We recall a characterization of the weighted harmonic Bergman space over the half-space in terms of the fractional derivative: Let $\alpha > -1$ and $1 \le p < \infty$. If $(1+\alpha)/p + \gamma > 0$, then

$$\|f\|_{b^p_\alpha(\mathbf{H})} \approx \|z_n \mathcal{D}^\gamma f\|_{L^p_\alpha(\mathbf{H})} \tag{3.3}$$

as f ranges over all $b^p_\alpha(\mathbf{H})$-functions. Here, $\mathcal{D}$ is the fractional derivative. See Theorem 4.8 in [8] for the definition of $\mathcal{D}$ and the proof of (3.3). Note the condition "as f ranges over all $b^p_\alpha(\mathbf{H})$-functions"; the norm equivalence stated above would fail if f is allowed to vary over all harmonic functions on $\mathbf{H}$.

The special case $\gamma = 1$ of (3.3) and Proposition 3.2 yield the next norm equivalence.

Proposition 3.3. *Given $\alpha > -1$ and $1 \le p < \infty$, the norms*

$$\|f\|_{b^p_\alpha(\mathbf{H})}, \quad \|z_n \partial_n f\|_{L^p_\alpha(\mathbf{H})}, \quad \|z_n |\bigtriangledown f|\|_{L^p_\alpha(\mathbf{H})}$$

are comparable to one another for all $f \in b^p_\alpha(\mathbf{H})$.

Now, we are ready to prove the characterization below.

Theorem 3.4. *Let $\alpha > -1$ and $0 < p < \infty$. Suppose $f \in b^p_\alpha(\mathbf{H})$. Then there exists a positive continuous function $g \in L^p_\alpha(\mathbf{H})$ such that*

$$|f(z) - f(w)| \le \rho(z,w)[g(z) + g(w)]$$

for all $z, w \in \mathbf{H}$. Furthermore, if $1 \le p < \infty$, then the function g associated with f can be chosen in such a way that $\|g\|_{L^p_\alpha(\mathbf{H})}$ is comparable to $\|f\|_{b^p_\alpha(\mathbf{H})}$.

Proof. Let $\alpha > -1$ and $0 < p < \infty$. Suppose $f \in b^p_\alpha(\mathbf{H})$. Fix $r \in (0, 1/2)$ and consider any two points $z, w \in \mathbf{H}$ with $\rho(z,w) < r$. Since $E_r(z)$ is a convex set, it follows from Lemma 2.1 that

$$\begin{aligned} |f(z) - f(w)| &\le |z - w| \int_0^1 |\bigtriangledown f(t(z-w) + w)|\, dt \\ &\le \rho(z,w)|z - \overline{w}| \sup_{a \in E_r(z)} |\bigtriangledown f(a)| \\ &\le \rho(z,w) h(z), \end{aligned}$$

where

$$h(z) = C(r)\, z_n \sup_{a \in E_r(z)} |\bigtriangledown f(a)|.$$

If $\rho(z,w) \ge r$, then, by the triangle inequality, we get

$$|f(z) - f(w)| \le \rho(z,w) \left(\frac{|f(z)|}{r} + \frac{|f(w)|}{r} \right).$$

Now, letting $g(z) = |f(z)|/r + h(z)$, we know that

$$|f(z) - f(w)| \le \rho(z,w)(g(z) + g(w))$$

for all $z, w \in \mathbf{H}$. Since g is a positive continuous function on $\mathbf{H}$, we only need to prove that $h \in L^p_\alpha(\mathbf{H})$.

As stated earlier, pseudo-hyperbolic distance ρ satisfies the triangle inequality. So we choose $r' \in (0,1)$ such that $E_r(a) \subset E_{r'}(z)$ for every $a \in E_r(z)$. By proceeding as in the proof of Proposition 3.2, we conclude $\|h\|_{L^p_\alpha(\mathbf{H})} \lesssim \|f\|_{b^p_\alpha(\mathbf{H})}$. In fact, Lemma 2.3, (2.2) and (2.3) give us that

$$\begin{aligned} |h(z)|^p &\lesssim z_n^p \sup_{a \in E_r(z)} \frac{1}{d(a, \partial E_r(a))^{n+p}} \int_{E_r(a)} |f(w)|^p \, dw \\ &\lesssim z_n^{-(n+\alpha)} \int_{E_{r'}(z)} |f|^p \, dV_\alpha \end{aligned}$$

and thus Fubini's theorem and (3.2) yield

$$\begin{aligned} \|h\|^p_{L^p_\alpha(\mathbf{H})} &\lesssim \int_{\mathbf{H}} z_n^{-(n+\alpha)} \int_{E_{r'}(z)} |f(w)|^p \, dV_\alpha(w) \, dV_\alpha(z) \\ &= \int_{\mathbf{H}} |f(w)|^p \int_{E_{r'}(w)} z_n^{-n} \, dz \, dV_\alpha(w) \\ &\approx \|f\|^p_{b^p_\alpha(\mathbf{H})}. \end{aligned}$$

So far, we have proved that g is a positive continuous function such that

$$|f(z) - f(w)| \le \rho(z,w)[g(z) + g(w)] \tag{3.4}$$

and

$$\|g\|_{L^p_\alpha(\mathbf{H})} \lesssim \|f\|_{b^p_\alpha(\mathbf{H})}. \tag{3.5}$$

Now, assume $1 \le p < \infty$. We need to prove the reverse inequality of (3.5) to complete the proof of the theorem. From (3.4), we know

$$\frac{|f(z) - f(w)|}{|z - w|} \le \left[\frac{g(z)}{z_n} + \frac{g(w)}{w_n}\right], \qquad z \ne w. \tag{3.6}$$

Supposing $z' = w'$ and letting w approach z, we have

$$z_n |\partial_n f(z)| \le 2g(z).$$

Since $1 \le p < \infty$, Proposition 3.3 implies

$$\|f\|_{b^p_\alpha(\mathbf{H})} \approx \|z_n \partial_n f\|_{L^p_\alpha(\mathbf{H})} \lesssim \|g\|_{L^p_\alpha(\mathbf{H})} \tag{3.7}$$

as desired. Consequently, (3.4), (3.5) and (3.7) complete the proof of the theorem. □

Since $\rho \le \beta$, Theorem 3.4 yields the next theorem.

Theorem 3.5. *Let $\alpha > -1$ and $0 < p < \infty$. Suppose $f \in b^p_\alpha(\mathbf{H})$. Then there exists a positive continuous function $g \in L^p_\alpha(\mathbf{H})$ such that*

$$|f(z) - f(w)| \le \beta(z,w)[g(z) + g(w)]$$

for all $z, w \in \mathbf{H}$. Furthermore, if $1 \le p < \infty$, then the function g associated with f can be chosen in such a way that $\|g\|_{L^p_\alpha(\mathbf{H})}$ is comparable to $\|f\|_{b^p_\alpha(\mathbf{H})}$.

Also, Theorem 3.4 implies the following theorem.

Theorem 3.6. *Let $\alpha > -1$ and $0 < p < \infty$. Suppose $f \in b^p_\alpha(\mathbf{H})$. Then there exists a positive continuous function $g \in L^p_{p+\alpha}(\mathbf{H})$ such that*

$$|f(z) - f(w)| \le |z - w|[g(z) + g(w)] \tag{3.8}$$

for all $z, w \in \mathbf{H}$. Furthermore, if $1 \le p < \infty$, then the function g associated with f can be chosen in such a way that $\|g\|_{L^p_{p+\alpha}(\mathbf{H})}$ is comparable to $\|f\|_{b^p_\alpha(\mathbf{H})}$.

Proof. Let $\alpha > -1$ and $0 < p < \infty$. Suppose $f \in b^p_\alpha(\mathbf{H})$. Then the proof of Theorem 3.4 yields that there exists a positive continuous function h such that $\|h\|_{L^p_\alpha(\mathbf{H})} \lesssim \|f\|_{b^p_\alpha(\mathbf{H})}$ and

$$|f(z) - f(w)| \le \rho(z, w)[h(z) + h(w)] \tag{3.9}$$

for all $z, w \in \mathbf{H}$. In particular, $\|h\|_{L^p_\alpha(\mathbf{H})} \approx \|f\|_{b^p_\alpha(\mathbf{H})}$ when $1 \le p < \infty$. Note that the right-hand side of (3.9) is less than or equal to

$$|z - w| \left[\frac{h(z)}{z_n} + \frac{h(w)}{w_n} \right]$$

for all $z, w \in \mathbf{H}$. Now, let $g(z) = h(z)/z_n$. Then g satisfies (3.8) and $\|g\|_{L^p_{p+\alpha}(\mathbf{H})} = \|h\|_{L^p_\alpha(\mathbf{H})}$. Consequently, the proof is complete. $\square$

3.2. A difference quotient of harmonic function on H

We now introduce a difference quotient of harmonic function on $\mathbf{H}$. Given $f \in h(\mathbf{H})$, we define

$$Lf(z, w) := \frac{f(z) - f(w)}{|z - w|}, \qquad z \ne w$$

for $z, w \in \mathbf{H}$. Then one can apply Theorem 3.4 to get the following theorem.

We denote by $L^p(m)$ the Lebesgue space associated with the measure m.

Theorem 3.7. *Let $\alpha > -1$, $n + \alpha < p < \infty$ and $\gamma = (p + \alpha - n)/2$. Then L maps $b^p_\alpha(\mathbf{H})$ boundedly into $L^p_\alpha(\mathbf{H})$.*

Proof. Let $f \in b^p_\alpha(\mathbf{H})$. Since $p > n + \alpha > 1$, Theorem 3.4 and the triangle inequality yield that there exists a positive continuous function $g \in L^p_\alpha(\mathbf{H})$ such that $\|g\|_{L^p_\alpha(\mathbf{H})} \approx \|f\|_{b^p_\alpha(\mathbf{H})}$ and

$$|Lf(z, w)|^p \lesssim \frac{|g(z)|^p + |g(w)|^p}{|z - \overline{w}|^p}, \qquad z \ne w$$

for z, w in $\mathbf{H}$. It follows that from Fubini's theorem

$$\|Lf\|^p_{L^p(V_\gamma \times V_\gamma)} \le 2 \int_{\mathbf{H}} |g(z)|^p \int_{\mathbf{H}} \frac{dV_\gamma(w)}{|z - \overline{w}|^p}\, dV_\gamma(z). \tag{3.10}$$

Note $p - n - \gamma = (p - n - \alpha)/2 > 0$ and thus we obtain by Lemma 3.1

$$\int_{\mathbf{H}} \frac{dV_\gamma(w)}{|z - \overline{w}|^p} \approx z_n^{n+\gamma-p}.$$

Since $\gamma = (p+\alpha-n)/2$, we have from (3.10)

$$\|Lf\|_{L^p(V_\gamma \times V_\gamma)} \lesssim \|g\|_{L^p_\alpha(\mathbf{H})} \approx \|f\|_{b^p_\alpha(\mathbf{H})}.$$

The constants suppressed above are independent of f. The proof is complete. □

Remark. The holomorphic case over the unit disk in $\mathbf{C}$ of Theorem 3.7 is investigated in [16]. In the same paper, the case $0 < p < 2+\alpha$ is also considered:

Suppose $\alpha > -1$ and $0 < p < 2+\alpha$. Then L (which is called the symmetric lifting operator) maps $A^p_\alpha(D)$ boundedly into $A^p(dA_\alpha \times dA_\alpha)$. Moreover, this is no longer true when $p > 2+\alpha$. Here $A^p_\alpha(D)$ is the weighted Bergman space on the unit disk D in $\mathbf{C}$.

For the higher dimensional case, we refer to [9] and [10].

Since $\mathbf{H}$ is unbounded, one cannot have the harmonic analogue over the half-space. Actually, there is an example which shows that L never maps $b^p_\alpha(\mathbf{H})$ into $L^p(V_\alpha \times V_\alpha)$ for all possible α and p. See Example 4.4 in [4].

4. Characterization of harmonic Bergman space over the unit Ball

In this section we consider the harmonic case on the setting of the unit ball in $\mathbf{R}^n$. For a real α, we denote by v_α the weighted measure on $\mathbf{B}$ given by $dv_\alpha(x) = (1-|x|^2)^\alpha\, dx$.

The following property is the ball analogue of Proposition 3.3. The case $n = 2$ is proved by Hardy and Littlewood [7]. For a proof of the case $n \geq 3$, see [13].

Proposition 4.1. *Given $\alpha > -1$ and $0 < p < \infty$, we have*

$$\int_{\mathbf{B}} |f|^p\, dv_\alpha \approx |f(0)|^p + \int_{\mathbf{B}} |\nabla f(x)|^p (1-|x|^2)^p\, dv_\alpha(x) \tag{4.1}$$

for all $f \in h(\mathbf{B})$.

Let $\zeta(x,y)$ be the hyperbolic distance on $\mathbf{B}$ defined by

$$\zeta(x,y) := \frac{1}{2}\log\frac{1+\delta(x,y)}{1-\delta(x,y)},$$

where the pseudohyperbolic distance δ on $\mathbf{B}$ is defined by

$$\delta(x,y) = \frac{|x-y|}{[x,y]}$$

for $x, y \in \mathbf{B}$. Here

$$[x,y] := \sqrt{1 - 2x\cdot y + |x|^2|y|^2}, \qquad x, y \in \mathbf{B}$$

and $x \cdot y$ is the dot product of x and y in $\mathbf{R}^n$. Note that $[x,y] \geq 1 - |x||y|$ for $x, y \in \mathbf{B}$.

A straightforward calculation shows that the pseudohyperbolic ball $D_r(x)$ is a Euclidean ball $B_s(y)$ with

$$y = \frac{(1-r^2)}{1-|x|^2r^2}\, x \qquad \text{and} \qquad s = \frac{(1-|x|^2)r}{1-|x|^2r^2} \tag{4.2}$$

and thus we have

$$d\big(x, \partial D_r(x)\big) = \frac{r(1-|x|^2)}{1+r|x|}. \tag{4.3}$$

The next lemma is the ball version of Lemma 2.1. See [2] for the proof.

Lemma 4.2. *The inequality*

$$\frac{1-\delta(l,x)}{1+\delta(l,x)} \le \frac{[l,y]}{[x,y]} \le \frac{1+\delta(l,x)}{1-\delta(l,x)}$$

holds for all $l, x, y \in \mathbf{B}$.

From this lemma, we have

$$1-|x|^2 \approx 1-|y|^2, \qquad y \in D_r(x). \tag{4.4}$$

In particular, given $\alpha > -1$ and $0 < r < 1$, we have by (4.2) and (4.4)

$$v_\alpha\big(D_r(x)\big) \approx (1-|x|^2)^{n+\alpha}$$

for all $x \in \mathbf{B}$.

The following lemma is the ball version of Lemma 3.1. See Lemma 2.5 in [2] for the proof.

Lemma 4.3. *Given* $\alpha > -1$ *and* c *real, the estimates*

$$\int_{\mathbf{B}} \frac{dv_\alpha(y)}{[x,y]^{n+\alpha+c}} \approx \begin{cases} (1-|x|^2)^{-c} & \text{if } \; c>0 \\ 1+\log(1-|x|^2)^{-1} & \text{if } \; c=0 \\ 1 & \text{if } \; c<0 \end{cases}$$

hold for all $x \in \mathbf{B}$. *The constants suppressed above are independent of* x.

The next lemma is needed to prove the implication (c) $\Longrightarrow$ (a) of Theorem 4.5.

Lemma 4.4. *Let* $1 \le k \le n$. *Suppose* $x, y \in \mathbf{B}$, $y = (x_1, \dots, x_{k-1}, tx_k, x_{k+1}, \dots, x_n)$ *where* t *is a scalar. Then*

$$\lim_{y\to x} \frac{\delta(x,y)}{|x-y|} = \lim_{y\to x} \frac{\zeta(x,y)}{|x-y|} = \frac{1}{1-|x|^2}.$$

Proof. Let $1 \le k \le n$ and $x, y \in \mathbf{B}$ with $y = (x_1, \dots, x_{k-1}, tx_k, x_{k+1}, \dots, x_n)$ where t is a scalar. First, we have

$$\lim_{y\to x} \frac{\delta(x,y)}{|x-y|} = \lim_{t\to 1} \frac{1}{g(t)} = \frac{1}{1-|x|^2},$$

where $g(t) = \sqrt{1-2(|x|^2+(t-1)x_k^2)+|x|^2(|x|^2+(t^2-1)x_k^2)}$.

Meanwhile, by the definition of $\zeta(x,y)$ and the simple calculation, we have

$$\begin{aligned}\lim_{y\to x}\frac{\zeta(x,y)}{|x-y|} &= \lim_{y\to x}\frac{1}{2|x-y|}\log\frac{[x,y]+|x-y|}{[x,y]-|x-y|}\\ &= \frac{1}{2x_k}\lim_{t\to 1}\frac{1}{|1-t|}\log\frac{g(t)+|1-t|x_k}{g(t)-|1-t|x_k}\\ &= \lim_{t\to 1}\frac{1}{g(t)}\end{aligned}$$

as desired. The proof is complete. □

Now, having Proposition 4.1, Lemmas 4.3 and 4.4, we modify the proof of Theorems 3.4, 3.5 and 3.6 for getting the following characterization.

Theorem 4.5. *Let $\alpha > -1$ and $0 < p < \infty$. Suppose $f \in h(\mathbf{B})$. Then the following conditions are equivalent.*

(a) $f \in b^p_\alpha(\mathbf{B})$;

(b) *There exists a positive continuous function $g \in L^p_\alpha(\mathbf{B})$ such that*

$$|f(x)-f(y)| \le \delta(x,y)[g(x)+g(y)]$$

for all $x, y \in \mathbf{B}$;

(c) *There exists a positive continuous function $h \in L^p_\alpha(\mathbf{B})$ such that*

$$|f(x)-f(y)| \le \zeta(x,y)[h(x)+h(y)]$$

for all $x, y \in \mathbf{B}$;

(d) *There exists a positive continuous function $k \in L^p_{\alpha+p}(\mathbf{B})$ such that*

$$|f(x)-f(y)| \le |x-y|[k(x)+k(y)]$$

for all $x, y \in \mathbf{B}$.

Furthermore, when this is the case, the functions g, h and k associated with f can be chosen in such a way that $\|g\|_{L^p_\alpha(\mathbf{B})}$, $\|h\|_{L^p_\alpha(\mathbf{B})}$ and $\|k\|_{L^p_{\alpha+p}(\mathbf{B})}$ are comparable to $\|f\|_{b^p_\alpha(\mathbf{B})}$.

Proof. Having Lemma 4.4 and Proposition 4.1, the equivalence (a) $\Longleftrightarrow$ (b) is proved by modifying the proof of Theorem 3.4. Since $\delta \le \zeta$, the implication (b) $\Longrightarrow$ (c) is trivial. Also, using $[x,y] \ge 1-|x||y| \ge 1-|x|, 1-|y|$, implication (b) $\Longrightarrow$ (d) is easily proved.

Now, we prove the implication (c) $\Longrightarrow$ (a). Let $\alpha > -1$, $0 < p < \infty$ and $f \in h(\mathbf{B})$. Suppose that there exists a positive continuous function $h \in L^p_\alpha(\mathbf{B})$ such that

$$|f(x)-f(y)| \le \zeta(x,y)[h(x)+h(y)] \tag{4.5}$$

for all $x, y \in \mathbf{B}$. Let $1 \le k \le n$ and assume $y = (x_1, \dots, x_{k-1}, tx_k, x_{k+1}, \dots, x_n)$ where t is a scalar. Dividing the above inequality by $|x-y|$ and taking the limit $y \to x$, Lemma 4.4 yields

$$|\partial_k f(x)| \le \frac{2h(x)}{1-|x|^2}, \qquad k = 1, \dots, n$$

and thus, we have

$$|\bigtriangledown f(x)| \lesssim \frac{h(x)}{1-|x|}. \tag{4.6}$$

Since $h \in L^p_\alpha(\mathbf{B})$, (4.6) and Proposition 4.1 imply that the condition (a) holds.
Similarly, Proposition 4.1 yields the implication (d) $\Longrightarrow$ (a). □

We consider the difference quotient L of harmonic function for the ball case. Given $f \in h(\mathbf{B})$, L is given by

$$Lf(x,y) := \frac{f(x)-f(y)}{|x-y|}, \qquad x \neq y$$

for $x, y \in \mathbf{B}$.

Theorem 4.5 makes it possible that if we imitate the proof of Theorem 3.7, then we can have the following ball analogue.

Theorem 4.6. *Let $\alpha > -1$, $n+\alpha < p < \infty$ and $\gamma = (p+\alpha-n)/2$. Then L maps $b^p_\alpha(\mathbf{B})$ boundedly into $L^p(v_\gamma \times v_\gamma)$.*

Note that the case $c < 0$ of Lemma 4.3 is different from that of Lemma 3.1. It makes possible that one may extend the above mentioned holomorphic case for $0 < p < 2+\alpha$ to the harmonic case over the real ball.

Theorem 4.7. *Let $\alpha > -1$ and $0 < p < n+\alpha$. Then L maps $b^p_\alpha(\mathbf{B})$ boundedly into $L^p(v_\alpha \times v_\alpha)$.*

Proof. Let $f \in b^p_\alpha(\mathbf{B})$. By Theorem 4.5 and the triangle inequality, there exists a positive continuous function $g \in L^p_\alpha(\mathbf{B})$ such that $\|g\|_{L^p_\alpha(\mathbf{B})} \approx \|f\|_{b^p_\alpha(\mathbf{B})}$ and

$$\begin{aligned} \|Lf\|^p_{L^p(v_\alpha\times v_\alpha)} &\lesssim \int_{\mathbf{B}}\int_{\mathbf{B}} \frac{\delta(x,y)^p(|g(x)|^p+|g(y)|^p)}{|x-y|^p}\, dv_\alpha(x)\, dv_\alpha(y) \\ &= 2\int_{\mathbf{B}} |g(x)|^p \int_{\mathbf{B}} \frac{dv_\alpha(y)}{[x,y]^p}\, dv_\alpha(x). \end{aligned} \tag{4.7}$$

Since $p-n-\alpha < 0$, Lemma 4.3 implies

$$\int_{\mathbf{B}} \frac{dv_\alpha(y)}{[x,y]^p} \approx 1.$$

Consequently, we have from (4.7)

$$\|Lf\|_{L^p(v_\alpha\times v_\alpha)} \lesssim \|g\|_{L^p_\alpha(\mathbf{B})} \approx \|f\|_{b^p_\alpha(\mathbf{B})}.$$

The constants suppressed above are independent of f. The proof is complete. □

Remark. For $p \geq n+\alpha$, see Example 3.9 in [4] showing that L never maps $b^p_\alpha(\mathbf{B})$ into $L^p(v_\alpha \times v_\alpha)$.

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