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ON CONSTANT MEAN CURVATURE GRAPHS WITH CONVEX BOUNDARY

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Abstract. We give area and height estimates for cmc-graphs over a bounded planar $C^{2,\alpha}$ domain $\Omega \subset \mathbb{R}^3$. For a constant H satisfying $H^2|\Omega| \leq 9\pi/16$, we show that the height h of H-graphs over Ω with vanishing boundary satisfies $|h| < (\tilde{r}/2\pi)H|\Omega|$, where $\tilde{r}$ is the middle zero of $(x-1)(H^2|\Omega|(x+2)^2 - 9\pi(x-1))$. We use this height estimate to prove the following existence result for cmc H-graphs: for a constant H satisfying $H^2|\Omega| < (\sqrt{297}-13)\pi/8$, there exists an H-graph with vanishing boundary.

1. Introduction

The existence of disk-type minimal or constant mean curvature (cmc) surface spanning a given Jordan curve Riemannian manifolds is a classical problem in the surface theory. Douglas and Radó's solution for the Plateau's problem says that every rectifiable Jordan curve spans a minimal disk. Wente showed the existence of cmc surfaces spanning a rectifiable Jordan curve [15]. After Wente's result, various existence results for disk-type cmc surfaces spanning a closed rectifiable curve $\Gamma \subset \mathbb{R}^3$ was obtained under different geometric assumptions relating Γ and H ([4], [13], [14]). For example, Steffen [13] showed the following: Let $D(X_0)$ be the area of the area minimizing minimal surface spanning Γ. For a real number H satisfying $H^2D(X_0) < 2\pi/3$, there is a disk-type cmc H surface spanning Γ. When Γ is a plane curve, the condition was extended for H satisfying $H^2D(X_0) < 3\pi/4$. It is conjectured that it suffices to assume that $H^2D(X_0) < \pi$, which is optimal if Γ is a circle.

If Γ is convex, then there is a graph surface with cmc H spanning Γ for small $|H|$. It is interesting to find the upper bound of $|H|$ for which there exist an H-graph having Γ as boundary. More precisely, let Ω be a $C^{2,\alpha}$ bounded convex domain in $\mathbb{R}^2$. For a constant H, an H-graph over Ω is the graph of

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the solution of the constant mean curvature (cmc) boundary value problem:

$$\begin{aligned} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= -2H \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Let $|\Omega| = \text{Area } \Omega$. Following Setffen's result, it is reasonable to assume that $H^2|\Omega| < 3\pi/4$. Recently, Ripoll and Lopez showed that (1) has a solution $u \in C^{2,\alpha}(\bar{\Omega})$ for H satisfying $H^2|\Omega| < \pi/2$ ([11], [6]). It is conjectured by Montiel [9] that it suffices to assume that $H^2|\Omega| < \pi$.

We show that if H satisfies $H^2|\Omega| < (\sqrt{297} - 13)\pi/8$, then there is a classical solution $u \in C^{2,\alpha}(\bar{\Omega})$ of (1). We use an isoperimetric type inequality for cmc surfaces (12) to get an area estimate (13) and a height estimate (14) for H-graphs, where H satisfies $H^2|\Omega| \le 9\pi/16$. It is also shown that the area of an H-graph is less than $2|\Omega|$ for H satisfying $H^2|\Omega| \le 9\pi/16$. In §3, we use the height estimate (14) to show that if H satisfies $H^2|\Omega| < (\sqrt{297} - 13)\pi/8$, then (1) has a solution $u \in C^{2,\alpha}(\bar{\Omega})$.

2. Height estimate for *H*-graphs

In this section, we derive a height estimate for H-graphs using an isoperimetric type inequality for cmc surfaces in $\mathbb{R}^3$. Let Ω be a bounded C^2 domain in the plane $z = 0$ and suppose that $u \in C^2(\bar{\Omega})$ is a solution of (1) for given H. We may assume that $H > 0$ and $u \ge 0$ [8]. Let Σ be the graph of u and let h be the height of Σ. We use $|\ |$ to denote the area. We recall the following height estimate for cmc graphs by Lopez and Montiel [7]

$$h \le \frac{|\Sigma| H}{2\pi}, \tag{2}$$

where the equality holds if and only if $\partial\Omega$ is a circle.

Let $\Sigma_t = \{(x, y, z) \in \Sigma \mid z \ge t\}$ and let ν_t be the inward co-normal of Σ_t along $\partial\Sigma_t$. Let π_3 be the projection onto the plane $z = 0$ and let $e_3 = (0, 0, 1)$. Let ϕ be the immersion of Σ. It is shown in [7] that the critical points of the function $f = \langle \phi, e_3 \rangle$ has measure zero and $|\Omega_t|$, where $\Omega_t = \pi_3(\Sigma_t)$, is a continuous function of t for almost all t. Let V be the algebraic volume enclosed by Σ and Ω:

$$V = -(1/3) \int_\Sigma \langle \phi, N \rangle dA,$$

where N is the unit normal vector field for Σ with $N_3 < 0$. The following lemma is proved using the divergence theorem for cmc surface Σ with planar boundary ([6], [7]). We give a new proof based on the balancing formula and the co-area formula for H-graphs.

Lemma 1. *Let V be the volume of the domain enclosed by Σ and the plane $z = 0$. Then we have*

$$2HV = \int_\Sigma |\nabla\langle\phi, e_3\rangle|^2 dA. \tag{3}$$

Proof. The balancing formula for cmc surfaces immersed in $\mathbb{R}^3$ with planar boundary [5] implies that

$$\int_{\partial\Sigma_t} \langle\nu_t, e_3\rangle ds_t = 2H|\Omega_t|,$$

where ds_t is the line element of $\partial\Sigma_t$. Integrating this equation between $t = 0$ and $t = \infty$, we obtain

$$2HV = \int_0^\infty \int_{\partial\Sigma_t} \langle\nu_t, e_3\rangle ds_t dt.$$

The co-area formula [10] implies that

$$\int_0^\infty \int_{\partial\Sigma_t} \langle\nu_t, e_3\rangle ds_t = \int_\Sigma \langle\nu_t, e_3\rangle|\nabla f| dAdt.$$

Since $f = \langle\phi, e_3\rangle$ satisfies

$$\langle\nabla f_p, v\rangle = \langle(d\phi)_p(v), e_3\rangle,$$

for each $p \in \Sigma$ and each $v \in T_p\Sigma$, we have

$$|\nabla f|^2 = 1 - \langle N, e_3\rangle^2 = \langle\nu_t, e_3\rangle^2 \tag{4}$$

along $\partial\Sigma_t$. Hence $\langle\nu_t, e_3\rangle = |\nabla f|$ and the lemma follows. □

From (3) and (4), we have

$$2HV = \int_\Sigma (1 - \langle N, e_3\rangle^2) dA = |\Sigma| - \int_\Sigma \langle N, e_3\rangle^2 dA. \tag{5}$$

Since $|\langle N, e_3\rangle|$ is the Jacobian of π_3, we have $\int_\Sigma |\langle N, e_3\rangle| dA = |\Omega|$. Since $|\langle N, e_3\rangle| < 1$, we have

$$|\Sigma| < |\Omega| + 2HV < |\Omega| + 2hH|\Omega|. \tag{6}$$

This area estimate of Σ and Serrin's height estimate $h \leq 1/H$ for H-graphs [12] show that

$$|\Sigma| < 3|\Omega|.$$

On the other hand, the relative isoperimetric inequality [1]

$$18\pi V^2 \leq |\Sigma|^3 \tag{7}$$

gives the relation between the surface area $|\Sigma|$ and the enclosed volume V.

Theorem 1. *For H satisfying $0 < H^2|\Omega| < 2\pi/3$, we have*

$$|\Sigma| < r|\Omega|,$$

where r is the second largest (real) zero of the cubic polynomial

$$2H^2|\Omega|x^3 - 9\pi(x-1)^2. \tag{8}$$

In particular, if $H^2|\Omega| \le 9\pi/16$, then

$$|\Sigma| \le 2|\Omega|.$$

Proof. It is straightforward to see that the polynomial $2H^2|\Omega|x^3 - 9\pi(x-1)^2$ has three different real zeros for H satisfying $H^2|\Omega| < 2\pi/3$. From (6) and (7), we have

$$|\Sigma| - |\Omega| < 2HV \le 2H\left(\frac{|\Sigma|^3}{18\pi}\right)^{1/2}. \tag{9}$$

Define x by $|\Sigma| = x|\Omega|$. Then we have $1 < x < 3$. From (9), we have

$$2H^2|\Omega|x^3 - 9\pi(x-1)^2 > 0.$$

For H satisfying $H^2|\Omega| < 2\pi/3$, (8) has three real zeros and the smallest zero is < 1 and the biggest zero is > 3. Hence x satisfies $1 < x < r$ for the middle zero r of (8). We have $r = 2$ when $H^2|\Omega| = 9\pi/16$. Because r is an increasing function of $H^2|\Omega|$ for $H^2|\Omega| < 2\pi/3$, we have $|\Sigma| \le 2|\Omega|$ for H-graphs with H satisfying $H^2|\Omega| \le 9\pi/16$. □

For an H-graph with $H^2|\Omega| \le 9\pi/16$, we get from (2) the following height estimate:

$$h \le \frac{r|\Omega|}{2\pi}H \le \frac{|\Omega|}{\pi}H. \tag{10}$$

This gives a better height estimate than Serrin's $h \le 1/H$. Serrin gives a different height estimate using the geometry of Ω.

Lemma 2. *Let Ω be a bounded domain in $\mathbb{R}^2$ and let $2R$ be the diameter of Ω. If a positive constant H satisfies $H < 1/R$, then every H-graph over Ω satisfies the height estimate*

$$h \le \frac{1}{H} - \sqrt{\frac{1}{H^2} - R^2}.$$

The equality holds if and only if the H-graph is a spherical cap.

We improve (10) to prove the existence result for (1). For simplicity, we assume that $|\Omega| = \pi$ and Σ and $z = 0$ enclose volume V. We also assume that H satisfies $H^2|\Omega| \le 9\pi/16$. Since $|\Sigma| \le 2|\Omega|$ by Theorem 1, we have $V \le 2\pi/3$ from (7). Let Σ' be the rotational surface obtained from the Steiner symmetrization of Σ about the z-axis [1]. Then we have $|\Sigma'| \le |\Sigma|$ and Σ' and $z = 0$ encloses volume V. Let Ω' be the unit disk bounded by $\partial\Sigma'$. Since $V < 2\pi/3$, there is a small spherical cap Σ_s over Ω' which encloses volume V with $z = 0$. Among rotational surfaces over Ω' that encloses volume V, Σ_s has the least area. Hence we have $|\Sigma_s| \leq |\Sigma'| \leq |\Sigma|$. Let H_s be the mean curvature of Σ_s.

Lemma 3. *For H-graph with H satisfying $H^2|\Omega| \leq 9\pi/16$, the enclosed volume V and the surface area $|\Sigma|$ satisfy*

$$\frac{|\Sigma|^3}{V^2} \geq \frac{36\pi}{(1-\pi/|\Sigma|)\,(1+2\pi/|\Sigma|)^2}.$$

Proof. Let C be the center of the spherical cap Σ_s and let θ be the angle between the z-axis and $\partial\Sigma_s$ measured from C. Then we have $H_s = \sin\theta$ and $|\Sigma_s| = (2\pi/H_s^2)(1-\cos\theta)$. Plugging $\sin\theta = H_s$ into the later, we have

$$1+\cos\theta = \frac{2\pi}{|\Sigma_s|}. \tag{11}$$

We have

$$\begin{aligned} V = & \frac{2\pi}{3H_s^3}(1-\cos\theta) - \frac{\pi}{3H_s^3}\sin^2\theta\cos\theta \\ = & \frac{\pi}{3H_s^3}(1-\cos\theta)^2(2+\cos\theta)). \end{aligned}$$

Then we have

$$\frac{|\Sigma|^3}{V^2} \geq \frac{36\pi}{(1-\pi/|\Sigma_s|)\,(1+2\pi/|\Sigma_s|)^2}.$$

Let

$$g(x) = \frac{1}{(1-x)(1+2x)^2}.$$

Note that $g(x)$ is monotonically increasing on $[1/2, 1)$. Since $|\Sigma_s| \leq |\Sigma| < 2|\Omega| = 2\pi$, we have $g(\pi/|\Sigma_s|) \geq g(\pi/|\Sigma|)$. Therefore we have

$$|\Sigma|^3/V^2 \geq 36\pi/(1-\pi/|\Sigma|)(1+2\pi/|\Sigma|)^2. \tag{12}$$

□

From (6) and Lemma 3, we have

$$|\Sigma| - \pi \leq 2H\left(\frac{1}{36\pi}(|\Sigma|-\pi)(|\Sigma|+2\pi)^2\right)^{1/2}.$$

Define x by $|\Sigma| = \pi x$. Then we have

$$9(x-1)^2 \leq H^2(x-1)(x+2)^2.$$

Let

$$p(x) := (x-1)\left(H^2(x+2)^2 - 9(x-1)\right).$$

For H satisfying $0 < H^2 \leq 9/16$, $p(x)$ is negative for $x < 1$ and has three different real zeros. Moreover the middle zero $\tilde{r}$ of $p(x)$ is an increasing function of H^2 for $H^2 \leq 9/16$. We have

$$|\Sigma| \leq \tilde{r}|\Omega| \tag{13}$$

for H-graphs over a bounded planar domain Ω with H satisfying $H^2|\Omega| \leq 9\pi/16$. From (2) and (13), we have the following height estimate

$$h \leq \frac{\tilde{r}}{2\pi}|\Omega|H. \tag{14}$$

We note that $\tilde{r}$ is bigger than r of Theorem 1 for $H^2|\Omega| \leq 9\pi/16$.

3. Existence theorems for H-graphs

We use the height estimate (14) for H-graphs to prove the following.

Theorem 2. *For a given positive number H satisfying $H^2|\Omega| < (\sqrt{297}-13)\pi/8$, there is a constant τ such that any solution $u_{H'} \in C^2(\bar{\Omega})$ to (1), for given $H' \in [0, H]$, satisfies*

$$|u_{H'}| \leq \tau < 1/(2H).$$

There is a solution $u \in C^{2,\alpha}(\bar{\Omega})$ to (1) for a bounded convex $C^{2,\alpha}$ domain Ω in the plane.

Proof. We refer Theorem 3 of [11] to prove the above theorem: if there exists $\tau < 1/(2H)$ such that any solution $u \in C^2(\bar{\Omega})$ to (1) for given $H' \in [0, H]$ satisfies the a priori height estimate $|u| \leq \tau < 1/(2H)$, then there is a solution $u \in C^{2,\alpha}(\bar{\Omega})$ to (1).

Using homothety, we may assume that $|\Omega| = \pi$ and $|\Sigma| = |\Omega|x = \pi x$. Suppose that $H^2 \leq c$ for some c with $c < 9\pi/16$. Then $x \in [1, 2)$. The height estimate (14) implies that the height h of Σ satisfies

$$h < \frac{\tilde{r}H^2|\Omega|}{2\pi H}.$$

Note that $\tilde{r}$ depends on H^2. Due to Theorem 3 of [11], we have only to show that if $c < (\sqrt{297} - 13)\pi/8$ is given, then $H^2\tilde{r} \leq 1 - \epsilon$ for all H with $H^2 < c$, where $\epsilon > 0$ depends only on c.

Since $\tilde{r}$ is an increasing function of H^2, we assume that ${H_m}^2 = c$ and $c\tilde{r} = 1$. Then $\tilde{r}$ is given by

$$\tilde{r} = \frac{9 - 4{H_m}^2 - \sqrt{81 - 108{H_m}^2}}{2{H_m}^2}$$

and ${H_m}^2$ is $(\sqrt{297} - 13)/8$. Therefore for H satisfying $H^2 < c$ with $c < (\sqrt{297} - 13)/8$, we have $H^2\tilde{r} < 1 - \epsilon$ for ϵ depending only on c. This completes the proof. □

Remark. If we use (9) instead of (12), then we get $(3 - \sqrt{2})\pi/3$ instead of $(\sqrt{297} - 13)\pi/8$.

We can use Theorem 2 to prove existence of H-graphs over convex bounded domain in the plane. Let Ω be a convex bounded domains in $z = 0$ and H satisfies $H^2|\Omega| < (\sqrt{297} - 13)\pi/8$. For each point $x \in \partial\Omega$, we choose a bounded convex $C^{2,\alpha}$ domain Ω_x containing Ω and meeting Ω at x. Since

Ω is convex, we may further assume that $H^2|\Omega_x| < (\sqrt{297}-13)\pi/8$. From Theorem 2, we see that there is a $C^{2,\alpha}$ solution u_x of (1) for the domain Ω_x. Strong maximum principle shows that $u_x \geq 0$ on $\partial\Omega$. Hence u_x is a super-solution of (1). Let S_0 be the set of all super-solutions of (1). The function $u(x) := \inf_{v\in S_0} v(x)$ satisfies $M_H(u) = 0$ in Ω, $u|_{\partial\Omega} = 0$ and $u \in C^2(\Omega)\cap C^0(\bar{\Omega})$ [2] (Here $M_H(u) = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + 2H$). This proves the following.

Theorem 3. *For a convex bounded domain Ω in $\mathbb{R}^2$ and a constant H satisfying $H^2|\Omega| < (\sqrt{297}-13)\pi/8$, there is a function $u \in C^2(\Omega)\cap C^0(\bar{\Omega})$.*

When $\partial\Omega$ is a circle, the constant $(\sqrt{297}-13)\pi/8$ is far from the optimal constant π. In the proof of Theorem 3 of [11], quarter cylinder of mean curvature H was used as a barrier to obtain the a priori gradient estimate. Instead of the quarter cylinders, we may use parts of nodoids as a barrier to get a better constant. Suppose that $|\Omega| = \pi$. Let κ be the minimum of the curvature of $\partial\Omega'$ which is smaller than 1. Let Ξ be a circle on the plane $z = 0$ with radius $1/\kappa$. There is a one parameter family of nodoids, for which Ξ is the circle of the largest radius. We restrict the range of H to $(\kappa, 1]$. Let N_H be the nodoid of mean curvature H in the family. Let B_H be the part of N_H between $z = 0$ and $z = h_H$, where $h_H = h_H(\kappa, H)$ is the smallest positive value of z for which N_H has horizontal tangent plane. For fixed κ, h_H is a continuous decreasing function of H and $\lim_{H\to\kappa} h_H = 1/\kappa$. On the other hand, as a function of $\kappa \in [H, \infty)$ for fixed H, h_H is decreasing with $\lim_{\kappa\to\infty} h_H = 1/2H$.

Theorem 4. *Let Ω be a given bounded convex $C^{2,\alpha}$ domain in $\mathbb{R}^2$ and let h_H be given as above. Let $H_d^2|\Omega| \leq 2\pi/3$ be a positive constant depending on κ such that $2\pi h_{H_d}/H_d|\Omega|$ is the second largest zero of*

$$2{H_d}^2|\Omega|x^3 - 9\pi(x-1)^2.$$

Then (1) *is solvable for H satisfying $H < H_d$.*

Proof. We repeat the argument of proofs of Theorem 2. But we have to use $2{H_d}^2|\Omega|x^3 - 9\pi(x-1)^2$ instead of $p(x)$ and assume that $H_d^2|\Omega| \leq 2\pi/3$. □

Finally we raise the following question.

Question. *Does it hold that $|G| < 2|\Omega|$ for every H-graph G over a bounded planar domain Ω and does the equality hold if and only if G is a half sphere?*

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