

## ON SOME MODULAR EQUATIONS AND THEIR APPLICATIONS II

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ABSTRACT. We first derive some modular equations of degrees 3 and 9 and present their concise proofs based on algebraic computations. We then use these modular equations to establish explicit relations and formulas for the parameterizations for the theta functions  $\varphi$  and  $\psi$ . In addition, we find specific values of the parameterizations to evaluate some numerical values of the cubic continued fraction.

### 1. Introduction

We begin this section by introducing Ramanujan's definition of his general theta function. For  $|ab| < 1$ , define

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

Note that two special cases of  $f(a, b)$  are defined by, for  $|q| < 1$ ,

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty}$$

and

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

where

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n).$$

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Let  $a, b,$  and  $c$  be arbitrary complex numbers except that  $c$  cannot be a non-positive integer. Then, for  $|z| < 1,$  the Gaussian or ordinary hypergeometric function  ${}_2F_1(a, b; c; z)$  is defined by

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n,$$

where  $(a)_0 := 1$  and  $(a)_n := a(a + 1)(a + 2) \cdots (a + n - 1)$  for each positive integer  $n.$

Now the complete elliptic integral of the first kind  $K(k)$  is defined by

$$(1.1) \quad K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \frac{\pi}{2} \varphi^2\left(e^{-\pi \frac{K'}{K}}\right),$$

where  $0 < k < 1, K' = K(k'),$  and  $k' = \sqrt{1 - k^2}.$  The number  $k$  is called the modulus of  $K$  and  $k'$  is called the complementary modulus.

Let  $K, K', L,$  and  $L'$  denote complete elliptic integrals of the first kind associated with the moduli  $k, k', l,$  and  $l',$  respectively, where  $0 < k < 1$  and  $0 < l < 1.$  Suppose that

$$(1.2) \quad \frac{L'}{L} = n \frac{K'}{K}$$

holds for some positive integer  $n.$  A relation between  $k$  and  $l$  induced by (1.2) is called a modular equation of degree  $n.$

If we set

$$q = \exp\left(-\pi \frac{K'}{K}\right) \quad \text{and} \quad q' = \exp\left(-\pi \frac{L'}{L}\right),$$

we see that (1.2) is equivalent to the relation  $q^n = q'.$  Hence a modular equation can be viewed as an identity involving theta functions at the arguments  $q$  and  $q^n.$

Note that the definition of a modular equation mentioned above is the one used by Ramanujan, but we emphasize that there are several definitions of a modular equation in the literature. For example, refer the books by R. A. Rankin in [5] and B. Schoeneberg in [6] for other definitions of a modular equation. Following Ramanujan, set  $\alpha = k^2$  and  $\beta = l^2,$  then we say that  $\beta$  has degree  $n$  over  $\alpha.$  By the relationship between complete elliptic integrals of the first kind and hypergeometric function, we have

$$n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}.$$

Let  $z_n = \varphi^2(q^n).$  Then the multiplier  $m$  for degree  $n$  is defined by

$$m := \frac{\varphi^2(q)}{\varphi^2(q^n)} = \frac{z_1}{z_n}.$$

Next we introduce the definitions of 4 parameterizations for the theta functions  $\varphi$  and  $\psi$  from [7, 8, 10]. For any positive real numbers  $k$  and  $n,$  define

$h_{k,n}$  by

$$(1.3) \quad h_{k,n} := \frac{\varphi(q)}{k^{1/4}\varphi(q^k)},$$

where  $q = e^{-\pi\sqrt{n/k}}$ , define  $h'_{k,n}$  by

$$(1.4) \quad h'_{k,n} := \frac{\varphi(-q)}{k^{1/4}\varphi(-q^k)},$$

where  $q = e^{-2\pi\sqrt{n/k}}$ , define  $l_{k,n}$  by

$$(1.5) \quad l_{k,n} := \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)},$$

where  $q = e^{-\pi\sqrt{n/k}}$ , define  $l'_{k,n}$  by

$$(1.6) \quad l'_{k,n} := \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)},$$

where  $q = e^{-\pi\sqrt{n/k}}$ .

In [7, 8, 10], several new modular equations for the theta functions were derived, some explicit relations and formulas for the parameterizations were offered, and some values of the parameterizations were determined. Moreover, in [9], some new modular equations of degrees 3 and 9 for the theta functions  $\varphi$  and  $\psi$  were derived in order to establish explicit relations and formulas for the parameterizations for  $h_{k,n}$ ,  $h'_{k,n}$ ,  $l_{k,n}$ , and  $l'_{k,n}$  and show some applications of those modular equations to evaluations of the cubic continued fraction.

In this paper, we further derive some more modular equations of degrees 3 and 9 for the theta functions  $\varphi$  and  $\psi$  and present their concise proofs based on algebraic computations as in [9]. Furthermore, we find explicit relations and formulas for the corresponding parameterizations, evaluate some numerical values of  $h_{k,n}$ ,  $h'_{k,n}$ ,  $l_{k,n}$ , and  $l'_{k,n}$  for some positive real numbers  $k$  and  $n$  by employing the relations and formulas established earlier, and evaluate some numerical values of the cubic continued fraction.

## 2. Preliminary results

In this section, we introduce fundamental theta function identities that will play key roles in deriving some modular equations. We also recall some useful explicit relations for the parameterizations of  $h_{k,n}$ ,  $l_{k,n}$ , and  $l'_{k,n}$  for some positive real numbers  $k$  and  $n$ . Let  $k$  be the modulus as in (1.1). Set  $x = k^2$  and also set

$$(2.1) \quad k^2 = x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}.$$

Then

$$(2.2) \quad \varphi^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) =: z,$$

where

$$(2.3) \quad q = e^{-y} := \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}\right) = \exp\left(-\pi \frac{K(k')}{K(k)}\right).$$

**Lemma 2.1** ([1], Theorem 5.4.1). *If  $x$ ,  $q$ , and  $z$  are related by (2.1), (2.2), and (2.3), then*

- (i)  $\varphi(q) = \sqrt{z}$ ,
- (ii)  $\varphi(-q) = \sqrt{z}(1-x)^{1/4}$ .

**Lemma 2.2** ([1], Theorem 5.4.2). *If  $x$ ,  $q$ , and  $z$  are related by (2.1), (2.2), and (2.3), then*

- (i)  $\psi(q) = \sqrt{\frac{1}{2}z} \left(\frac{x}{q}\right)^{1/8}$ ,
- (ii)  $\psi(q^2) = \frac{1}{2}\sqrt{z} \left(\frac{x}{q}\right)^{1/4}$ .

**Lemma 2.3** ([2], Entry 1(ii), Chapter 20). *For  $|q| < 1$ , we have*

$$1 + \frac{\psi(-q^{1/3})}{q^{1/3}\psi(-q^3)} = \left(1 + \frac{\psi^4(-q)}{q\psi^4(-q^3)}\right)^{1/3}.$$

**Lemma 2.4** ([2], Entry 5, Chapter 19). *Let  $\beta$  be the third degree and  $m = \frac{z_1}{z_3}$ . Then*

- (i)  $\sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{1-\beta}{1-\alpha}} - \sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = m^2$ ,
- (ii)  $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{1-\alpha}{1-\beta}} - \sqrt{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}} = \left(\frac{3}{m}\right)^2$ .

**Lemma 2.5** ([2], Entry 3, Chapter 20). *Let  $\gamma$  be the ninth degree and  $m = \frac{z_1}{z_9}$ . Then*

- (i)  $\left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8} - \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{m}$ ,
- (ii)  $\left(\frac{\alpha}{\gamma}\right)^{1/8} + \left(\frac{1-\alpha}{1-\gamma}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8} = \frac{3}{\sqrt{m}}$ .

Next two results will be useful in evaluating specific values of  $h_{k,n}$ ,  $h'_{k,n}$ ,  $l_{k,n}$ , and  $l'_{k,n}$ .

**Lemma 2.6** ([8], Theorem 2.2). *For any positive real number  $k$ ,*

$$h_{k,1} = 1.$$

**Lemma 2.7** ([10], Theorem 2.3). *For any positive real number  $k$ ,*

$$l_{k,1} = 1.$$

**Lemma 2.8** ([9], Corollary 3.14). *For every positive real number  $n$ , we have*

$$(l_{9,n} - l'_{9,n} + \sqrt{3}) \left( \frac{1}{l_{9,n}} - \frac{1}{l'_{9,n}} + \sqrt{3} \right) = 1.$$

We recall that the cubic continued fraction  $G(q)$  is defined by

$$G(q) := \frac{q^{1/3}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} + \dots = q^{1/3} \frac{\chi(-q)}{\chi^3(-q^3)},$$

for  $|q| < 1$ , where  $\chi(q) := (-q; q^2)_\infty$ .

The following result exhibits some general formulas for the values of  $G(e^{-\pi\sqrt{n}})$  and  $G(-e^{-\pi\sqrt{n}})$  in terms of  $l'_{3,n}$ ,  $l'_{9,n}$ , and  $l_{9,n}$ .

**Lemma 2.9** ([10], Theorem 6.2). *For any positive real number  $n$ , we have*

- (i)  $G(e^{-\pi\sqrt{n}}) = \frac{1}{\sqrt{3}l'_{9,n} - 1},$
- (ii)  $G^3(e^{-\pi\sqrt{n/3}}) = \frac{1}{3l'^4_{3,n} - 1},$
- (iii)  $G(-e^{-\pi\sqrt{n}}) = \frac{-1}{\sqrt{3}l_{9,n} + 1}.$

**Lemma 2.10** ([7], Lemma 6.3.6). *We have*

$$G(e^{-2\pi\sqrt{n}}) = -G(e^{-\pi\sqrt{n}})G(-e^{-\pi\sqrt{n}})$$

for any positive real number  $n$ .

### 3. Modular equations

Note that Ramanujan’s 23 eta function identities, which are certain types of modular equations, are given in [3]. In this section, we derive some modular equations of degrees 3 and 9 and present their proofs based on concise algebraic computations. In addition, we establish some explicit relations and formulas for  $h_{k,n}$ ,  $h'_{k,n}$ ,  $l_{k,n}$ , and  $l'_{k,n}$  by employing these modular equations.

**Theorem 3.1.** *If  $P = \frac{\psi(q)}{q^{1/4}\psi(q^3)}$  and  $Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)}$ , then*

$$(3.1) \quad P^4 + \frac{9}{P^4} = \left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4 + 8.$$

*Proof.* By Lemma 2.2,

$$P = \sqrt{\frac{z_1}{z_3}} \left(\frac{\alpha}{\beta}\right)^{1/8} \quad \text{and} \quad Q = \sqrt{\frac{z_1}{z_3}} \left(\frac{\alpha}{\beta}\right)^{1/4},$$

where  $\beta$  has degree 3 over  $\alpha$ . Thus

$$\frac{P^2}{Q} = \sqrt{\frac{z_1}{z_3}} \quad \text{and} \quad \frac{Q}{P} = \left(\frac{\alpha}{\beta}\right)^{1/8}.$$

By Lemma 2.4,

$$\left(\frac{P}{Q}\right)^4 + \sqrt{\frac{1-\beta}{1-\alpha}} - \left(\frac{P}{Q}\right)^4 \sqrt{\frac{1-\beta}{1-\alpha}} = \frac{P^8}{Q^4}$$

and

$$\left(\frac{Q}{P}\right)^4 + \sqrt{\frac{1-\alpha}{1-\beta}} - \left(\frac{Q}{P}\right)^4 \sqrt{\frac{1-\alpha}{1-\beta}} = \frac{9Q^4}{P^8}.$$

Combining and rearranging above two equations in terms of  $P$  and  $Q$ , we deduce that

$$\left(1 - \frac{P^4}{Q^4}\right) \left(1 - \frac{Q^4}{P^4}\right) = (P^4 - 1) \left(\frac{9}{P^4} - 1\right),$$

which is equivalent to (3.1). Hence we complete the proof. □

Using the definition of  $l'_{k,n}$ , we have the following:

**Corollary 3.2.** *For every positive real number  $n$ , we have*

$$(3.2) \quad 3 \left( l_{3,n}^4 + \frac{1}{l_{3,n}^4} \right) = \left( \frac{l'_{3,n}}{l'_{3,4n}} \right)^4 + \left( \frac{l'_{3,4n}}{l'_{3,n}} \right)^4 + 8.$$

*Proof.* Letting  $q = e^{-\pi\sqrt{n/3}}$  in (1.6), we find that  $P = 3^{1/4} l'_{3,n}$  and  $Q = 3^{1/4} l'_{3,4n}$  in Theorem 3.1. Rewriting (3.1) in terms of  $l'_{3,n}$  and  $l'_{3,4n}$ , we complete the proof. □

**Theorem 3.3.** *If  $P = \frac{\varphi(q)}{\varphi(q^9)}$  and  $Q = \frac{\psi(q^2)}{q^2\psi(q^{18})}$ , then*

$$(3.3) \quad \sqrt{PQ} + \frac{3}{\sqrt{PQ}} = \sqrt{\frac{Q}{P}} + \sqrt{\frac{P}{Q}} + 2.$$

*Proof.* By Lemma 2.1,

$$P = \sqrt{\frac{z_1}{z_9}} \quad \text{and} \quad Q = \sqrt{\frac{z_1}{z_9}} \left(\frac{\alpha}{\gamma}\right)^{1/4},$$

where  $\gamma$  has degree 9 over  $\alpha$ . Thus

$$\frac{Q}{P} = \left(\frac{\alpha}{\gamma}\right)^{1/4}.$$

By Lemma 2.5,

$$\sqrt{\frac{P}{Q}} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8} - \sqrt{\frac{P}{Q}} \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8} = P$$

and

$$\sqrt{\frac{Q}{P}} + \left(\frac{1-\alpha}{1-\gamma}\right)^{1/8} - \sqrt{\frac{Q}{P}} \left(\frac{1-\alpha}{1-\gamma}\right)^{1/8} = \frac{3}{P}.$$

Combining and rearranging above two equations in terms of  $P$  and  $Q$ , we deduce that

$$\left(1 - \sqrt{\frac{P}{Q}}\right) \left(1 - \sqrt{\frac{Q}{P}}\right) = \left(P - \sqrt{\frac{P}{Q}}\right) \left(\frac{3}{P} - \sqrt{\frac{Q}{P}}\right),$$

which is equivalent to (3.3). Hence we complete the proof. □

Using the definitions of  $h_{k,n}$  and  $l'_{k,n}$ , we have the following:

**Corollary 3.4.** *For every positive real number  $n$ , we have*

$$(3.4) \quad \sqrt{3} \left( \sqrt{h_{9,n}l'_{9,4n}} + \frac{1}{\sqrt{h_{9,n}l'_{9,4n}}} \right) = \sqrt{\frac{h_{9,n}}{l'_{9,4n}}} + \sqrt{\frac{l'_{9,4n}}{h_{9,n}}} + 2.$$

*Proof.* Letting  $q = e^{-\pi\sqrt{n/9}}$  in (1.3) and (1.6), we find that  $P = \sqrt{3}h_{9,n}$  and  $Q = \sqrt{3}l'_{9,4n}$  in Theorem 3.3. Rewriting (3.3) in terms of  $h_{9,n}$  and  $l'_{9,4n}$ , we complete the proof. □

**Theorem 3.5.** *If  $P = \frac{\psi(-q)}{q\psi(-q^9)}$  and  $Q = \frac{\psi(-q^3)}{q^3\psi(-q^{27})}$ , then*

$$(3.5) \quad \left(P + 3 + \frac{3}{P}\right) \left(Q + 3 + \frac{3}{Q}\right) = \left(\frac{Q}{P}\right)^2.$$

*Proof.* For simplicity, let  $A = q\psi(-q^9)$  and  $B = q^3\psi(-q^{27})$ . Then  $\psi(-q) = AP$  and  $\psi(-q^3) = BQ$ . Thus by Lemma 2.3,

$$P + 1 = \left(\frac{qB^4Q^4}{A^4} + 1\right)^{1/3} \quad \text{and} \quad Q + 1 = \left(\frac{A^4}{qB^4} + 1\right)^{1/3}.$$

Combining and rewriting above two equations in terms of  $P$  and  $Q$ , we deduce that

$$((P + 1)^3 - 1) ((Q + 1)^3 - 1) = Q^4,$$

which is equivalent to (3.3). Hence we complete the proof. □

Using the definition of  $l_{k,n}$ , we have the following:

**Corollary 3.6.** *For every positive real number  $n$ , we have*

$$(3.6) \quad 3 \left( l_{9,n} + \sqrt{3} + \frac{1}{l_{9,n}} \right) \left( l_{9,9n} + \sqrt{3} + \frac{1}{l_{9,9n}} \right) = \left( \frac{l_{9,9n}}{l_{9,n}} \right)^2.$$

*Proof.* Letting  $q = e^{-\pi\sqrt{n/9}}$  in (1.5), we find that  $P = \sqrt{3}l_{9,n}$  and  $Q = \sqrt{3}l_{9,9n}$  in Theorem 3.5. Rewriting (3.5) in terms of  $l_{9,n}$  and  $l_{9,9n}$ , we complete the proof. □

#### 4. Evaluations of $l_{k,n}$ and $l'_{k,n}$

In this section, we evaluate specific values of  $l_{k,n}$  and  $l'_{k,n}$  for some positive real numbers  $k$  and  $n$  by using the explicit relations and formulas established in Section 3.

The following results exhibit a general method for evaluating the values of  $l'_{3,4^n}$  for all positive integers  $n$ . We show the case when  $n = 1$ ,  $n = 2$ , and  $n = 3$ .

**Theorem 4.1.** *We have*

$$\begin{aligned} \text{(i)} \quad l'_{3,4} &= \frac{1 + \sqrt{3}}{\sqrt{2}}, \\ \text{(ii)} \quad l'_{3,16} &= 1 + \sqrt{3} + \sqrt{2 + \sqrt{3}}, \\ \text{(iii)} \quad l'_{3,64} &= \left( \frac{3a^8 - 8a^4 + 3 + \sqrt{3}(a^4 - 1)\sqrt{(3a^4 - 1)(a^4 - 3)}}{2} \right)^{1/4}, \end{aligned}$$

where

$$a = 1 + \sqrt{3} + \sqrt{2 + \sqrt{3}}.$$

*Proof.* For (i), letting  $n = 1$  in (3.2) and putting the value of  $l'_{3,1} = (2 + \sqrt{3})^{1/4}$  from Theorem 4.3(i) in [9], we find that

$$(2 - \sqrt{3})l'_{3,4}{}^8 - 4l'_{3,4}{}^4 + 2 + \sqrt{3} = 0.$$

Solving for  $l'_{3,4}$  and using the fact that  $l'_{3,4}$  has a real value greater than 1, we complete the proof.

For (ii), letting  $n = 4$  in (3.2) and putting the value of  $l'_{3,4}$  from the previous result of (i), we find that

$$l'_{3,16}{}^8 - 34(7 + 4\sqrt{3})l'_{3,16}{}^4 + 97 + 56\sqrt{3} = 0.$$

Solving for  $l'_{3,16}$  and using the fact that  $l'_{3,16}$  has a real value greater than 1, we complete the proof.

For (iii), letting  $n = 16$  in (3.2) and putting the value of  $l'_{3,16}$  from the previous result of (ii), we find that

$$l'_{3,64}{}^8 - (3a^8 - 8a^4 + 3)l'_{3,64}{}^4 + a^8 = 0,$$

where

$$a = 1 + \sqrt{3} + \sqrt{2 + \sqrt{3}}.$$

Solving for  $l'_{3,64}$  and using the fact that  $l'_{3,64}$  has a real value greater than 1, we complete the proof.  $\square$

See Theorem 4.10(ix) in [10] for an alternative proof for Theorem 4.1(i). By repeating the same argument as in the proof of Theorem 4.1, we can evaluate the values of  $l'_{3,4^n}$  for  $n = 4, 5, 6, \dots$

**Theorem 4.2.** *We have*



$$(i) \quad l'_{9,4} = 1 + \sqrt{3} + \sqrt{3 + 2\sqrt{3}},$$

$$(ii) \quad l'_{9,36} = \frac{\sqrt{3}a^2 - 2a + \sqrt{3} + 2\sqrt{\sqrt{3}a(a^2 - \sqrt{3}a + 1)}}{(\sqrt{3}a - 1)^2},$$

where

$$a = 2 - \sqrt{3} + (38 - 22\sqrt{3})^{1/3} + \frac{2(5 - 3\sqrt{3})}{(38 - 22\sqrt{3})^{1/3}},$$

*Proof.* For (i), letting  $n = 1$  in (3.4) and using  $h_{9,1} = 1$  from Lemma 2.7, we find that

$$l'^2_{9,4} - 2(1 + \sqrt{3})l'_{9,4} + 1 = 0.$$

Solving for  $l'_{9,4}$  and noting the fact that  $l'_{9,4} > 1$ , we complete the proof.

For (ii), let  $n = 9$  in (3.4), then we have

$$(\sqrt{3}h_{9,9} - 1)l'_{9,36} - 2\sqrt{h_{9,9}l'_{9,36}} - h_{9,9} + \sqrt{3} = 0,$$

where

$$h_{9,9} = 2 - \sqrt{3} + (38 - 22\sqrt{3})^{1/3} + \frac{2(5 - 3\sqrt{3})}{(38 - 22\sqrt{3})^{1/3}}$$

from Theorem 4.2(ii) in [9]. Now solving for  $l'_{9,36}$  and using the fact that  $l'_{9,36} > 1$ , we complete the proof. □

See Theorem 4.3(iii) in [9] for an alternative proof for Theorem 4.2(i).

**Corollary 4.3.** *Let  $a$  be as in Theorem 4.2(ii). Then we have*

$$l_{9,36} = \frac{1}{2} \left( b - \sqrt{3} + \sqrt{\frac{\sqrt{3}(b^2 + 1)(b - \sqrt{3})}{\sqrt{3}b - 1}} \right),$$

where

$$b = \frac{\sqrt{3}a^2 - 2a + \sqrt{3} + 2\sqrt{\sqrt{3}a(a^2 - \sqrt{3}a + 1)}}{(\sqrt{3}a - 1)^2}.$$

*Proof.* Let  $n = 36$  in Lemma 2.8 and  $b = l'_{9,36}$ . Then we find that

$$(\sqrt{3}b - 1)l^2_{9,36} - (\sqrt{3}b^2 - 4b + \sqrt{3})l_{9,36} - b^2 + \sqrt{3}b = 0.$$

Putting the value of  $b$  from Theorem 4.2(ii), solving for  $l_{9,36}$ , and using the fact that  $l_{9,36} > 1$ , we complete the proof. □

The following results exhibit a general method for evaluating the values of  $l_{9,9^n}$  for all positive integers  $n$ . We show the case when  $n = 1$  and  $n = 2$ .

**Theorem 4.4.** *We have*

$$(i) \quad l_{9,9} = 2 + \sqrt{3} + (38 + 22\sqrt{3})^{1/3} + \frac{2(5 + 3\sqrt{3})}{(38 + 22\sqrt{3})^{1/3}},$$

$$(ii) \quad l_{9,81} = ((a^2 + 1)(a + \sqrt{3}))^{2/3} b^{1/3} + ((a^2 + 1)(a + \sqrt{3}))^{1/3} b^{2/3} + b,$$

where

$$a = 2 + \sqrt{3} + (38 + 22\sqrt{3})^{1/3} + \frac{2(5 + 3\sqrt{3})}{(38 + 22\sqrt{3})^{1/3}},$$

$$b = a(a^2 + \sqrt{3}a + 1).$$

*Proof.* For (i), letting  $n = 1$  in (3.6) and using  $l_{9,1} = 1$  from Lemma 2.7, we find that

$$l_{9,9}^3 - 3(2 + \sqrt{3})l_{9,9}^2 - 3(3 + 2\sqrt{3})l_{9,9} - 3(2 + \sqrt{3}) = 0.$$

Solving for  $l_{9,9}$  and noting the fact that  $l_{9,9}$  is real valued, we complete the proof.

For (ii), let  $n = 9$  in (3.6), then we have

$$3 \left( l_{9,9} + \sqrt{3} + \frac{1}{l_{9,9}} \right) \left( l_{9,81} + \sqrt{3} + \frac{1}{l_{9,81}} \right) = \left( \frac{l_{9,81}}{l_{9,9}} \right)^2.$$

Putting the value of  $l_{9,9}$  from the previous result of (i), solving for  $l_{9,81}$ , and using the fact that  $l_{9,81}$  is real valued, we complete the proof.  $\square$

See Theorem 4.4(ii) in [9] for an alternative proof for Theorem 4.4(i). By repeating the same argument as in the proof of Theorem 4.4, we can evaluate the values of  $l_{9,9^n}$  for  $n = 3, 4, 5, \dots$

**Corollary 4.5.** *Let  $a$  and  $b$  be as in Theorem 4.4(ii). Then we have*

$$l'_{9,81} = \frac{1}{2} \left( c + \sqrt{3} + \sqrt{\frac{\sqrt{3}(c^2 + 1)(c + \sqrt{3})}{\sqrt{3}c + 1}} \right),$$

where

$$c = \left( (a^2 + 1)(a + \sqrt{3}) \right)^{2/3} b^{1/3} + \left( (a^2 + 1)(a + \sqrt{3}) \right)^{1/3} b^{2/3} + b.$$

*Proof.* The result follows directly from Lemma 2.8 and Theorem 4.4(ii).  $\square$

### 5. Evaluations of $G(q)$

We now turn to an application of some numerical values of  $l_{k,n}$  and  $l'_{k,n}$  for some positive real numbers  $k$  and  $n$  to evaluations of the cubic continued fraction. In [7], the values of  $G(e^{-\pi/\sqrt{3}})$ ,  $G(-e^{-\pi/\sqrt{3}})$ , and  $G(e^{-2\pi/\sqrt{3}})$  were evaluated. In particular,  $G(e^{-2\pi/\sqrt{3}})$  was evaluated by multiplying both  $-G(e^{-\pi/\sqrt{3}})$  and  $G(-e^{-\pi/\sqrt{3}})$  as in Lemma 2.9. In this section, we evaluate  $G(e^{-2\pi/\sqrt{3}})$  by putting the value of  $l'_{3,4}$  in Lemma 2.9(ii). Moreover, we evaluate the numerical values of  $G(e^{-4\pi/\sqrt{3}})$ ,  $G(e^{-8\pi/\sqrt{3}})$ ,  $G(-e^{-2\pi/\sqrt{3}})$ , and  $G(-e^{-4\pi/\sqrt{3}})$ .

**Theorem 5.1.** *We have*

$$(i) \quad G(e^{-2\pi/\sqrt{3}}) = \frac{1}{2}(-5 + 3\sqrt{3})^{1/3},$$

$$(ii) \quad G(e^{-4\pi/\sqrt{3}}) = \frac{1}{2} \left( 7 + 6\sqrt{3} - 9\sqrt{2 + \sqrt{3}} \right)^{1/3},$$

$$(iii) \quad G(e^{-8\pi/\sqrt{3}}) = \frac{1}{2^{4/3}} \left( 3a^4 - 7 - 3\sqrt{3}(a^4 - 1)\sqrt{\frac{a^4 - 3}{3a^4 - 1}} \right)^{1/3},$$

where

$$a = 1 + \sqrt{3} + \sqrt{2 + \sqrt{3}}.$$

*Proof.* For (i), letting  $n = 4$  in Lemma 2.9(ii) and putting the value of  $l'_{3,4}$  from Theorem 4.1(i), we complete the proof.

For (ii), letting  $n = 16$  in Lemma 2.9(ii) and putting the value of  $l'_{3,16}$  from Theorem 4.1(ii), we complete the proof.

For (iii), letting  $n = 64$  in Lemma 2.9(ii) and putting the value of  $l'_{3,64}$  from Theorem 4.1(iii), we complete the proof. □

**Corollary 5.2.** *We have*

$$(i) \quad G(-e^{-2\pi/\sqrt{3}}) = - \left( \frac{1}{2}(5 + 3\sqrt{3}) \left( 7 + 6\sqrt{3} - 9\sqrt{2 + \sqrt{3}} \right) \right)^{1/3},$$

$$(ii) \quad G(-e^{-4\pi/\sqrt{3}}) = - \left( \frac{(3a^4 - 7)\sqrt{3a^4 - 1} - 3\sqrt{3}(a^4 - 1)\sqrt{a^4 - 3}}{2(7 + 6\sqrt{3} - 9\sqrt{2 + \sqrt{3}})\sqrt{3a^4 - 1}} \right)^{1/3},$$

where

$$a = 1 + \sqrt{3} + \sqrt{2 + \sqrt{3}}.$$

*Proof.* Parts (i) and (ii) follow directly from Lemma 2.10 and Theorem 5.1. □

Note that the numerical values of  $G(e^{-\pi})$ ,  $G(e^{-2\pi})$ , and  $G(-e^{-\pi})$  were evaluated in [4]. Note also that the numerical values of  $G(e^{-\pi})$ ,  $G(e^{-2\pi})$ ,  $G(e^{-3\pi})$ ,  $G(-e^{-2\pi})$ , and  $G(-e^{-3\pi})$  were evaluated in [9]. Hence the numerical values of  $G(e^{-\pi})$  and  $G(e^{-2\pi})$  were given in both [4] and [9], but they were evaluated by different proofs. We close this section by evaluating the numerical values of  $G(e^{-6\pi})$ ,  $G(e^{-9\pi})$ ,  $G(e^{-12\pi})$ ,  $G(e^{-18\pi})$ ,  $G(-e^{-6\pi})$ , and  $G(-e^{-9\pi})$ .

**Theorem 5.3.** *Let  $a$  and  $b$  be as in Corollary 4.3. Then we have*

$$(i) \quad G(e^{-6\pi}) = \frac{(\sqrt{3}a - 1)^2}{2 \left( 1 + 3^{3/4}\sqrt{a(a^2 - \sqrt{3}a + 1)} \right)},$$

$$(ii) \quad G(-e^{-6\pi}) = -\frac{1}{4} \left( (\sqrt{3}b - 1)^2 - 3^{3/4}\sqrt{(b^2 + 1)(b - \sqrt{3})(\sqrt{3}b - 1)} \right).$$

*Proof.* Part (i) follows directly from Lemma 2.9(i) and Theorem 4.2(ii). Part (ii) follows directly from Lemma 2.9(iii) and Corollary 4.3. □

**Corollary 5.4.** *Let  $a$  and  $b$  be as in Corollary 4.3. Then we have*

$$G(e^{-12\pi}) = \frac{(\sqrt{3}a - 1)^2 \left( (\sqrt{3}b - 1)^2 - 3^{3/4} \sqrt{(b^2 + 1)(b - \sqrt{3})(\sqrt{3}b - 1)} \right)}{8 \left( 1 + 3^{3/4} \sqrt{a(a^2 - \sqrt{3}a + 1)} \right)}.$$

*Proof.* The result follows directly from Lemma 2.10 and Theorem 5.3.  $\square$

**Theorem 5.5.** *Let  $a$  and  $b$  be as in Theorem 4.4(ii). Then we have*

$$G(-e^{-9\pi}) = -\frac{1}{1 + \sqrt{3}c},$$

where

$$c = \left( (a^2 + 1)(a + \sqrt{3}) \right)^{2/3} b^{1/3} + \left( (a^2 + 1)(a + \sqrt{3}) \right)^{1/3} b^{2/3} + b.$$

*Proof.* Letting  $n = 81$  in Lemma 2.9(iii) and putting the value of  $l_{9,81}$  from Theorem 4.4(ii), we complete the proof.  $\square$

**Theorem 5.6.** *Let  $c$  be as in Corollary 4.5. Then we have*

$$G(e^{-9\pi}) = \frac{1}{4} \left( 3^{3/4} \sqrt{(c^2 + 1)(c + \sqrt{3})(\sqrt{3}c + 1)} - (\sqrt{3}c + 1)^2 \right).$$

*Proof.* Letting  $n = 81$  in Lemma 2.9(i) and putting the value of  $l'_{9,81}$  from Corollary 4.5, we complete the proof.  $\square$

**Corollary 5.7.** *Let  $c$  be as in Corollary 4.5. Then we have*

$$G(e^{-18\pi}) = \frac{3^{3/4} \sqrt{(c^2 + 1)(c + \sqrt{3})(\sqrt{3}c + 1)} - (\sqrt{3}c + 1)^2}{4(\sqrt{3}c + 1)}.$$

*Proof.* The result is an immediate consequence of Lemma 2.10 and Theorems 5.5 and 5.6.  $\square$

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