

A NOTE ON THE HYPER-ORDER OF ENTIRE FUNCTIONS

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ABSTRACT. In the paper, we have two purposes. Firstly, we estimate the hyper-order of an entire function which shares two functions with its first derivative, and two examples are given to show the conclusion is sharp. Secondly, we generalize the Brück conjecture with the idea of sharing functions.

1. Introduction and main results

In the Nevanlinna theory, the order and the hyper-order of a meromorphic function are two important concepts. So, it is meaningful to discuss the properties of the order and the hyper-order for a meromorphic function. Let us recall the definitions of the order and hyper-order of a meromorphic function f , which are respectively defined as (see [12])

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r},$$
$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \log \log M(r, f)}{\log r}.$$

We also define the hyper-lower-order as follows.

$$\lambda(f) = \liminf_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r} = \liminf_{r \rightarrow \infty} \frac{\log \log \log M(r, f)}{\log r}.$$

In 1982, Bank and Laine [1] investigated the solutions of a differential equation and obtained the following result.

Theorem A. *Let $A(z)$ be a nonconstant polynomial of degree n , and let $f \neq 0$ be a solution of the equation $f'' + A(z)f = 0$. Then the order of growth of f is $\frac{n+2}{2}$.*

Since then, to study properties of the order and hyper-order for the solutions of some differential equations becomes a hot topic and is discussed by many experts.

In 2008, Li and Gao [8] deduced a result as follows.

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Theorem B. Let Q_1 and Q_2 be two nonzero polynomials, and let P be a polynomial. If f is a nonconstant solution of the equation

$$f^{(k)} - Q_1 = e^P(f - Q_2),$$

then $\sigma(f) = n$, where, and in the sequel, n denotes the degree of P .

It is well-known that the problem of the differential equations is closely related with the problem of meromorphic functions sharing values. So, it is an interesting attempt to derive the properties of order and hyper-order for meromorphic functions by considering the situation of sharing values.

In 2009, Lü and Yi [11] obtained a result.

Theorem C. Let f be a nonconstant meromorphic function with finitely many poles, and let $Q_1, Q_2 (\neq Q_1)$ be two polynomials. If

$$f(z) = Q_1 \Rightarrow f'(z) = Q_1 \text{ and } f(z) = Q_2 \Rightarrow f'(z) = Q_2,$$

then f is of finite order.

In 2009, Chen, Lü and Yi [4] improved the above theorem and deduced the following result.

Theorem D. Let f be a nonconstant meromorphic function with finitely many poles, and let $R_3(z) = c_1 e^z$ and $R_4(z) = c_2 e^z$, where c_1, c_2 are two nonzero constants. If

$$f(z) = R_3 \Rightarrow f'(z) = R_3 \text{ and } f(z) = R_4 \Rightarrow f'(z) = R_4,$$

then f is of finite order.

From Theorem C and D, we see that f and f' share two functions with finite order. So, it is natural to ask what will happen if they share functions with infinite order. In the work, we discuss the problem and derive the following result.

Theorem 1.1. Let $Q_1 \neq 0$ and Q_2 be two distinct polynomials, let f, γ be two entire functions. If

$$f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z) \text{ and } f(z) = \beta(z) \Rightarrow f'(z) = \beta(z),$$

where $\alpha = Q_1 e^\gamma, \beta = Q_2 e^\gamma$, and if $\alpha - \alpha'$ or $\beta - \alpha'$ has at most finitely many zeros, then $\sigma(f) \leq \sigma(\alpha) = \rho(\gamma)$.

Remark 1. The following examples show that the conclusion $\sigma(f) \leq \rho(\gamma)$ is sharp.

Example 1. Let $f(z) = Ae^z$, where A is a nonzero constant. Let $\alpha(z) = e^{e^{-z}+z}$, and let $\beta(z) = P(z)e^{e^{-z}+z}$, where $P(z)$ is a polynomial. Noting that $f = f'$, we have

$$f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z) \text{ and } f(z) = \beta(z) \Rightarrow f'(z) = \beta(z).$$

Obviously, $\alpha(z) - \alpha'(z) = e^{e^{-z}}$ has no zeros. Thus, it satisfies the assumptions of Theorem 1.1 and $\sigma(f) = 0 < \sigma(\alpha) = 1$.

Example 2. Let $f(z) = ze^{z^2} + e^{z/2}$, $\alpha(z) = (4z^2 - z + 2)e^{z^2}$, and $\beta(z) = ze^{z^2}$. Then

$$\frac{f' - \alpha}{f - \alpha} = \frac{1}{2} \text{ and } f(z) - \beta(z) \neq 0.$$

It satisfies the assumptions of Theorem 1.1 and $\sigma(f) = 0$.

Remark 2. The condition that $\alpha - \alpha'$ or $\beta - \alpha'$ has at most finitely many zeros is essential in our proof of Theorem 1.1. But we don't know whether it is necessary or not. If γ is a polynomial, then the above condition obviously holds.

Remark 3. In Theorem 1.1, if the order of γ is zero, for example γ is a polynomial, then $\sigma(f) = 0$, which is an important property for a meromorphic function f .

Remark 4. If f is a meromorphic function with finitely many poles and Q_1, Q_2 are two rational functions, the conclusion is still valid in Theorem 1.1.

2. Some results of the Brück conjecture

In 1996, Brück [3] posed the following conjecture which reveals the relationship between f and f' if an entire function f shares one value CM with its first derivative f' ?

Conjecture. Let f be a non-constant entire function such that the hyper-order $\sigma(f)$ of f is not a positive integer and $\sigma(f) < \infty$. If f and f' share a finite value a CM, then

$$\frac{f' - a}{f - a} = c,$$

where c is non-zero constant.

In [5], Chen generalized the Brück conjecture with the idea of sharing z and got the following result.

Theorem E. Let f be a nonconstant entire function such that the hyper order $\sigma(f) < \frac{1}{2}$, k be a positive integer. If f and $f^{(k)}$ share z CM, then

$$f^{(k)}(z) - z = c(f(z) - z),$$

where c is a nonzero constant.

In the following, due to the methods of Chen and Zhang [5], Zhang and Liao [13], we obtain a result which is an improvement of the previous theorems in some sense.

Theorem 2.1. Let f, α be two meromorphic functions with at most finitely many poles, and let $\sigma(f) < \frac{1}{2}$ and $\sigma(\alpha) < \lambda(f)$. If f and α have no common poles, f and $f^{(k)}$ share α CM, then $\frac{f^{(k)} - \alpha}{f - \alpha} = c$, where c is a nonzero constant.

Remark 5. In the proof of Theorem 2.1, we need the condition $\sigma(P) < \lambda(f)$ to confirm that $r \notin E$

$$\frac{M(r, P)}{M(r, f)} \rightarrow 0 \text{ as } r \rightarrow \infty,$$

where $E \subset [0, +\infty)$ is a set with finite logarithmic measure. If it is deleted, we do not know whether the above conclusion holds or not.

Remark 6. If P is a polynomial, then the condition $\sigma(P) < \lambda(f)$ can be deleted in Theorem 2.1. The result is listed as below, which can be proved with the same way in Theorem 2.1.

Corollary 2.2. *Let f be a meromorphic function with at most finitely many poles and hyper-order $\sigma(f) < \frac{1}{2}$, let P be a non-zero polynomial. If f and $f^{(k)}$ share P CM, then $\frac{f^{(k)} - P}{f - P} = c$, where c is a nonzero constant.*

3. Some lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 3.1 ([10, 9]). *Let $\{f_n\}$ be a family of functions meromorphic (analytic) on the unit disc Δ . If $a_n \rightarrow a$, $|a| < 1$, and $f_n^\sharp(a_n) \rightarrow \infty$, then there exist*

- (a) *a subsequence of f_n (which we still write as $\{f_n\}$);*
- (b) *points $z_n \rightarrow z_0$ $|z_0| < 1$;*
- (c) *positive numbers $\rho_n \rightarrow 0$*

such that $f_n(z_n + \rho_n \xi) = g_n(\xi) \rightarrow g(\xi)$ locally uniformly, where g is a nonconstant meromorphic (entire) function on \mathbb{C} , such that

$$\rho_n \leq \frac{M}{f_n^\sharp(a_n)},$$

where M is a constant which is independent of n .

With a similar method in [9, Lemma 2], we obtain the following result, which plays an important part in the proof of Theorem 1.1.

Lemma 3.2. *Let f be a meromorphic function of hyper-order $\sigma(f) > 0$. Then, for any $\epsilon > 0$, there exists a sequence $z_n \rightarrow \infty$ such that $f^\sharp(z_n) > e^{|z_n|^{\sigma(f)-\epsilon}}$ if n is large enough.*

Proof. On the contrary, there exist $\epsilon > 0$, $N > 0$ and $R > 0$ such that for all z , $|z| \geq R$ satisfying $f^\sharp(z) \leq e^{|z|^{\sigma(f)-\epsilon}}$. Thus,

$$\begin{aligned} S(r, f) &= \frac{1}{\pi} \iint_{|z| < r} f^\sharp(z)^2 d\sigma = \frac{1}{\pi} \iint_{R \leq |z| < r} f^\sharp(z)^2 d\sigma + O(1) \\ &\leq \frac{1}{\pi} \iint_{R \leq |z| < r} e^{2|z|^{\sigma(f)-\epsilon}} d\sigma + O(1) = \frac{1}{\pi} \int_0^{2\pi} d\theta \int_R^r e^{2|z|^{\sigma(f)-\epsilon}} dt + O(1) \\ &\leq 2re^{2r^{\sigma(f)-\epsilon}} [1 + o(1)]. \end{aligned}$$

By the definition of Ahlfors characteristic of f , we have

$$T(r, f) = \int_0^r \frac{S(t, f)}{t} dt \leq 2re^{2r^{\sigma(f)-\epsilon}} [1 + o(1)].$$

Then, the hyper-order of f is

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r} \leq \limsup_{r \rightarrow \infty} \frac{(\sigma(f) - \epsilon) \log r}{\log r} = \sigma(f) - \epsilon,$$

a contradiction. Thus, the proof is completed. \square

Lemma 3.3 ([2]). *Let $h(z)$ be an entire function with order $\rho(h) = \rho < \frac{1}{2}$, set*

$$A(r) = \inf_{|z|=r} \log |h(z)|, \quad B(r) = \sup_{|z|=r} \log |h(z)|.$$

If $\rho < \alpha < 1$, then

$$\underline{\log dens}\{r : A(r) > (\cos \pi \alpha)B(r)\} \geq 1 - \frac{\rho}{\alpha}$$

where the lower logarithmic density $\underline{\log dens}H$ of subset $H \subset (1, \infty)$ is defined by

$$\underline{\log dens}H = \liminf \left(\int_1^r (\mathcal{X}_H(t)/t) dt \right) / \log r,$$

where $\mathcal{X}_H(t)$ is the characteristic function of set H .

Remark 7. From the definitions of logarithmic measure and logarithmic density, it is easy to see that the lower logarithmic density $\underline{\log dens}H > 0$, then the logarithmic measure $lmH = +\infty$.

Lemma 3.4 ([7, p. 51 Theorem 3.2]). *Let f be a transcendental entire function and $0 < \delta < \frac{1}{4}$. Suppose that at the point z with $|z| = r$ the inequality*

$$|f(z)| > M(r, f)\nu(r, f)^{-\frac{1}{4}+\delta}$$

holds, then there exists a set $F \in R^+$ of finite logarithmic measure, i.e., $\int_F \frac{1}{t} dt < +\infty$, such that

$$f^{(m)}(z) = \left(\frac{\nu(r, f)}{z}\right)^m (1 + o(1))f(z)$$

holds for all $r \notin F$.

4. The Proof of Theorem 1.1

With the similar way of [6, 10], we prove Theorem 1.1 as follows.

Noting that $\alpha = Q_1 e^\gamma$, thus $\sigma(\alpha) = \rho(\gamma)$. So, we just need to obtain $\sigma(f) \leq \rho(\gamma)$.

On the contrary, assume that $\sigma(f) = d > c = \rho(\gamma)$. Set $H = f - \alpha$. Then

(I) $H(z) = 0 \Rightarrow H'(z) = \alpha(z) - \alpha'(z)$,

(II) $H(z) = \beta(z) - \alpha(z) \Rightarrow H'(z) = \beta(z) - \alpha'(z)$.

Let $P = \beta - \alpha$ and $F = \frac{H}{P}$. Obviously, $\sigma(F) = \sigma(f) = d$. By Lemma 3.2, then for $0 < \epsilon < \frac{d-c}{2}$, there exists a sequence $w_n \rightarrow \infty$ as $n \rightarrow \infty$ such that

$$F^\sharp(w_n) > e^{|w_n|^{\sigma(F)-\epsilon}} = e^{|w_n|^{d-\epsilon}}.$$

Noting that $P = \beta - \alpha$ has at most finitely many zeros, then there exists a positive number r such that F has no poles in $D = \{z : |z| > r\}$.

In view of $w_n \rightarrow \infty$ as $n \rightarrow \infty$, without loss of generality, we may assume $|w_n| \geq r + 1$ for all n . Define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z).$$

Then every F_n are analytic in D_1 and $F_n^\sharp(0) \rightarrow \infty$ as $n \rightarrow \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Therefore, we can apply Lemma 3.1. Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$ with $|z_n| < r < 1$ and $\rho_n \rightarrow 0$ such that the sequence $(g_n)_n$ defined by

$$(4.1) \quad g_n(\zeta) = F_n(z_n + \rho_n\zeta) = \frac{H(w_n + z_n + \rho_n\zeta)}{P(w_n + z_n + \rho_n\zeta)} \rightarrow g(\zeta)$$

locally uniformly in \mathbb{C} with a nonconstant entire function g and

$$(4.2) \quad \rho_n \leq \frac{M}{F_n^\sharp(0)} = \frac{M}{F^\sharp(w_n)} \leq M e^{-|w_n|^{d-\epsilon}}$$

for a positive number M .

From (4.1), we have

$$(4.3) \quad \begin{aligned} g_n'(\zeta) &= \rho_n \frac{H'(w_n + z_n + \rho_n\zeta)}{P(w_n + z_n + \rho_n\zeta)} - \rho_n \frac{H(w_n + z_n + \rho_n\zeta)P'(w_n + z_n + \rho_n\zeta)}{P^2(w_n + z_n + \rho_n\zeta)} \\ &\rightarrow g'(\zeta). \end{aligned}$$

Noting that $P = \alpha - \beta = Qe^\gamma$, where $Q = Q_1 - Q_2$ is a nonzero polynomial, we have $\frac{P'}{P} = \frac{Q' + Q\gamma'}{Q}$ and $\rho(\gamma') = \rho(\gamma) = c$. In view of the definition of order, we have

$$(4.4) \quad \begin{aligned} \left| \frac{P'}{P} \Big|_{z=w_n+z_n+\rho_n\zeta} \right| &= \left| \frac{Q' + Q\gamma'}{Q} \Big|_{z=w_n+z_n+\rho_n\zeta} \right| \\ &\leq |w_n|^q M(|w_n + z_n + \rho_n\zeta|, \gamma') \\ &\leq |w_n|^q M(2|w_n|, \gamma') \leq |w_n|^q e^{A|w_n|^{c+\epsilon}}, \end{aligned}$$

where A is a positive constant and q is an integer. Noting that $0 < \epsilon < \frac{d-c}{2}$, we have $d - \epsilon > c + \epsilon$. Then, combining (4.2) and (4.4) yields

$$(4.5) \quad \begin{aligned} \left| \rho_n \frac{H(w_n + z_n + \rho_n\zeta)P'(w_n + z_n + \rho_n\zeta)}{P^2(w_n + z_n + \rho_n\zeta)} \right| &= \left| \rho_n g_n(\zeta) \frac{P'(w_n + z_n + \rho_n\zeta)}{P(w_n + z_n + \rho_n\zeta)} \right| \\ &\leq M |g_n(\zeta)| |w_n|^q e^{A|w_n|^{c+\epsilon} - |w_n|^{d-\epsilon}} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

From (4.3) and (4.5), we deduce that

$$(4.6) \quad \rho_n \frac{H'(w_n + z_n + \rho_n \zeta)}{P(w_n + z_n + \rho_n \zeta)} \rightarrow g'(\zeta).$$

We claim that

- (1) $g(\zeta) = 0 \Rightarrow g'(\zeta) = 0$ and
- (2) $g(\zeta) = 1 \Rightarrow g'(\zeta) = 0$.

Suppose that $g(\zeta_0) = 0$, then by Hurwitz's theorem there exist $\zeta_n, \zeta_n \rightarrow \zeta_0$, such that (for n sufficiently large)

$$g_n(\zeta_n) = \frac{H(w_n + z_n + \rho_n \zeta_n)}{P(w_n + z_n + \rho_n \zeta_n)} = 0.$$

By the assumption of (I) we have

$$(4.7) \quad H'(w_n + z_n + \rho_n \zeta_n) = (\alpha - \alpha')(w_n + z_n + \rho_n \zeta_n).$$

Observing that $\alpha - \alpha' = (Q_1 - Q'_1 - Q_1 \gamma')e^\gamma$, similarly as (4.5), we have

$$\begin{aligned} & \rho_n \left| \frac{(\alpha - \alpha')(w_n + z_n + \rho_n \zeta_n)}{P(w_n + z_n + \rho_n \zeta_n)} \right| \\ &= \rho_n \left| \frac{[Q_1 - Q'_1 - Q_1 \gamma'](w_n + z_n + \rho_n \zeta_n)}{[Q_1 - Q_2](w_n + z_n + \rho_n \zeta_n)} \right| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Then,

$$g'(\zeta_0) = \lim_{n \rightarrow \infty} \rho_n \frac{H'(w_n + z_n + \rho_n \zeta_n)}{P(w_n + z_n + \rho_n \zeta_n)} = \lim_{n \rightarrow \infty} \rho_n \frac{(\alpha - \alpha')(w_n + z_n + \rho_n \zeta_n)}{P(w_n + z_n + \rho_n \zeta_n)} = 0.$$

Thus $g(\zeta) = 0 \Rightarrow g'(\zeta) = 0$, which is (1). Similarly, we can prove (2). Thus, we finish the proof of the claims.

Noting that either $\alpha - \alpha'$ or $\beta - \alpha'$ has at most finitely many zeros, without loss of generality, we assume that $\alpha - \alpha'$ has at most finitely many zeros. Next, we shall prove that $g(\zeta) \neq 0$.

Suppose ξ_0 is a zero of $g(\zeta)$ with multiplicity $m(\geq 2)$. Then $g^{(m)}(\xi_0) \neq 0$. Thus there exists a positive number δ , such that

$$(4.8) \quad g(\zeta) \neq 0, \quad g'(\zeta) \neq 0, \quad g^{(m)}(\zeta) \neq 0$$

on $D_\delta^g = \{z : 0 < |\zeta - \xi_0| < \delta\}$.

Noting that $g \neq 0$, by Rouché theorem there exist $\zeta_{n,j}$ ($j = 1, 2, \dots, m$) on $D_{\delta/2} = \{\xi : |\zeta - \xi_0| < \delta/2\}$ such that

$$g_n(\zeta_{n,j}) = H(w_n + z_n + \rho_n \zeta_{n,j}) = 0 \quad (j = 1, \dots, m).$$

Observing that $\alpha - \alpha'$ has at most finitely many zeros, then (for n large enough) we have

$$\begin{aligned} & H'(w_n + z_n + \rho_n \zeta_{n,j}) \\ &= [\alpha - \alpha'](w_n + z_n + \rho_n \zeta_{n,j}) \neq 0, \end{aligned}$$

which implies that each $\zeta_{n,j}$ is a simple zero of $H(w_n + z_n + \rho_n \zeta)$, that is $\zeta_{n,j} \neq \zeta_{n,i}$ ($1 \leq i \neq j \leq m$).

As in (4.5), we deduce

$$(4.9) \quad \rho_n \frac{[\alpha - \alpha'](w_n + z_n + \rho_n \zeta)}{P(w_n + z_n + \rho_n \zeta)} \rightarrow 0.$$

It follows from (4.6) and (4.9) that

$$(4.10) \quad K_n(\zeta) = \rho_n \frac{H'(w_n + z_n + \rho_n \zeta)}{P(w_n + z_n + \rho_n \zeta)} - \rho_n \frac{[\alpha - \alpha'](w_n + z_n + \rho_n \zeta)}{P(w_n + z_n + \rho_n \zeta)} \rightarrow g'(\zeta)$$

and $K_n(\zeta_{n,j}) = 0$ ($j = 1, \dots, m$). By (4.8) we obtain

$$\lim_{n \rightarrow \infty} \zeta_{n,j} = \xi_0 \quad (j = 1, 2, \dots, m).$$

Noting that (4.8), (4.10) and $K_n(\zeta)$ has m zeros $\zeta_{n,j}$ ($j = 1, 2, \dots, m$) in $D_{\delta/2}$, we obtain from the Hurwitz's theorem that ξ_0 is a zero of g' with multiplicity m , and thus $g^{(m)}(\xi_0) = 0$. This is a contradiction. Hence $g(\zeta) \neq 0$.

We have shown that g is a non-vanishing, entire function that takes the value 1 always with multiplicity at least 2. But this contradicts Nevanlinna's second fundamental theorem that the sum of the defects is at most 2.

Thus, we complete the proof of Theorem 1.1.

5. Proof of Theorem 2.1

From the assumptions of Theorem 2.1, we set

$$(5.1) \quad \frac{f^{(k)} - \alpha}{f - \alpha} = \frac{1}{Q} e^\phi,$$

where ϕ is an entire function and Q is a polynomial. Noting that f and α have only finitely many poles, then there exists a polynomial P such that fP and αP are two entire functions. Set

$$F = fP \text{ and } A = \alpha P.$$

Obviously, $\sigma(F) = \sigma(f) < \frac{1}{2}$ and $\sigma(A) = \sigma(\alpha) < \lambda(F) = \lambda(f)$. We also have

$$(5.2) \quad f^{(k)} = \left(\frac{F}{P}\right)^{(k)} = \sum_{j=0}^k C_{k,j} F^{(k-j)} \left(\frac{1}{P}\right)^{(j)},$$

where $C_{k,j} = \frac{k!}{j!(k-j)!}$ is a constant.

In the following, we will prove ϕ is a constant.

Rewriting (5.1) as

$$(5.3) \quad \begin{aligned} e^\phi &= Q \frac{f^{(k)} - \alpha}{f - \alpha} = Q \frac{\sum_{j=0}^k C_{k,j} F^{(k-j)} \left(\frac{1}{P}\right)^{(j)} - \frac{A}{P}}{\frac{F}{P} - \frac{A}{P}} \\ &= Q \frac{\sum_{j=0}^k C_{k,j} \frac{F^{(k-j)}}{F} \left(\frac{1}{P}\right)^{(j)} P - \frac{A}{F}}{1 - \frac{A}{F}}. \end{aligned}$$

From (5.3), we have

$$(5.4) \quad \phi = \log Q \frac{\sum_{j=0}^k C_{k,j} \frac{F^{(k-j)}}{F} (\frac{1}{P})^{(j)} \frac{1}{P} - \frac{A}{F}}{1 - \frac{A}{F}},$$

where $\log g$ is a principle branch of $\text{Log } g$. By Lemma 3.4, we get

$$(5.5) \quad \frac{F^{(k-j)}(z_r)}{F(z_r)} = (\frac{\nu(r, F)}{z_r})^{k-j} (1 + o(1)) \quad (j = 0, \dots, k)$$

possibly outside a set of finite logarithmic measure E , where $|F(z_r)| = M(r, F)$. Observing that $\sigma(A) < \lambda(F)$, for $0 < \epsilon < \frac{\lambda(F) - \sigma(A)}{2}$, we have

$$\sigma(A) = \limsup_{r \rightarrow \infty} \frac{\log \log \log M(r, A)}{\log r} \leq \sigma(A) + \epsilon,$$

which implies that (for r large enough)

$$(5.6) \quad M(r, A) \leq e^{e^{r^{\sigma(A)+\epsilon}}}.$$

Similarly, we deduce that (for r large enough)

$$(5.7) \quad M(r, F) \geq e^{e^{r^{\lambda(F)-\epsilon}}}.$$

Observing that $\epsilon < \frac{\lambda(F) - \sigma(A)}{2}$, we have $\sigma(A) + \epsilon < \lambda(F) - \epsilon$. Then, combining (5.6) and (5.7) yields that

$$(5.8) \quad \frac{M(r, A)}{M(r, F)} \leq \frac{e^{e^{r^{\sigma(A)+\epsilon}}}}{e^{e^{r^{\lambda(F)-\epsilon}}}} \leq \frac{1}{e^{[e^{r^{\lambda(F)-\epsilon}} - e^{r^{\sigma(A)+\epsilon}}]}} \rightarrow 0 \text{ as } r \rightarrow \infty.$$

It indicates that

$$(5.9) \quad \frac{M(r, A)}{M(r, F)} \rightarrow 0 \text{ as } r \rightarrow \infty.$$

So, substituting (5.5) (5.9) into (5.4), we deduce

$$(5.10) \quad \begin{aligned} |\phi(z_r)| &= \left| \log Q \frac{\sum_{j=0}^k C_{k,j} \frac{F^{(k-j)}}{F} (\frac{1}{P})^{(j)} P - \frac{A}{F}}{1 - \frac{A}{F}} \right| \\ &\leq \left| \log \left| Q \frac{\sum_{j=0}^k C_{k,j} \frac{F^{(k-j)}}{F} (\frac{1}{P})^{(j)} P - \frac{A}{F}}{1 - \frac{A}{F}} \right| \right| + 2\pi \\ &\leq A[1 + o(1)] \log \nu(r, F), \end{aligned}$$

for $|z_r| = r \notin E$, where A is a positive number. Now, we consider into two cases.

Case 1. ϕ is a polynomial with degree $n \geq 1$.

Noting that the definition of hyper-order of f , for $\epsilon < \frac{1}{2}$, we have

$$(5.11) \quad \log \nu(r, F) \leq r^{(\frac{1}{2}+\epsilon)}.$$

Then, from (5.10) we obtain that

$$(5.12) \quad B(1 + o(1))|r|^n = |\phi(z_r)| \leq A[1 + o(1)]r^{\frac{1}{2} + \epsilon},$$

where B is a positive constant. This is a contradiction.

Case 2. ϕ is a transcendental entire function.

Since $\frac{\log \log \nu(r, F)}{\log r} < \frac{1}{2}$ for sufficiently large r , from (5.10),

$$(5.13) \quad |\phi(z_r)| \leq A[1 + o(1)]r^{\frac{1}{2}}.$$

Obviously, $\rho(\phi) \leq \sigma(f) < \frac{1}{2}$. Then, by Lemma 3.3, we know that there exists a set $H \subset (1, \infty)$ that have logarithmic measure $lmH = \infty$, such that for all $|z| = r \in H$, we have

$$(5.14) \quad |\phi(z)| \geq M(r, \phi)^c,$$

where $0 < c < 1$. Now, for all z_r satisfying $|z_r| = r \in H \setminus (E \cup [0, 1])$ and $|F(z_r)| = M(r, F)$, by (5.13) and (5.14), we get

$$(5.15) \quad \frac{M(r, \phi)^c}{r^{\frac{1}{2}}} \leq A.$$

Since ϕ is transcendental, we derive that

$$\frac{M(r, \phi)^c}{r^{\frac{1}{2}}} \rightarrow \infty,$$

which contradicts with (5.15). Thus, ϕ is a constant.

Now, in the same way as [13, Theorem 1.1] we prove that Q is a constant. On the contrary, we suppose that $q = \deg Q \geq 1$ and $e^\phi = 1$ without loss of generality.

Rewriting (5.9) as

$$(5.16) \quad 1 = Q \frac{\sum_{j=0}^k C_{k,j} \frac{F^{(k-j)}}{F} (\frac{1}{P})^{(j)} P - \frac{A}{F}}{1 - \frac{A}{F}}.$$

Since P is a polynomial, we have

$$(5.17) \quad |(\frac{1}{P})^{(j)}(z_r)P(z_r)| = r^{-j}(d_1 + o(1)),$$

where d_1 is a positive number. We know that any zero of Q comes from a pole of f with multiplicity k . Thus $q \geq k$. Then, from the equations (5.5), (5.9), and (5.17) we derive that ($|z| = r \notin E$)

$$r^{q-k} \nu(r, F)^k (d_2 + o(1)) = 1,$$

where d_2 is a positive number. Thus, it implies that $\nu(r, F)$ is bounded. While, for a transcendental entire function F , we have

$$\nu(r, F) \rightarrow \infty \text{ as } r \rightarrow \infty.$$

So, we derive a contradiction. Hence, Q is also a constant, and $\frac{e^\phi}{Q}$ becomes a constant c .

Hence, we complete the proof of Theorem 2.1.

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