Bull. Korean Math. Soc. **50** (2013), No. 4, pp. 1193–1200
http://dx.doi.org/10.4134/BKMS.2013.50.4.1193

INVARIANT RINGS AND REPRESENTATIONS OF SYMMETRIC GROUPS

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Abstract. The center of the Lie group $SU(n)$ is isomorphic to $\mathbb{Z}_n$. If d divides n, the quotient $SU(n)/\mathbb{Z}_d$ is also a Lie group. Such groups are locally isomorphic, and their Weyl groups $W(SU(n)/\mathbb{Z}_d)$ are the symmetric group Σ_n. However, the integral representations of the Weyl groups are not equivalent. Under the mod p reductions, we consider the structure of invariant rings $H^*(BT^{n-1};\mathbb{F}_p)^W$ for $W = W(SU(n)/\mathbb{Z}_d)$. Particularly, we ask if each of them is a polynomial ring. Our results show some polynomial and non–polynomial cases.

Let W be a finite group. For a modular representation

$$\rho : W \longrightarrow GL(n;\mathbb{F}_p),$$

the group $\rho(W)$ acts on the polynomial algebra $S(V) = \mathbb{F}_p[t_1,\dots,t_n]$. The set of invariants $S(V)^{\rho(W)}$ has a ring structure, and it is said to be the ring of invariants, [8] and [7]. In this paper, we discuss the invariant rings for various representations of the symmetric group Σ_n, along the line of work of [6].

The following is a topological aspect of our results, [8, Chapter 10]. Suppose G is a compact connected Lie group. It is well-known that the cohomology of the classifying space $H^*(BG;\mathbb{Q})$ is isomorphic to the ring of invariants $H^*(BT^n;\mathbb{Q})^{W(G)}$, which is a polynomial ring. We recall that $\mathbb{Q}$ can be replaced by a finite field $\mathbb{F}_p$ when the prime p is large. Here we note $S(V) \cong H^*(BT^n;\mathbb{F}_p)$.

We consider the integral representations of symmetric groups, which is the Weyl group of $SU(n)$. If d divides n, the quotient $SU(n)/\mathbb{Z}_d$ is also a Lie group. The integral representations of Σ_n induced by the actions of the Weyl groups of $SU(n)/\mathbb{Z}_d$ on maximal tori are $\mathbb{Z}$-inequivalent, [2]. In fact, the $\mathbb{Z}$-representation of $W(SU(n)/\mathbb{Z}_d)$ on T^{n-1}, up to $\mathbb{Z}$-equivalence, is given by $\phi_d^{-1}W(SU(n))\phi_d$ for a non-singular matrix ϕ_d. The representation of $\Sigma_n = W(SU(n))$ is generated

Received August 3, 2012; Revised October 5, 2012.
2010 *Mathematics Subject Classification.* Primary 55R35; Secondary 13A50, 55P60.
Key words and phrases. invariant theory, unstable algebra, pseudoreflection group, Lie group, p-compact group, classifying space.

by the permutation matrices together with the following $(n-1)\times(n-1)$ matrix:

$$\begin{pmatrix} 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & \vdots \\ & & & -1 \end{pmatrix}.$$

In other words, each column vector is one of the set of the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n-1}\}$ and the vector $\boldsymbol{b} = {}^t(-1, -1, \ldots, -1)$.

Let $W_{n,d}$ denote $\phi_d^{-1} W(SU(n))\phi_d$. We use the same symbol for both integral and modular representations. If p does not divide d, then $W_{n,d} \cong W(SU(n))$ at p. It is known that $H^*(BSU(n); \mathbb{F}_p) = H^*(BT^{n-1}; \mathbb{F}_p)^{W_{n,1}}$, and that $H^*(BT^2; \mathbb{F}_3)^{W_{3,3}}$ is a polynomial ring which is not realizable, [5]. In the case of $n = 4$ and $d = 2$, the following result shows that the invariant ring $H^*(BT^3; \mathbb{F}_2)^{W_{4,2}}$ is a polynomial ring which is realizable.

Theorem 1. *Let $H^*(BT^3; \mathbb{F}_2) = \mathbb{F}_2[t_1, t_2, t_3]$ with $\deg(t_i) = 2$. The following hold*:

(1) $H^*(BT^3; \mathbb{F}_2)^{W_{4,2}} = \mathbb{F}_2[x_2, x_4, x_6]$, *where* $x_2 = t_3$, $x_4 = t_1^2 + t_2^2 + t_1t_2 + t_1t_3 + t_2t_3$ *and* $x_6 = t_1t_2(t_1 + t_2 + t_3)$.

(2) $H^*(BT^3; \mathbb{F}_2)^{W_{4,2}} \cong H^*(BSU(3) \times BS^1; \mathbb{F}_2)$.

We define the homomorphism $\rho_d : \Sigma_4 \longrightarrow GL(3; \mathbb{F}_2)$ by $\rho_d(x) = \phi_d^{-1} x \phi_d$ for $x \in \Sigma_4$. Then $\mathrm{Im}\, \rho_2 = W_{4,2}$ and $\ker \rho_2 \cong \mathbb{Z}_2$. If $d = 4$, on the other hand, then ρ_4 is a faithful representation. The structure of $H^*(BT^4; \mathbb{F}_2)^{W_{4,4}}$ is as follows:

Theorem 2. *For $n = d = 4$, the following hold*:

(1) $H^*(BT^3; \mathbb{F}_2)^{W_{4,4}} = \mathbb{F}_2[x_2, x_8, x_{12}]$, *where* $x_2 = t_3$, $x_8 = t_1^4 + t_2^4 + t_1^2t_2^2 + t_1^2t_3^2 + t_2^2t_3^2 + t_1^2t_2t_3 + t_1t_2^2t_3 + t_1t_2t_3^2$ *and* $x_{12} = t_1t_2(t_1+t_2)(t_1+t_3)(t_2+t_3)(t_1 + t_2 + t_3)$.

(2) $H^*(BT^3; \mathbb{F}_2)^{W_{4,4}}$ *is not realizable.*

We note here that, in the case of $n = d$, the representation $W_{n,n}$ is equivalent to the dual representation $W(SU(n))^*$. It is known, [3], that for $p \geq 5$, the invariant ring $H^*(BT^{p-1}; \mathbb{F}_p)^{W(SU(p))^*}$ is not polynomial. We will show some analogous results for $p = 2, 3$.

Theorem 3. *The following hold*:

(1) *Let* $n = 6, 8$. *Then* $H^*(BT^{n-1}; \mathbb{F}_2)^{W_{n,n}}$ *is not a polynomial ring.*

(2) *Let* $n = 6, 9$. *Then* $H^*(BT^{n-1}; \mathbb{F}_3)^{W_{n,n}}$ *is not a polynomial ring.*

We use a result of Dwyer–Wilkerson [3, Theorem 1.4]. Suppose that V is a finite dimensional vector space over the field $\mathbb{F}_p$, and that W is a subgroup of $\mathrm{Aut}(V)$. Let U be a subset of V, and W_U the subgroup of W consisting of elements which fix U pointwise. Then if $S(V)^{W^*}$ is a polynomial ring over $\mathbb{F}_p$, then W_U must be a pseudoreflection group.

1. Invariant rings that are polynomial rings

As sated in the introduction, the representation of $\Sigma_n = W(SU(n))$ is generated by the permutation matrices Σ_{n-1} together with the following $(n-1)\times(n-1)$ matrix:

$$\begin{pmatrix} 1 & 0 & \cdots & -1 \\ 0 & 1 & & -1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$

We write $W_{n,d} = \phi_d^{-1} W(SU(n))\phi_d$ for the following matrix ϕ_d:

$$\phi_d = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ d-1 & \cdots & d-1 & d \end{pmatrix}.$$

We will prove Theorem 1 and Theorem 2 in this section. We consider the case of $n = 4$. The representation $W(SU(4))$ is generated by 3 reflections.

$$W(SU(4)) = \Sigma_4 = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

First take $d = 2$. Notice that ϕ_2 and ϕ_2^{-1} can be expressed as follows:

$$\phi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}, \ \phi_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Consequently, the reflection group $W_{4,2} = \phi_2^{-1} W(SU(4))\phi_2$ is generated by the following:

$$W_{4,2} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Next consider the mod 2 reduction.

$$W_{4,2} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle.$$

Hence, we see $W_{4,2} \cong \Sigma_3$ at $p = 2$.

We recall how to see if a ring of invariants $H^*(BT^n; \mathbb{F}_p)^W$ is polynomial, [4], [7] and [8]. A set of n elements $x_1, x_2, \ldots, x_n \in H^*(BT^n; \mathbb{F}_p)^W$ is said to be a system of parameters if the solution of the following system of equations

$$\begin{cases} x_1(t_1, t_2, \ldots, t_n) = 0 \\ x_2(t_1, t_2, \ldots, t_n) = 0 \\ \qquad\vdots \\ x_n(t_1, t_2, \ldots, t_n) = 0 \end{cases}$$

is trivial. Namely $t_1 = t_2 = \cdots = t_n = 0$. As usual, we write $H^*(BT^n; \mathbb{F}_p) = \mathbb{F}_p[t_1, t_2, \ldots, t_n]$. Let $d(x)$ denote $\frac{1}{2}\deg(x)$ so that $d(t_i) = 1$ for $1 \le i \le n$. According to [8, Proposition 5.5.5], for a finite group W, if we can find a system of parameters $\{x_1, x_2, \ldots x_n\}$ with $\prod_{i=1}^n d(x_i) = |W|$, then $H^*(BT^n; \mathbb{F}_p)^W = \mathbb{F}_p[x_1, x_2, \ldots x_n]$.

Proof of Theorem 1. (1) Suppose $x_2 = t_3$, $x_4 = t_1^2 + t_2^2 + t_1t_2 + t_1t_3 + t_2t_3$ and $x_6 = t_1t_2(t_1 + t_2 + t_3)$. It is easy to check that the element x_2, x_4, x_6 are $W_{4,2}$-invariant, and that $\{x_2,x_4,x_6\}$ is a system of parameters. Consequently $H^*(BT^3\mathbb{F}_2)^{W_{4,2}}$ is the polynomial ring generated by x_2, x_4 and x_6, since $|W_{4,2}| = 6 = d(x_2) \cdot d(x_4) \cdot d(x_6)$.

(2) Recall that the mod 2 reduction of $W_{4,2}$ is generated by A and B, where $A = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$ and $B = \begin{pmatrix} 1&0&0\\1&1&0\\1&0&1 \end{pmatrix}$. Let $\psi = \begin{pmatrix} 1&1&0\\1&0&0\\0&1&1 \end{pmatrix}$. Then a calculation shows

$$\bar{A} = \psi^{-1}A\psi = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \bar{B} = \psi^{-1}B\psi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, both matrices $\bar{A}$ and $\bar{B}$ are elements of $W(SU(3)) \times W(S^1)$. Notice that the mod 2 reduction of $W(SU(3))$ is generated by $\begin{pmatrix} 0&1\\1&0 \end{pmatrix}$ and $\begin{pmatrix} 1&1\\0&1 \end{pmatrix}$. This means that $W_{4,2}$ is equivalent to $W(SU(3)) \times W(S^1)$. Therefore, we see $H^*(BT^3; \mathbb{F}_2)^{W_{4,2}} \cong H^*(BT^3; \mathbb{F}_2)^{W(SU(3))\times W(S^1)} = H^*(BSU(3) \times BS^1; \mathbb{F}_2)$. This completes the proof. □

Next consider the case of $d = 4$. For $\phi_4 = \begin{pmatrix} 1&0&0\\0&1&0\\3&3&4 \end{pmatrix}$, we see $W_{4,4} = \phi_4^{-1}W(SU(4))\phi_4$. The reflection group $W_{4,4}$ is generated by the matrices $\begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$, $\begin{pmatrix} 1&0&0\\3&3&4\\-3&-2&-3 \end{pmatrix}$ and $\begin{pmatrix} -2&-3&-4\\-3&-2&-4\\3&3&5 \end{pmatrix}$. Hence its mod 2 reduction is as follows:

$$W_{4,4} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle.$$

Proof of Theorem 2. (1) Notice that $\{x_2, x_8, x_{12}\}$ is a system of parameters. Furthermore $|W_{4,4}| = 24 = d(x_2) \cdot d(x_8) \cdot d(x_{12})$. An argument similar to the one in Theorem 1 shows the desired result.

(2) If the unstable algebra $H^*(BT^3;\mathbb{F}_2)^{W_{4,4}}$ is realizable, there is a 2-compact group X such that $H^*(BT^3;\mathbb{F}_2)^{W_{4,4}} \cong H^*(BX;\mathbb{F}_2)$. Since the polynomial algebra is generated by even-degree elements, the classifying space BX is 2-torsion free. So the 2-adic cohomology is also a polynomial algebra generated by elements of the same degree. We can find, [1], a compact connected Lie group G such that $H^*(BX;\mathbb{Z}_2^\wedge) \cong H^*(BG;\mathbb{Z}_2^\wedge)$. However, any Lie group G does not satisfy the condition that $H^*(BG;\mathbb{F}_2) = \mathbb{F}_2[x_2, x_8, x_{12}]$, since this cohomology does not contain a generator of degree 4. Thus, $H^*(BT^3;\mathbb{F}_2)^{W_{4,4}}$ is not realizable. □

Remark 1.1. Recall that in the case of $n = d$, the representation $W_{n,n}$ is equivalent to the dual representation $W(SU(n))^*$. In fact, for the following $(n-1)\times(n-1)$ matrix,

$$\phi = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}$$

we see [5] that $\phi^{-1}\sigma\phi = {}^t\sigma$ for each of the generators σ of the reflection group Σ_{n-1}. We claim that $\phi^{-1}\Sigma_{n-1}\phi = \Sigma_{n-1}^*$. Consequently, we see $\psi^{-1}W_{n,n}\psi = W(SU(n))^*$ for $\psi = \phi_n^{-1}\phi$.

2. Non-polynomial cases

We will prove Theorem 3 in this section. To do so, we need a few basic results. As sated before, according to a result of Dwyer-Wilkerson, we will find a subset U such that the subgroup W_U is not generated by pseudoreflections. The representation $W(SU(n))$ as a subgroup of $GL(n-1,\mathbb{F}_p)$ is generated by the permutation representation of Σ_{n-1} together with the following $(n-1)\times(n-1)$ matrix:

$$\begin{pmatrix} 1 & & 0 & -1 \\ & \ddots & & \vdots \\ 0 & & 1 & -1 \\ 0 & \cdots & 0 & -1 \end{pmatrix}$$

In other words, if we let $\sum_{i=1}^n t_i = 0$, then the representation of $W(SU(n))$ can be regarded as the permutation representation of Σ_n. For instance, when $n = 4$, the transposition $(1,2)$ corresponds to the matrix $\left(\begin{smallmatrix} 0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right)$, $(2,3)$ to $\left(\begin{smallmatrix} 1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right)$, and $(3,4)$ to $\left(\begin{smallmatrix} 1&0&-1\\0&1&-1\\0&0&-1\end{smallmatrix}\right)$, respectively. We will use this convention.

Proof of Theorem 3. (1) First we consider the case of $n = 6$. Let $\boldsymbol{x} = {}^t(1,1,1,0,0)$ and $\boldsymbol{y} = {}^t(1,1,0,1,1)$, and let $U = \{\boldsymbol{x}, \boldsymbol{y}\}$. Recall that any element of $W(SU(6))$ is a 5×5 matrix such that each column is one of the set of the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5\}$ and the vector $\boldsymbol{b} = {}^t(1,1,1,1,1)$, since $p = 2$.

Take a matrix $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5) \in W(SU(6))$ such that $A\boldsymbol{x} = \boldsymbol{x}$. Notice that $A\boldsymbol{x} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3$ and $\boldsymbol{x} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$. If the first column $\boldsymbol{a}_1$ is $\boldsymbol{e}_1$, then $\boldsymbol{b} \notin \{\boldsymbol{a}_2, \boldsymbol{a}_3\}$, and hence $\{\boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{e}_2, \boldsymbol{e}_3\}$. Similarly, if $\boldsymbol{a}_1 = \boldsymbol{e}_4$, then one can show $\{\boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{e}_5, \boldsymbol{b}\}$. It turns out that all possible combinations are either

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} \text{ and } \{\boldsymbol{a}_4, \boldsymbol{a}_5\} \subset \{\boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{b}\}$$

or

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{b}\} \text{ and } \{\boldsymbol{a}_4, \boldsymbol{a}_5\} \subset \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}.$$

Furthermore, we have $A\boldsymbol{y} = \boldsymbol{y}$ if $A \in W_U$. Again, notice that $A\boldsymbol{y} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_4 + \boldsymbol{a}_5$ and $\boldsymbol{y} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_4 + \boldsymbol{e}_5$. One can show that $\boldsymbol{a}_3 = \boldsymbol{e}_3$ or $\boldsymbol{b}$, and hence $W_U \cong D_8$ as follows:

$$W_U = \left\{ \begin{matrix} e, & (1,2), & (4,5), & (1,2)(4,5) \\ (1,4)(2,5)(3,6), & (1,5,2,4)(3,6), & (1,4,2,5)(3,6), & (1,5)(2,4)(3,6) \end{matrix} \right\}.$$

Here we regard, for example, as follows:

$$(1,2) = (\boldsymbol{e}_2, \boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5) \text{ and } (1,5,2,4)(3,6) = (\boldsymbol{e}_5, \boldsymbol{e}_4, \boldsymbol{b}, \boldsymbol{e}_1, \boldsymbol{e}_2).$$

Since W_U is not a pseudoreflection group, we see that $H^*(BT^5; \mathbb{F}_2)^{W_{6,6}}$ is not a polynomial ring by [3].

Next consider the case of $n = 8$. Let $U = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ for $\boldsymbol{x} = {}^t(1,1,1,1,0,0,0)$, $\boldsymbol{y} = {}^t(1,1,0,0,1,1,0)$ and $\boldsymbol{z} = {}^t(1,0,1,0,1,0,1)$. Any element of $W(SU(8))$ is a 7×7 matrix such that each column is one of the set of the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_7\}$ and the vector $\boldsymbol{b} = {}^t(1,1,1,1,1,1,1)$. Take a matrix $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_7) \in W(SU(8))$ such that $A\boldsymbol{x} = \boldsymbol{x}$. If the first column $\boldsymbol{a}_1$ is $\boldsymbol{e}_1$, then $\{\boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\} = \{\boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$. Similarly, if $\boldsymbol{a}_1 = \boldsymbol{e}_5$, then $\{\boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\} = \{\boldsymbol{e}_6, \boldsymbol{e}_7, \boldsymbol{b}\}$. All possible combinations are either

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\} \text{ and } \{\boldsymbol{a}_5, \boldsymbol{a}_6, \boldsymbol{a}_7\} \subset \{\boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7, \boldsymbol{b}\}$$

or

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\} = \{\boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7, \boldsymbol{b}\} \text{ and } \{\boldsymbol{a}_5, \boldsymbol{a}_6, \boldsymbol{a}_7\} \subset \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}.$$

Furthermore, we have $A\boldsymbol{y} = \boldsymbol{y}$ and $A\boldsymbol{z} = \boldsymbol{z}$ if $A \in W_U$. One can show that W_U is expressed as follows:

$$W_U = \mathbb{Z}/2\langle\alpha\rangle \times \mathbb{Z}/2\langle\beta\rangle \times \mathbb{Z}/2\langle\gamma\rangle,$$

where $\alpha = (1,2)(3,4)(5,6)(7,8)$, $\beta = (1,3)(2,4)(5,7)(6,8)$ and $\gamma = (1,5)(2,6)(3,7)(4,8)$. This group is not a pseudoreflection group, hence $H^*(BT^7; \mathbb{F}_2)^{W_{8,8}}$ is not a polynomial ring.

(2) We consider the case of $n = 6$ at $p = 3$. Let $U = \{\boldsymbol{x}, \boldsymbol{y}\}$ for $\boldsymbol{x} = {}^t(1,1,-1,-1,0)$ and $\boldsymbol{y} = (1,-1,0,1,-1)$. Take a matrix $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5) \in W(SU(6))$ such that $A\boldsymbol{x} = \boldsymbol{x}$. It the first column $\boldsymbol{a}_1$ is $\boldsymbol{e}_1$, then $\boldsymbol{a}_2 = \boldsymbol{e}_2$. Similarly, if $\boldsymbol{a}_1 = \boldsymbol{e}_3$, then $\boldsymbol{a}_2 = \boldsymbol{e}_4$, and if $\boldsymbol{a}_1 = \boldsymbol{e}_5$, then $\boldsymbol{a}_2 = \boldsymbol{b}$. Here $\boldsymbol{b} = {}^t(-1,-1,-1,-1,-1)$. One of the following holds:

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}, \{\boldsymbol{a}_3, \boldsymbol{a}_4\} = \{\boldsymbol{e}_3, \boldsymbol{e}_4\} \text{ and } \boldsymbol{a}_5 \in \{\boldsymbol{e}_5, \boldsymbol{b}\}$$

or

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2\} = \{\boldsymbol{e}_3, \boldsymbol{e}_4\}, \{\boldsymbol{a}_3, \boldsymbol{a}_4\} = \{\boldsymbol{e}_5, \boldsymbol{b}\} \text{ and } \boldsymbol{a}_5 \in \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$$

or

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2\} = \{\boldsymbol{e}_5, \boldsymbol{b}\}, \{\boldsymbol{a}_3, \boldsymbol{a}_4\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\} \text{ and } \boldsymbol{a}_5 \in \{\boldsymbol{e}_3, \boldsymbol{e}_4\}.$$

Furthermore, we have $A\boldsymbol{y} = \boldsymbol{y}$ if $A \in W_U$. It follows that

$$W_U = \mathbb{Z}/3\langle\alpha\rangle,$$

where $\alpha = (1,3,5)(2,4,6)$. Therefore, this group is not a pseudoreflection group, and hence $H^*(BT^5; \mathbb{F}_3)^{W_{6,6}}$ is not a polynomial ring.

Next consider the case of $n = 9$. Let $U = \{\boldsymbol{x}, \boldsymbol{y}\}$ for $\boldsymbol{x} = {}^t(1,1,1,-1,-1,-1,0,0)$ and $\boldsymbol{y} = {}^t(1,-1,0,1,-1,0,1,-1)$. Take a matrix $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_8) \in W(SU(9))$ such that $A\boldsymbol{x} = \boldsymbol{x}$. It the first column $\boldsymbol{a}_1$ is $\boldsymbol{e}_1$, then we obtain $\{\boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{e}_2, \boldsymbol{e}_3\}$. Similarly, if $\boldsymbol{a}_1 = \boldsymbol{e}_4$, then $\{\boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{e}_5, \boldsymbol{e}_6\}$, and if $\boldsymbol{a}_1 = \boldsymbol{e}_7$, then $\{\boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{e}_8, \boldsymbol{b}\}$. Here $\boldsymbol{b} = {}^t(-1,-1,-1,-1,-1,-1,-1,-1)$. We see that one of the following holds:

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}, \{\boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6\} = \{\boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6\} \text{ and } \{\boldsymbol{a}_7, \boldsymbol{a}_8\} \subset \{\boldsymbol{e}_7, \boldsymbol{e}_8, \boldsymbol{b}\}$$

or

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6\}, \{\boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6\} = \{\boldsymbol{e}_7, \boldsymbol{e}_8, \boldsymbol{b}\} \text{ and } \{\boldsymbol{a}_7, \boldsymbol{a}_8\} \subset \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$$

or

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{e}_7, \boldsymbol{e}_8, \boldsymbol{b}\}, \{\boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} \text{ and } \{\boldsymbol{a}_7, \boldsymbol{a}_8\} \subset \{\boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6\}.$$

Furthermore, we have $A\boldsymbol{y} = \boldsymbol{y}$ if $A \in W_U$. Consequently, we see that

$$W_U = \mathbb{Z}/3\langle\alpha\rangle \times \mathbb{Z}/3\langle\beta\rangle,$$

where $\alpha = (1,2,3)(4,5,6)(7,8,9)$ and $\beta = (1,4,7)(2,5,8)(3,6,9)$. Once again $H^*(BT^8; \mathbb{F}_3)^{W_{9,9}}$ is not a polynomial ring. □

Remark 2.1. In the proof of Theorem 3, we have found subgroups W_U which are not pseudoreflection groups. In the case of $n = 6$ and $p = 2$, the group W_U can be made smaller. Namely, if $\boldsymbol{z} = {}^t(1,0,1,0,1)$ and $U = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$, then W_U can be expressed as follows:

$$W_U = \mathbb{Z}/2\langle\alpha\rangle,$$

where $\alpha = (1,4)(2,5)(3,6)$. This is not a pseudoreflection group.

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