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VALUE DISTRIBUTION AND UNIQUENESS ON q-DIFFERENCES OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, by using the q-difference analogue of lemma on the logarithmic derivative lemma to re-establish some estimates of Nevanlinna characteristics of $f(qz)$, we deal with the value distribution and uniqueness of certain types of q-difference polynomials.

1. Introduction

In this paper, we assume that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, such as the proximity function $m(r, f)$, counting function $N(r, f)$, characteristic function $T(r, f)$ for a meromorphic function $f(z)$ in the complex plane (see e.g. [7, 14]). We also use $\overline{N}_{(2}(r, \frac{1}{f})$ to denote the counting function of zeros of $f(z)$ such that the multiple zeros are counted once and the simple zeros are not counted in $\{z : |z| \leq r\}$. We now recall that a meromorphic function $a(z)$ is said to be a small function of $f(z)$ if $T(r, a) = S(r, f)$, where $S(r, f)$ is used to denote any quantity satisfying $S(r, f) = o(\{T(r, f)\}$ as $r \to \infty$, possibly outside of a set of finite logarithmic measure, furthermore, possibly outside of a set of logarithmic density 0, i.e., outside of a set E such that $\lim_{r\to\infty} \int_{[1,r]\cap E} \frac{dt}{t} / \log r = 0$. The family of all small functions related to $f(z)$ is denoted by $\mathscr{F}(f)$.

Recently, a number of fundamental results on difference operators and difference polynomials have been derived. For examples, the difference analogue of lemma on the logarithmic derivative [2, 5], the difference counterpart of Clunie and Mohon'ko lemma [5, 9], Nevanlinna characteristics of $f(z + c)$ for $c \in \mathbb{C}\backslash\{0\}$ in the complex plane [2] and Nevanlinna theory to difference operators, especially the difference analogue of the second main theorem [6]. Using these results, the value distribution and uniqueness of difference operators and difference polynomials of meromorphic functions have been dealt with in the

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past five years (see e.g. [3, 4, 8, 10, 11]). However, there are only few papers concerning with the value distribution and uniqueness of q-difference operators and q-difference polynomials (see [12, 16]).

The purpose of this paper is to study the value distribution and uniqueness of q-differences of meromorphic function of zero order. The main tool is to use the q-difference analogue of lemma on the logarithmic derivative [1] to re-establish some estimates on the Nevanlinna characteristics of $f(qz)$, which are somewhat different from Nevanlinna characteristics of $f(qz)$ obtained by Zhang and Korhonen in [16].

This paper is organized as follows. In Section 2, we present some results on value distribution of q-difference polynomials of meromorphic functions of zero order. In Section 3, we investigate uniqueness of q-difference polynomials of meromorphic functions of zero order.

2. Value distribution of q-difference polynomials

Laine and Yang [10] investigated the value distribution of difference products of entire functions and obtained the following result.

Theorem 2.A ([10, Theorem 2])**.** *Let $f(z)$ be a transcendental entire function of finite order, and c be a nonzero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Subsequently, a parallel result for the q-difference case has been proved in [16].

Theorem 2.B ([16, Theorem 4.1])**.** *Let $f(z)$ be a transcendental meromorphic (resp. entire) function of zero order and q be nonzero complex constant. Then for $n \geq 6$ (resp. $n \geq 2$), $f(z)^n f(qz)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

In addition, we also recall the following related result.

Theorem 2.C ([16, Theorem 4.3])**.** *Let $f(z)$ be a transcendental meromorphic (resp. entire) function of zero order and q be nonzero complex constant. Then for $n \geq 6$ (resp. $n \geq 2$), $f(z)^n(f(z)-1)f(qz)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

In this section, we will establish an improvement of Theorem 2.B and Theorem 2.C, which is stated as follows.

Theorem 2.1. *Let $f(z)$ be a transcendental meromorphic (resp. entire) function of zero order and q be nonzero complex constant, and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n (\neq 0)$, and m be the number of the distinct zeros of $P(z)$. Then for $n > 2m+3$ (resp. $n > m$), $P(f(z))f(qz) - a(z)$ has infinitely many zeros, where $a(z) \in \mathscr{F}(f)\backslash\{0\}$.*

The restriction in Theorem 2.1 to $a(z) \in \mathscr{F}(f)\backslash\{0\}$ is essential.

Example 2.1. Let $q \in \mathbb{C}$ such that $0 < |q| < 1$. The q-Gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},$$

where $(a;q)_\infty = \Pi_{k=0}^{\infty}(1-aq^k)$. By defining

$$\gamma_q(z) := (1-q)^{x-1}\Gamma_q(x), z = q^x,$$

and $\gamma_q(0) := (q;q)_\infty$, we see that $\gamma_q(z)$ is meromorphic of zero order with no zero. By taking $P(z) = z$ and $f(z) = \gamma_q(z)$. If $a(z) \equiv 0$, then $P(f(z))f(qz) - a(z) = \gamma_q(z) \cdot \gamma_q(qz)$ has no zero.

Example 2.2. The zero order growth restriction in Theorem 2.1 can not be extended to finite order. This can be seen by taking $P(z) = z^n + 1, f(z) = e^z$ and $q = -n$. Then $P(f(z))f(qz) - 1$ has no zero.

In order to prove Theorem 2.1, we need some preliminaries as follows.

Lemma 2.1 ([1, Lemma 5.2]). *If $T : \mathbb{R}^+ \to \mathbb{R}^+$ is a piecewise continuous increasing function such that*

$$\lim_{r\to\infty} \frac{\log T(r)}{\log r} = 0,$$

then the set

$$E := \{r : T(C_1 r) \geq C_2 T(r)\}$$

has logarithmic density 0 for all $C_1 > 1$ and $C_2 > 1$.

Lemma 2.2. *Let $f(z)$ be a nonconstant meromorphic function of zero order, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$\begin{aligned}
&N\left(r, \frac{1}{f(qz)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r,f),\\
&N(r, f(qz)) \leq N(r, f(z)) + S(r,f),\\
&\overline{N}\left(r, \frac{1}{f(qz)}\right) \leq \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r,f),\\
&\overline{N}(r, f(qz)) \leq \overline{N}(r, f(z)) + S(r,f),
\end{aligned}$$

on a set of logarithmic density 1.

Proof. We will use the similar method used in [16]. Here, we only prove the case $|q| > 1$. By a simple geometric observation, we obtain

$$N\left(r, \frac{1}{f(qz)}\right) \leq N\left(|q|r, \frac{1}{f(z)}\right).$$

Since the order of $f(z)$ is zero, we conclude from Lemma 2.1 that,

$$N\left(|q|r, \frac{1}{f(z)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r,f),$$

on a set of logarithmic density 1. Therefore,

$$N\left(r, \frac{1}{f(qz)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r,f),$$

on a set of logarithmic density 1.

Similarly, we can prove the remainders. Here, we omit their proofs. □

Now, we recall the q-difference analogue of lemma on the logarithmic derivative as follows.

Lemma 2.3 ([1, Theorem 1.2])**.** *Let $f(z)$ be a nonconstant zero order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = o(T(r,f))$$

on a set of logarithmic density 1.

Lemma 2.4. *Let $f(z)$ be a nonconstant meromorphic function of zero order, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$T(r, f(qz)) \leq T(r, f(z)) + S(r,f)$$

on a set of logarithmic density 1.

Proof. By Lemma 2.2 and Lemma 2.3, we obtain

$$\begin{aligned} T(r, f(qz)) &= m(r, f(qz)) + N(r, f(qz)) \\ &\leq m\left(r, \frac{f(qz)}{f(z)}\right) + m(r, f(z)) + N(r, f(z)) + S(r,f) \\ &= T(r, f(z)) + S(r,f) \end{aligned}$$

on a set of logarithmic density 1. □

Remark 2.1. In [16], the authors showed that the conclusion in Lemma 2.4 holds on a set of lower logarithmic density 1.

Lemma 2.5. *Let $f(z)$ be an entire function of zero order and q be nonzero constant, and let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n (\neq 0)$. Then*

$$T(r, P(f(z))f(qz)) = T(r, P(f(z))f(z)) + S(r,f)$$

on a set of logarithmic density 1.

Proof. Since $f(z)$ is entire of zero order, we obtain, by Lemma 2.3,

$$\begin{aligned} T(r, P(f(z))f(qz)) &= m(r, P(f(z))f(qz)) \\ &\leq m(P(f(z))f(z)) + m\left(r, \frac{f(qz)}{f(z)}\right) + S(r,f) \\ &= T(r, P(f(z))f(z)) + S(r,f) \end{aligned}$$

on a set of logarithmic density 1. Similarly, we also have

$$T(r, P(f(z))f(z)) \leq T(r, P(f(z))f(qz)) + S(r, f)$$

on a set of logarithmic density 1. Therefore,

$$T(r, P(f(z))f(qz)) = T(r, P(f(z))f(z)) + S(r, f)$$

on a set of logarithmic density 1. □

Now, we give a proof of Theorem 2.1 completely.

Proof of Theorem 2.1. Suppose that $P(f(z))f(qz) - a(z)$ has finitely many zeros only. If $f(z)$ is meromorphic of zero order, then we may apply the Valiron-Mohon'ko lemma, Nevanlinna main theorems, Lemma 2.4 and Lemma 2.2 to obtain

$$\begin{aligned}
nT(r,f) + S(r,f) &= T(r, P(f(z)) \\
&\leq T(r, P(f(z))f(qz)) + T\left(r, \frac{1}{f(qz)}\right) + S(r,f) \\
&\leq T(r, f(z)) + \overline{N}(r, P(f(z))f(qz)) + \overline{N}\left(r, \frac{1}{P(f(z))f(qz)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{P(f(z))f(qz) - a(z)}\right) + S(r,f) \\
&\leq T(r, f(z)) + \overline{N}(r, P(f)) + \overline{N}(r, f(qz)) \\
&\quad + \overline{N}\left(r, \frac{1}{P(f(z))}\right) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + S(r,f) \\
&\leq T(r, f(z)) + m\overline{N}(r, f(z)) + \overline{N}(r, f(z)) \\
&\quad + m\overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r,f) \\
&\leq (2m+3)T(r,f) + S(r,f),
\end{aligned}$$

on a set of logarithmic density 1, contradicting $n > 2m + 3$.

If, on the other hand, $f(z)$ is entire of order zero, then

$$\begin{aligned}
T(r, P(f(z))f(qz)) &\leq \overline{N}\left(r, \frac{1}{P(f(z))f(qz)}\right) + \overline{N}\left(r, \frac{1}{P(f(z))f(qz) - a(z)}\right) \\
&\quad + S(r,f) \\
&= \overline{N}\left(r, \frac{1}{P(f(z))f(qz)}\right) + S(r,f) \\
&\leq (m+1)T(r, f(z)) + S(r,f)
\end{aligned}$$

on a set of logarithmic density 1. Taking using of the Valiron-Mohon'ko lemma and Lemma 2.5, we conclude that

$$(n+1)T(r, f(z)) = T(r, P(f(z))f(z)) + S(r,f)$$

$$= T(r, P(f(z))f(qz)) + S(r, f)$$
$$\leq (m+1)T(r, f(z)) + S(r, f)$$

on a set of logarithmic density 1, contradicting $n > m$. The proof of Theorem 2.1 is completed. □

3. Shared common values of q-difference polynomials

Suppose that $f(z)$ and $g(z)$ are meromorphic functions, and $a \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We say $f(z)$ and $g(z)$ share a CM(counting multiplicities) if $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities. If $f(z) - a$ and $g(z) - a$ have the same zeros, we say $f(z)$ and $g(z)$ share a IM(ignoring multiplicities).

Corresponding to the results on uniqueness in [15, 16], Zhang and Korhonen further obtained.

Theorem 3.A ([16, Theorem 5.1])**.** *Let $f(z)$ and $g(z)$ be two transcendental meromorphic* (*resp. entire*) *functions of zero order. Suppose that q is a nonzero complex constant and n is an integer satisfying $n \geq 8$* (*resp. $n \geq 4$*). *If $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share $1, \infty$ CM, then $f(z) \equiv tg(z)$ for $t^{n+1} = 1$.*

Theorem 3.B ([16, Theorem 5.2])**.** *Let $f(z)$ and $g(z)$ be two transcendental entire functions of zero order. Suppose that q is a nonzero complex constant and $n \geq 6$ is an integer. If $f(z)^n(f(z)-1)f(qz)$ and $g(z)^n(g(z)-1)g(qz)$ share 1 CM, then $f(z) \equiv g(z)$.*

In this section, we firstly deduce more details about Theorem 3.A. Then, by combining all results above and the uniqueness of difference products on transcendental entire functions of finite order in [12], we further investigate the uniqueness of q-difference polynomials of meromorphic functions of zero order.

Theorem 3.1. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic* (*resp. entire*) *functions of zero order. Suppose that q is a nonzero complex constant and n is an integer satisfying $n \geq 14$* (*resp. $n \geq 6$*). *If $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$.*

Remark 3.1. Under the assumption of Theorem 3.1, if $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share $a \in \mathbb{C}\backslash\{0\}$ CM, we also have $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$. In its proof, we only set

$$F_0(z) = \frac{f(z)^n f(qz)}{a} \quad \text{and} \quad G_0(z) = \frac{g(z)^n g(qz)}{a}.$$

Then $F_0(z)$ and $G_0(z)$ share 1 CM. But the conclusion is not true if $a = 0$. For example, let $f(z) = z$ and $g(z) = \frac{1}{2}z$. Then for all $q \neq 0$, $f(z)^6 f(qz) = qz^7$ and $g(z)^6 g(qz) = \frac{q}{2^7}z^7$ share 0 CM. However, $f(z) = 2g(z), 2^7 \neq 1$ and $f(z)g(z) = \frac{1}{2}z^2$.

Theorem 3.2. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic (resp. entire) functions of zero order. Suppose that q is a nonzero complex constant and n is an integer satisfying $n \geq 26$ (resp. $n \geq 12$). If $f(z)^n f(qz)$ and $g(z)^n g(qz)$ share 1 IM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$.*

Theorem 3.3. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic (resp. entire) functions of zero order and q be nonzero complex constant, and let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n (\neq 0)$, and m be the number of the distinct zeros of $P(z)$. If $n > 3m + 4$ (resp. $n > 2m + 1$) and $P(f(z))f(qz)$ and $P(g(z))g(qz)$ share $1, \infty$ CM, then one of the following two results holds:*

(1) $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = LCM\{\lambda_j : j = 0, 1, \ldots, n\}$ denotes the lowest common multiple of $\lambda_j (j = 0, 1, \ldots, n)$, and

$$\lambda_j = \begin{cases} j+1, a_j \neq 0, \\ n+1, a_j = 0. \end{cases}$$

(2) $f(z)$ and $g(z)$ satisfy algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = P(w_1)w_1(qz) - P(w_2)w_2(qz).$$

In order to prove these theorems, we need some lemmas.

Lemma 3.1 ([15, Lemma 3])**.** *Let $F(z)$ and $G(z)$ be two nonconstant meromorphic functions. If $F(z)$ and $G(z)$ share 1 CM, one of the following three cases holds:*

$$\begin{aligned} (1)\ T(r, F(z)) \leq\ & \overline{N}(r, F(z)) + \overline{N}_{(2}(r, F(z)) + \overline{N}(r, G(z)) \\ & + \overline{N}_{(2}(r, G(z)) + \overline{N}\left(r, \frac{1}{F(z)}\right) + \overline{N}_{(2}\left(r, \frac{1}{F(z)}\right) \\ & + \overline{N}\left(r, \frac{1}{G(z)}\right) + \overline{N}_{(2}\left(r, \frac{1}{G(z)}\right) + S(r, F) + S(r, G), \end{aligned} \tag{3.1}$$

and similarly for $T(r, G(z))$;

(2) $F(z) \equiv G(z)$;

(3) $F(z)G(z) \equiv 1$,

where $\overline{N}_{(2}(r, \frac{1}{F(z)}) = \overline{N}(r, \frac{1}{F(z)}) - N_{(1}(r, \frac{1}{F(z)})$ and $N_{(1}(r, \frac{1}{F(z)})$ is the counting function of the simple zeros of $F(z)$ in $\{z : |z| \leq r\}$.

Remark 3.2. Set

$$N_2\left(r, \frac{1}{F(z)}\right) = \overline{N}\left(r, \frac{1}{F(z)}\right) + N_{(2}\left(r, \frac{1}{F(z)}\right).$$

Then we can find that $N_2(r, \frac{1}{F(z)})$ denotes the counting function of zeros of $F(z)$ such that the simple zeros are counted once and the multiple zeros are counted twice, and the inequality (3.1) turns into

$$T(r, F(z)) \leq N_2(r, F(z)) + N_2\left(r, \frac{1}{F(z)}\right) + N_2(r, G(z))$$

$$+ N_2\left(r, \frac{1}{G(z)}\right) + S(r,F) + S(r,G). \tag{3.2}$$

Lemma 3.2 ([13, Lemma 2.3]). *Let $F(z)$ and $G(z)$ be two nonconstant meromorphic functions sharing the value 1 IM. Let*

$$H(z) = \left(\frac{F''(z)}{F'(z)} - 2\frac{F'(z)}{F(z)-1}\right) - \left(\frac{G''(z)}{G'(z)} - 2\frac{G'(z)}{G(z)-1}\right).$$

If $H(z) \not\equiv 0$, then

$$\begin{aligned}
&T(r,F(z)) + T(r,G(z)) \\
\leq\ & 2\left[N_2(r,F(z)) + N_2(r,G(z)) + N_2\left(r,\frac{1}{F(z)}\right) + N_2\left(r,\frac{1}{G(z)}\right)\right] \\
&+ 3\left[\overline{N}(r,F(z)) + \overline{N}(r,G(z)) + \overline{N}\left(r,\frac{1}{F(z)}\right) + \overline{N}\left(r,\frac{1}{G(z)}\right)\right] \\
&+ S(r,F) + S(r,G).
\end{aligned}$$

In the follows, Theorems 3.1-3.3 will be proved.

Proof of Theorem 3.1. Let $F(z) = f(z)^n f(qz)$ and $G(z) = g(z)^n g(qz)$. Thus, $F(z)$ and $G(z)$ share 1 CM. Suppose first that $F(z) \neq G(z)$ and $F(z)G(z) \neq 1$.

If $f(z)$ and $g(z)$ are meromorphic of zero order, then we deduce from the first main theorem and Lemma 2.4 that

$$\begin{aligned}
nT(r,f(z)) + S(r,f) &= T(r,P(f(z))) \\
&\leq T(r,P(f(z))f(qz)) + T\left(r,\frac{1}{f(qz)}\right) + S(r,f) \\
&\leq T(r,P(f(z))f(qz)) + T(r,f(z)) + S(r,f).
\end{aligned}$$

Therefore

$$(n-1)T(r,f(z)) \leq T(r,P(f(z))f(qz)) + S(r,f) = T(r,F(z)) + S(r,f). \tag{3.3}$$

Similarly,

$$(n-1)T(r,g(z)) \leq T(r,G(z)) + S(r,g). \tag{3.4}$$

By using Lemma 2.4 again, we also have

$$\begin{aligned}
&T(r,F(z)) \leq (n+1)T(r,f(z)) + S(r,f) \quad \text{and} \\
&T(r,G(z)) \leq (n+1)T(r,g(z)) + S(r,g).
\end{aligned} \tag{3.5}$$

Now, we conclude from Nevanlinna main theorems, Lemma 2.4 and (3.5) that

$$\begin{aligned}
T(r,F(z)) &\leq \overline{N}(r,F(z)) + \overline{N}\left(r,\frac{1}{F(z)}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + S(r,F) \\
&\leq \overline{N}\left(r,f(z)\right) + \overline{N}\left(r,f(qz)\right) + \overline{N}\left(r,\frac{1}{f(z)}\right)
\end{aligned}$$

$$+\overline{N}\left(r, \frac{1}{f(qz)}\right) + \overline{N}\left(\frac{1}{G-1}\right) + S(r,f)$$
$$\leq 4T(r,f(z)) + T(r,G(z)) + S(r,f)$$
$$\leq 4T(r,f(z)) + (n+1)T(r,g(z)) + S(r,f) + S(r,g).$$

Thus, the above inequality and (3.3) yield

$$(n-5)T(r,f(z)) \leq (n+1)T(r,g(z)) + S(r,f) + S(r,g). \tag{3.6}$$

Similarly,

$$(n-5)T(r,g(z)) \leq (n+1)T(r,f(z)) + S(r,f) + S(r,g). \tag{3.7}$$

It follows from Remark 3.2 and Lemma 2.4 that

$$N_2\left(r, \frac{1}{F(z)}\right) \leq 2\overline{N}\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(qz)}\right) + S(r,f) \leq 3T(r,f(z)) + S(r,f). \tag{3.8}$$

Similarly, we also have

$$N_2(r,F(z)) \leq 3T(r,f(z)) + S(r,f), \tag{3.9}$$

$$N_2\left(r, \frac{1}{G(z)}\right) \leq 3T(r,g(z)) + S(r,g), \tag{3.10}$$

$$N_2\left(r, G(z)\right) \leq 3T(r,g(z)) + S(r,g). \tag{3.11}$$

Therefore, (3.2), (3.5), (3.8)−(3.11) yield

$$T(r,F(z)) + T(r,G(z)) \leq 2N_2\left(r,F(z)\right) + 2N_2\left(r, \frac{1}{F(z)}\right) + 2N_2\left(r,G(z)\right)$$
$$+ 2N_2\left(r, \frac{1}{G(z)}\right) + S(r,F) + S(r,G)$$
$$\leq 12[T(r,f(z)) + T(r,g(z))] + S(r,f) + S(r,g). \tag{3.12}$$

Thus, we deduce from (3.3), (3.4) and (3.12) that

$$(n-13)[T(r,f(z)) + T(r,g(z))] \leq S(r,f) + S(r,g),$$

contradicting $n \geq 14$.

If, on the other hand, $f(z)$ and $g(z)$ are entire of zero order. Replacing (3.3) and (3.4) by Lemma 2.5, and using the similar method above, we obtain

$$(n-5)[T(r,f(z)) + T(r,g(z))] \leq S(r,f) + S(r,g),$$

contradicting $n \geq 6$.

So, by Lemma 3.1, we obtain either $F(z) \equiv G(z)$ or $F(z)G(z) \equiv 1$.

If $F(z) \equiv G(z)$, i.e., $f(z)^n f(qz) = g(z)^n g(qz)$, by denoting $h(z) = \frac{f(z)}{g(z)}$, we obtain

$$h(z)^n h(qz) = 1. \tag{3.13}$$

It follows from Lemma 2.4 and (3.13) that

$$nT(r,h(z)) = T(r,h(z)^n) = T\left(r,\frac{1}{h(qz)}\right) \le T(r,h(z)) + S(r,h).$$

Then $h(z)$ must be nonzero constant since $n \ge 6$. Suppose that $h(z) = t$, we deduce from (3.13) that $t^{n+1} = 1$. Therefore, $f(z) = tg(z)$ and $t^{n+1} = 1$.

If $F(z)G(z) \equiv 1$, i.e.,

$$f(z)^n f(qz) g(z)^n g(qz) = 1. \tag{3.14}$$

Set $s(z) = f(z)g(z)$. Then $s(z)^n s(qz) = 1$. Similar to the discussion of (3.13), we also get $s(z)$ must be a nonzero constant, say t. Obviously, $t^{n+1} = 1$ from (3.14). Therefore, $f(z)g(z) = t$ and $t^{n+1} = 1$. The proof of Theorem 3.1 is completed. □

Proof of Theorem 3.2. Let $F(z) = f(z)^n f(qz)$ and $G(z) = g(z)^n g(qz)$. Similar to the proof of Theorem 3.1, we still obtain that (3.3)−(3.11) hold. Let $H(z)$ be defined as Lemma 3.2 and suppose that $H(z) \not\equiv 0$.

If $f(z)$ and $g(z)$ are meromorphic of zero order, then we deduce from Lemma 2.4 that

$$\overline{N}(r,F(z)) \le \overline{N}(r,f(z)) + \overline{N}(r,f(qz)) + S(r,f) \le 2T(r,f(z)) + S(r,f). \tag{3.15}$$

Similarly,

$$\overline{N}(r,G(z)) \le 2T(r,g(z)) + S(r,g), \tag{3.16}$$

$$\overline{N}\left(r,\frac{1}{F(z)}\right) \le 2T(r,f(z)) + S(r,f), \tag{3.17}$$

$$\overline{N}\left(r,\frac{1}{G(z)}\right) \le 2T(r,g(z)) + S(r,g). \tag{3.18}$$

It follows from Lemma 3.2, (3.8)−(3.11) and (3.15)−(3.18) that

$$T(r,F(z)) + T(r,G(z)) \le 24[T(r,f(z)) + T(r,g(z))] + S(r,f) + S(r,g).$$

Therefore, we deduce from (3.3) and (3.4) and above inequality that

$$(n-1)[T(,f(z)) + T(r,g(z))] \le 24[T(r,f(z)) + T(r,g(z))] + S(r,f) + S(r,g),$$

contradicting $n \ge 26$.

If, on the other hand, $f(z)$ and $g(z)$ are entire of zero order, then, replacing (3.3) and (3.4) by Lemma 2.5, and using the similar method above, we also get

$$(n-11)[T(r,f(z)) + T(r,g(z))] \le S(r,f) + S(r,g),$$

contradicting $n \ge 12$.

Thus, using Lemma 3.2 again, we get $H(z) \equiv 0$, i.e.,

$$\frac{F''(z)}{F'(z)} - 2\frac{F'(z)}{F(z)-1} = \frac{G''(z)}{G'(z)} - 2\frac{G'(z)}{G(z)-1}.$$

By integrating the above equality twice, we conclude that

$$F(z) = \frac{(b+1)G(z) + (a-b-1)}{bG(z) + (a-b)}, \tag{3.19}$$

where $a(\neq 0), b$ are two constants. In order to prove the conclusions of Theorem 3.2 are true, we will prove that either $F(z) = G(z)$ or $F(z)G(z) = 1$. Now, according to the coefficients of (3.19), we need to prove the following three cases.

Case 3.1. $b \neq 0, -1$.

If $a - b - 1 \neq 0$, we obtain from (3.19) that

$$\overline{N}\left(r, \frac{1}{F(z)}\right) = \overline{N}\left(r, \frac{1}{G(z) + \frac{a-b-1}{b+1}}\right).$$

Obviously, by Valiron-Mohon'ko lemma, (3.3), (3.4), (3.5) and (3.19) show that

$$\begin{cases} (n-1)T(r, f(z)) \leq (n+1)T(r, g) + S(r, f) + S(r, g), \\ (n-1)T(r, g(z)) \leq (n+1)T(r, g) + S(r, f) + S(r, g). \end{cases} \tag{3.20}$$

Thus, $S(r, f) = S(r, g)$.

Now, we may apply the second main theorem, Lemma 2.4, (3.4) and (3.20) to conclude that

$$\begin{aligned} (n-1)T(r, g(z)) &\leq T(r, G(z)) + S(r, g) \\ &\leq \overline{N}(r, G(z)) + \overline{N}\left(r, \frac{1}{G(z)}\right) + \overline{N}\left(r, \frac{1}{G(z) + \frac{a-b-1}{b+1}}\right) \\ &\quad + S(r, g) \\ &\leq \overline{N}(r, G(z)) + \overline{N}\left(r, \frac{1}{G(z)}\right) + \overline{N}\left(r, \frac{1}{F(z)}\right) + S(r, g) \\ &\leq \overline{N}(r, g(z)) + \overline{N}(r, g(qz)) + \overline{N}\left(r, \frac{1}{g(z)}\right) + \overline{N}\left(r, \frac{1}{g(qz)}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + S(r, f) + S(r, g) \\ &\leq 2T(r, f(z)) + 4T(r, g(z)) + S(r, g) \\ &\leq \left(\frac{2(n+1)}{n-1} + 4\right) T(r, g) + S(r, g). \end{aligned}$$

This implies that $n^2 - 8n + 3 \leq 0$, contradicting $n \geq 12$.

If $a - b - 1 = 0$, then (3.19) turns out to be

$$F(z) = \frac{(b+1)G(z)}{bG(z) + 1}. \tag{3.21}$$

Using a same method above, we also deduce a contradiction.

Case 3.2. $b = -1$ and $a \neq -1$.

Otherwise, if $b = -1$ and $a = -1$, we obtain $F(z)G(z) = 1$. Thus, we get $f(z)g(z) = t$ and $t^{n+1} = 1$ by using similar proof of (3.14). So, we only need to prove it is incorrect if $b = -1$ and $a \neq -1$. Here, (3.19) turns into

$$F(z) = \frac{a}{-G(z) + a + 1}.$$

Using a similar method of Case 3.1, we also deduce a contradiction.

Case 3.3. $b = 0$ and $a \neq 1$.

(3.19) turns into

$$F(z) = \frac{G(z) + a - 1}{a}.$$

Using a similar method of Case 3.1 again, we deduce a contradiction. Thus, $b = 0$ and $a = 1$. Therefore $F(z) = G(z)$. Similar to discuss (3.13), we deduce that $f(z) = tg(z)$ and $t^{n+1} = 1$. The Proof of Theorem 3.2 is completed. □

Proof of Theorem 3.3. Since $P(f(z))f(qz)$ and $P(g(z))g(qz)$ share $1, \infty$ CM, there exists an entire function $\alpha(z)$ such that

$$\frac{P(f(z))f(qz) - 1}{P(g(z))g(qz) - 1} = e^{\alpha(z)}. \tag{3.22}$$

We deduce that $e^{\alpha(z)} \equiv$ constant, say c, since $f(z)$ and $g(z)$ are both meromorphic of zero order. Rewriting (3.22), we obtain

$$cP(g(z))g(qz) = P(f(z))f(qz) - 1 + c. \tag{3.23}$$

We assert that $c = 1$.

If $c \neq 1$, $f(z)$ and $g(z)$ are meromorphic of zero order, then we may apply Nevanlinna main theorems, Lemma 2.2 and (3.23) to obtain

$$\begin{aligned} T(r, P(f(z)f(qz))) &\leq \overline{N}(r, P(f(z))f(qz)) + \overline{N}\left(r, \frac{1}{P(f(z))f(qz)}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{P(f(z))f(qz) - 1 + c}\right) + S(r, f) \\ &\leq \overline{N}(r, P(f(z))) + \overline{N}(r, f(qz)) + \overline{N}\left(r, \frac{1}{P(f(z))}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f(qz)}\right) + \overline{N}\left(r, \frac{1}{P(g(z))g(qz)}\right) + S(r, f) \\ &\leq (m+1)\overline{N}(r, f(z)) + (m+1)\overline{N}\left(r, \frac{1}{f(z)}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{P(g(z))}\right) + \overline{N}\left(r, \frac{1}{g(qz)}\right) + S(r, f) + S(r, g) \\ &\leq 2(m+1)T(r, f(z)) + (m+1)T(r, g(z)) \\ &\quad + S(r, f) + S(r, g). \end{aligned} \tag{3.24}$$

We also deduce from the first main theorem and Lemma 2.4 that

$$\begin{aligned} nT(r,f(z))+S(r,f) &= T(r,P(f(z))) \\ &\le T(r,P(f(z))f(qz))+T\left(r,\frac{1}{f(qz)}\right)+S(r,f) \\ &\le T(r,P(f(z))f(qz))+T(r,f(z))+S(r,f). \end{aligned}$$

Therefore

$$(n-1)T(r,f(z)) \le T(r,P(f(z))f(qz))+O(1). \tag{3.25}$$

Substituting (3.24) into (3.25), we conclude that

$$(n-2m-3)T(r,f(z)) \le (m+1)T(r,g(z))+S(r,f)+S(r,g).$$

Similarly,

$$(n-2m-3)T(r,g(z)) \le (m+1)T(r,f(z))+S(r,f)+S(r,g).$$

By combining the last two inequalities, we get

$$(n-3m-4)[T(r,f(z))+T(r,g(z))] \le S(r,f)+S(r,g),$$

contradicting $n > 3m+4$.

If $c \ne 1$, $f(z)$ and $g(z)$ are entire of zero order, then

$$\begin{aligned} T(r,P(f(z))f(qz)) &\le \overline{N}\left(r,\frac{1}{P(f(z))f(qz)}\right)+\overline{N}\left(r,\frac{1}{P(f(z))f(qz)-1+c}\right) \\ &\quad +S(r,f) \\ &\le \overline{N}\left(r,\frac{1}{P(f(z))f(qz)}\right)+\overline{N}\left(r,\frac{1}{P(g(z))g(qz)}\right) \\ &\le (m+1)T(r,f(z))+(m+1)T(r,g(z)) \\ &\quad +S(r,f)+S(r,g). \end{aligned}$$

Taking using of the Valiron-Mohon'ko lemma, Lemma 2.5 and above inequality, we deduce that

$$\begin{aligned} (n+1)T(r,f(z)) &= T(r,P(f(z))f(z))+S(r,f) \\ &= T(r,P(f(z))f(qz))+S(r,f) \\ &\le (m+1)T(r,f(z))+(m+1)T(r,g(z))+S(r,f)+S(r,g). \end{aligned}$$

Therefore,

$$(n-m)T(r,f(z)) \le (m+1)T(r,g(z))+S(r,f)+S(r,g). \tag{3.26}$$

Similarly,

$$(n-m)T(r,g(z)) \le (m+1)T(r,f(z))+S(r,f)+S(r,g). \tag{3.27}$$

(3.26) and (3.27) yield

$$(n-2m-1)[T(r,f(z))+T(r,g(z))] \le S(r,f)+S(r,g),$$

contradicting $n > 2m+1$.

Thus, $c = 1$ and (3.23) turns into

(3.28) $$P(f(z))f(qz) = P(g(z))g(qz).$$

Set $h(z) = \frac{f(z)}{g(z)}$. We will discuss the following two cases.

Case 3.A. Suppose that $h(z) \equiv$ constant, say h. Substituting $f(z) = hg(z)$ into (3.28), we obtain

$$g(qz)[a_n g(z)^n(h^{n+1} - 1) + a_{n-1}g(z)^{n-1}(h^n - 1) + \cdots + a_1 g(z)(h^2 - 1) + a_0(h - 1)] \equiv 0.$$

Since $g(z)$ is nonconstant meromorphic function, we have $g(qz) \not\equiv 0$. Hence, we get

(3.29)
$$a_n g(z)^n(h^{n+1} - 1) + a_{n-1}g(z)^{n-1}(h^n - 1) + \cdots + a_1 g(z)(h^2 - 1) + a_0(h - 1) \equiv 0.$$

We assert that $h^d = 1$, where d is defined as the assumption of Theorem 3.3. Therefore, $f(z) = tg(z)$ for a constant such that $t^d = 1$. So, we need to prove the following two subcases.

Subcase 3.A.1. Suppose that a_n is the only nonzero coefficient in (3.29). Since $g(z)$ is nonconstant meromorphic function, we obtain $h^{n+1} = 1$.

Subcase 3.A.2. Suppose that a_n is not the only nonzero coefficient in (3.29). If $h^{n+1} \neq 1$, by applying Valiron-Mohon'ko lemma to (3.29), we obtain $T(r, g(z)) = S(r, g)$. This is a impossible. Hence, $h^{n+1} = 1$. Similarly, we also deduce $h^{j+1} = 1$ if $a_j \neq 0$ for $j = 0, 1, \ldots, n$.

Case 3.B. Suppose that $h(z)$ is not a constant. we deduce from (3.28) that $f(z)$ and $g(z)$ satisfy algebraic equation $R(f(z), g(z)) = 0$, where $R(w_1, w_2) = P(w_1)w_1(qz) - P(w_2)w_2(qz)$.

The proof of Theorem 3.3 is completed. □

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