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ON STATISTICAL APPROXIMATION PROPERTIES OF MODIFIED q-BERNSTEIN-SCHURER OPERATORS

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Abstract. In this paper, a kind of modified q-Bernstein-Schurer operators is introduced. The Korovkin type statistical approximation property of these operators is investigated. Then the rates of statistical convergence of these operators are also studied by means of modulus of continuity and the help of functions of the Lipschitz class. Furthermore, a Voronovskaja type result for these operators is given.

1. Introduction

After Philips [13] introduced and studied q analogue of Bernstein polynomials, the applications of q-calculus in the approximation theory become one of the main areas of research. Recently the statistical approximation properties have also been investigated for q-analogue polynomials. For instance, in [1] Kantorovich type q-Bernstein operators; in [6] q-Baskakov-Kantorovich operators; in [12] Kantorovich type q-Szász-Mirakjan operators; in [4] q-Bleimann, Butzer and Hahn operators; in [2] Kantorovich type Lupaş operators based on q-integer were introduced and their statistical approximation properties were studied.

The goal of this paper is to introduce new modification of the q-Bernstein-Schurer operators which were defined by C.-V. Muraur [10] and to study the statistical approximation properties of these operators with the help of the Korovkin type approximation theorem. We also establish the rates of statistical convergence of these operators by means of the modulus of continuity and the help of functions of the Lipschitz class. Furthermore, we give a Voronovskaja type result for these operators.

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Before, proceeding further, let us give some basic definitions and notations from q-calculus. Details on q-integers can be found in [7] and [9].

Let $q > 0$, for each nonnegative integer k, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} (1-q^k)/(1-q), & q \neq 1, \\ k, & q = 1, \end{cases}$$

and

$$[k]_q! := \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k \geq 1, \\ 1, & k = 0, \end{cases}$$

respectively.

Then for $q > 0$ and integers n, k, $n \geq k \geq 0$, we have

$$[k+1]_q = 1 + q[k]_q \quad \text{and} \quad [k]_q + q^k[n-k]_q = [n]_q.$$

For the integers $n, k, n \geq k \geq 0$, the q-binomial coefficients is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Let $q > 0$, for nonnegative integer n, the q-analogue of $(x-a)^n$ is defined by

$$(x-a)_q^n := \begin{cases} 1, & n = 0, \\ (x-a)(x-qa)\cdots(x-q^{n-1}a), & n \geq 1. \end{cases}$$

2. Construction of the operators

Let $p \in \mathbb{N} \bigcup \{0\}$ be fixed. In 1962 F. Schurer [14] introduced and studied the linear positive operators $B_{n,p} : C[0, 1+p] \to C[0, 1]$ defined for any $n \in \mathbb{N}$ and any $f \in C[0, 1+p]$ as follows:

$$B_{n,p}(f; x) = \sum_{k=0}^{n+p} \binom{n+p}{k} x^k(1-x)^{n+p-k} f(k/n),\ x \in [0, 1].$$

Recently, C.-V. Muraur [10] introduced the q-analogue of the above Bernstein-Schurer operators $B_{n,p}(f; x)$ as follows:

$$S_{n,p}(f; q; x) = \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k(1-x)_q^{n+p-k} f([k]_q/[n]_q), \tag{1}$$

where $f \in C[0, 1+p]$, $p \in \mathbb{N} \bigcup \{0\}$ is fixed, $x \in [0, 1]$, $n \in \mathbb{N}$, $0 < q < 1$.

The moments of the operators $S_{n,p}(f; q; x)$ were obtained as follows (see [10]):

Remark 2.1. For $S_{n,p}(t^j; q; x)$, $j = 0, 1, 2$, we have

(i) $S_{n,p}(1; q; x) = 1$;

(ii) $S_{n,p}(t; q; x) = \frac{[n+p]_q x}{[n]_q}$;

(iii) $S_{n,p}(t^2; q; x) = \frac{[n+p]_q}{[n]_q^2}([n+p]_q x^2 + x(1-x))$.

Denoting $r_{n,p}(q,x)=\frac{[n]_q x}{[n+p]_q}$ and changing the scale of reference by replacing the term x by $r_{n,p}(q,x)$, in the definition of $S_{n,p}(f;q;x)$ given by (1), we can define modified q-Bernstein-Schurer operators as follows:

$$\widetilde{S}_{n,p}(f;q;x)=\sum_{k=0}^{n+p}\begin{bmatrix} n+p \\ k \end{bmatrix}_q r_{n,p}^k(q,x)(1-r_{n,p}(q,x))_q^{n+p-k}f([k]_q/[n]_q), \tag{2}$$

where $f\in C[0,1+p]$, $p\in\mathbb{N}\bigcup\{0\}$ is fixed, $x\in[0,1]$, $n\in\mathbb{N}$, $0<q<1$.

Now, we give some lemmas, which are necessary to prove our results.

Lemma 2.2. *Let* $r_{n,p}(q,x)=\frac{[n]_q x}{[n+p]_q}$ *for* $\widetilde{S}_{n,p}(t^j;q;x)$, $j=0,1,2,3,4$. *Then we have*

(i) $\widetilde{S}_{n,p}(1;q;x)=1$;

(ii) $\widetilde{S}_{n,p}(t;q;x)=x$;

(iii) $\widetilde{S}_{n,p}(t^2;q;x)=\frac{[n+p]_q}{[n]_q^2}\left[[n+p]_q r_{n,p}^2(q,x)+r_{n,p}(q,x)(1-r_{n,p}(q,x))\right]$;

(iv)

$$\begin{aligned}&\widetilde{S}_{n,p}(t^3;q;x)\\ &=\frac{[n+p]_q}{[n]_q^3}r_{n,p}(q,x)+\frac{2q+q^2}{[n]_q^3}[n+p]_q[n+p-1]_q r_{n,p}^2(q,x)\\ &\quad+\frac{q^3}{[n]_q^3}[n+p]_q[n+p-1]_q[n+p-2]_q r_{n,p}^3(q,x)\quad \textit{for } n+p\geq 2;\end{aligned}$$

(v)

$$\begin{aligned}&\widetilde{S}_{n,p}(t^4;q;x)\\ &=\frac{[n+p]_q}{[n]_q^4}r_{n,p}(q,x)+\frac{3q+3q^2+q^3}{[n]_q^4}[n+p]_q[n+p-1]_q r_{n,p}^2(q,x)\\ &\quad+\frac{3q^3+2q^4+q^5}{[n]_q^4}[n+p]_q[n+p-1]_q[n+p-2]_q r_{n,p}^3(q,x)\\ &\quad+\frac{q^6}{[n]_q^4}[n+p]_q[n+p-1]_q[n+p-2]_q[n+p-3]_q r_{n,p}^4(q,x)\quad \textit{for } n+p\geq 3.\end{aligned}$$

Proof. In view of the definition given by (2) and Remark 2.1, we can easily obtain identities (i), (ii), (iii) hold.

(iv) When $j=3$ and $n+p\geq 2$, in view of $[k+1]_q=1+q[k]_q$, we have

$$\begin{aligned}&\widetilde{S}_{n,p}(t^3;q;x)\\ &=\sum_{k=1}^{n+p}\begin{bmatrix} n+p \\ k \end{bmatrix}_q r_{n,p}^k(q,x)(1-r_{n,p}(q,x))_q^{n+p-k}([k]_q/[n]_q)^3\\ &=\frac{1}{[n]_q^3}\sum_{k=0}^{n+p-1}\frac{[n+p]_q!(1+2q[k]_q+q^2[k]_q^2)}{[k]_q![n+p-k-1]_q!}r_{n,p}^{k+1}(q,x)(1-r_{n,p}(q,x))_q^{n+p-k-1}\end{aligned}$$

$$= \frac{[n+p]_q}{[n]_q^3} r_{n,p}(q,x)$$
$$+ \frac{2q+q^2}{[n]_q^3} \sum_{k=1}^{n+p-1} \frac{[n+p]_q!}{[k-1]_q![n+p-k-1]_q!} r_{n,p}^{k+1}(q,x)(1-r_{n,p}(q,x))_q^{n+p-k-1}$$
$$+ \frac{q^3}{[n]_q^3} \sum_{k=2}^{n+p-1} \frac{[n+p]_q!}{[k-2]_q![n+p-k-1]_q!} r_{n,p}^{k+1}(q,x)(1-r_{n,p}(q,x))_q^{n+p-k-1}$$
$$= \frac{[n+p]_q}{[n]_q^3} r_{n,p}(q,x) + \frac{2q+q^2}{[n]_q^3}[n+p]_q[n+p-1]_q r_{n,p}^2(q,x)$$
$$+ \frac{q^3}{[n]_q^3}[n+p]_q[n+p-1]_q[n+p-2]_q r_{n,p}^3(q,x).$$

(v) When $j=4$ and $n+p \geq 3$, similar to the case of $j=3$ and $n+p \geq 2$, by simple calculation we can get the stated result. □

Lemma 2.3. *Let* $0<q<1$, $n \in \mathbb{N}$, $x \in [0,1]$. *Then we have*
(i) $\widetilde{S}_{n,p}(t-x;q;x)=0$;
(ii) $\widetilde{S}_{n,p}((t-x)^2;q;x) \leq \frac{1}{[n]_q}(1-\frac{[n]_q x}{[n+p]_q})$.

Proof. (i) By Lemma 2.2, it is clear that we have $\widetilde{S}_{n,p}(t-x;q;x)=0$.
(ii) In view of Lemma 2.2, for any $x \in [0,1]$, $n \in \mathbb{N}$, we have

$$\begin{aligned}\widetilde{S}_{n,p}((t-x)^2;q;x) &= \widetilde{S}_{n,p}(t^2;q;x) - 2x\widetilde{S}_{n,p}(t;q;x) + x^2 \\ &= \widetilde{S}_{n,p}(t^2;q;x)) - x^2 \\ &= \frac{x}{[n]_q}(1-\frac{[n]_q x}{[n+p]_q}) \leq \frac{1}{[n]_q}(1-\frac{[n]_q x}{[n+p]_q}).\end{aligned}$$ □

Let $q=\{q_n\}$, $0<q_n<1$ be a sequence satisfying the following two expressions:

(3) $$\lim_{n\to\infty} q_n = 1 \text{ and } \lim_{n\to\infty} q_n^n = a \ (a \text{ is a constant}).$$

Here, we can give the following results.

Lemma 2.4. *Let* $q=\{q_n\}$, $0<q_n<1$ *be a sequence satisfying the condition* (3), $x \in [0,1]$. *Then we have*
(i) $\lim_{n\to\infty}[n]_{q_n}\widetilde{S}_{n,p}((t-x)^2;q_n;x) = x(1-x)$;
(ii) $\lim_{n\to\infty}[n]_{q_n}\widetilde{S}_{n,p}((t-x)^4;q_n;x) = 0$.

Proof. (i) Let $q=\{q_n\}$, $0<q_n<1$ be a sequence satisfying the condition (3), we have $\lim_{n\to\infty}\frac{[n]_{q_n}}{[n+p]_{q_n}}=1$, hence, by the proof of Lemma 2.3(ii), we can obtain $\lim_{n\to\infty}[n]_{q_n}\widetilde{S}_{n,p}((t-x)^2;q_n;x) = \lim_{n\to\infty} x(1-\frac{[n]_{q_n}x}{[n+p]_{q_n}}) = x(1-x)$.

(ii) Let $n \geq 3$. In view of Lemma 2.2, using $[n+p]_{q_n} = [n]_{q_n} + q_n^n[p]_{q_n}$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$, we have

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}\widetilde{S}_{n,p}((t-x)^4;q_n;x)\\
=&\lim_{n\to\infty}[n]_{q_n}\Big[\widetilde{S}_{n,p}(t^4;q_n;x)-4x\widetilde{S}_{n,p}(t^3;q_n;x)\\
&+6x^2\widetilde{S}_{n,p}(t^2;q_n;x)-4x^3\widetilde{S}_{n,p}(t;q_n;x)+x^4\Big]\\
=&\lim_{n\to\infty}\Bigg[\frac{[n+p]_{q_n}([n+p]_{q_n}-1)([n+p]_{q_n}-[2]_{q_n})}{[n]^2_{q_n}}r^4_{n,p}(q_n,x)\\
&-4x\frac{[n+p]_{q_n}([n+p]_{q_n}-1)}{[n]_{q_n}}r^3_{n,p}(q_n,x)+6x^2\frac{[n+p]^2_{q_n}}{[n]_{q_n}}r^2_{n,p}(q_n,x)\\
&-3[n]_{q_n}x^4\Bigg]-x^4-3apx^4\\
=&\lim_{n\to\infty}\Bigg[\frac{[n+p]^2_{q_n}([n+p]_{q_n}-1)}{[n]^2_{q_n}}r^4_{n,p}(q_n,x)-4x\frac{[n+p]^2_{q_n}}{[n]_{q_n}}r^3_{n,p}(q_n,x)\\
&+6x^2\frac{[n+p]^2_{q_n}}{[n]_{q_n}}r^2_{n,p}(q_n,x)-3[n]_{q_n}x^4\Bigg]+x^4-3apx^4\\
=&\lim_{n\to\infty}\Bigg[\frac{([n+p]_{q_n}-1)[n]^2_{q_n}}{[n+p]^2_{q_n}}x^4-4\frac{[n]^2_{q_n}}{[n+p]_{q_n}}x^4+3[n]_{q_n}x^4\Bigg]+x^4-3apx^4\\
=&\lim_{n\to\infty}\frac{3[n]_{q_n}}{[n+p]_{q_n}}([n+p]_{q_n}-[n]_{q_n})x^4-3apx^4\\
=&\,0.
\end{aligned}$$

□

3. Statistical approximation of Korovkin type

Now, let us recall the concept of the statistical convergence which was introduced by Fast [5].

Let set $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$, the natural density of K is defined by $\delta(K) := \lim_{n\to\infty}\frac{1}{n}|K_n|$ if the limit exists (see [11], where $|K_n|$ denotes the cardinality of the set K_n).

A sequence $x = \{x_k\}$ is call statistically convergent to a number L, if for every $\varepsilon > 0$, $\delta\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} = 0$. This convergence is denoted as $st - \lim_k x_k = L$.

Note that any convergent sequence is statistically convergent, but not conversely. Details can be found in [3].

In approximation theory, the concept of statistically convergence was used by Gadjiev and Orhan [8]. They proved the following Bohman-Korovkin type approximation theorem for statistically convergence.

Theorem 3.1 (see [8])**.** *If the sequence of linear positive operators $A_n : C[a,b] \to C[a,b]$ satisfies the conditions*

$$st - \lim_n \|A_n(e_\upsilon;\cdot) - e_\upsilon\|_{C[a,b]} = 0$$

for $e_\upsilon(t) = t^\upsilon$, $\upsilon = 0,1,2$, then for any $f \in C[a,b]$,

$$st - \lim_n \|A_n(f;\cdot) - f\|_{C[a,b]} = 0.$$

Corollary 3.2. *Let interval $[c,d] \subseteq [a,b]$. If the sequence of linear positive operators $A_n : C[a,b] \to C[c,d]$ satisfies the conditions*

$$st - \lim_n \|A_n(e_\upsilon;\cdot) - e_\upsilon\|_{C[c,d]} = 0$$

for $e_\upsilon(t) = t^\upsilon$, $\upsilon = 0,1,2$, then for any $f \in C[a,b]$,

$$st - \lim_n \|A_n(f;\cdot) - f\|_{C[c,d]} = 0.$$

Proof. Similar to the proof of Theorem 3.1 (see [8]), we can get the desired conclusion. Here, the proof is omitted. □

Theorem 3.3. *Let $q = \{q_n\}$, $0 < q_n < 1$ be a sequence satisfying the following condition*

$$st - \lim_n q_n = 1,\ st - \lim_n q_n^n = c\,(c < 1). \tag{4}$$

Then for any $f \in C[0,1+p]$, we have

$$st - \lim_n \|\widetilde{S}_{n,p}(f;q_n;\cdot) - f\|_{C[0,1]} = 0.$$

Proof. By Corollary 3.2, for any $f \in C[0,1+p]$, enough to prove that $st - \lim_n \|\widetilde{S}_{n,p}(e_\upsilon;q_n;\cdot) - e_\upsilon\|_{C[0,1]} = 0$ for $e_\upsilon(t) = t^\upsilon$, $\upsilon = 0,1,2$.

By Lemma 2.2(i), we can easily get

$$st - \lim_n \|\widetilde{S}_{n,p}(e_0;q_n;\cdot) - e_0\|_{C[0,1]} = 0. \tag{5}$$

By Lemma 2.2(ii), we can easily get

$$st - \lim_n \|\widetilde{S}_{n,p}(e_1;q_n;\cdot) - e_1\|_{C[0,1]} = 0. \tag{6}$$

By Lemma 2.2(iii), we have

$$\|\widetilde{S}_{n,p}(e_2;q_n;\cdot) - e_2\|_{C[0,1]} = \|\frac{e_1}{[n]_{q_n}}(1 - \frac{[n]_{q_n} e_1}{[n+p]_{q_n}})\|_{C[0,1]} \le \frac{1}{[n]_{q_n}}.$$

Now for every given $\varepsilon > 0$, let us define the following sets: $T = \{k : \|\widetilde{S}_{k,p}(e_2;q_k;\cdot) - e_2\|_{C[0,1]} \ge \varepsilon\}$, $T_1 = \{k : \frac{1}{[k]_{q_k}} \ge \varepsilon\}$.

It is clear that $T \subseteq T_1$, so we get

$$\delta\{k \le n : \|\widetilde{S}_{k,p}(e_2;q_k;\cdot) - e_2\|_{C[0,1]} \ge \varepsilon\} \le \delta\{k \le n : \frac{1}{[k]_{q_k}} \ge \varepsilon\}.$$

By condition (4), we have

$$st - \lim_n \frac{1}{[n]_{q_n}} = 0,$$

so, we can get

$$st - \lim_n \|\widetilde{S}_{n,p}(e_2; q_n; \cdot) - e_2\|_{C[0,1]} = 0. \tag{7}$$

In view of the equalities (5), (6) and (7), the proof is complete. □

4. Rates of statistical convergence

In this section, we will give the rates of statistical convergence of the modified q-Bernstein-Schurer operators $\widetilde{S}_{n,p}(f; q; x)$.

Let $f \in C[0, 1 + p]$, for any $\delta > 0$, the usual modulus of continuity for f is defined as $\omega(f; \delta) = \sup\limits_{0<h\leq\delta} \sup\limits_{x,x+h\in[0,1+p]} |f(x + h) - f(x)|$.

By the property of the usual modulus of continuity, we have $\lim_{\delta\to 0^+} \omega(f; \delta) = 0$ for $f \in C[0, 1 + p]$.

Let $f \in C[0, 1 + p]$ for any $t \in [0, 1 + p]$ and $x \in [0, 1]$. Then we have $|f(t) - f(x)| \leq \omega(f; |t - x|)$, so for any $\delta > 0$, we get

$$\omega(f; |t - x|) \leq \begin{cases} \omega(f; \delta), & |t - x| < \delta, \\ \omega(f; \frac{(t-x)^2}{\delta}), & |t - x| \geq \delta. \end{cases}$$

In the light of $\omega(f; \lambda\delta) \leq (1 + \lambda)\omega(f; \delta)$ for $\lambda > 0$, it is clear that we have

$$|f(t) - f(x)| \leq (1 + \delta^{-2}(t - x)^2)\omega(f; \delta) \tag{8}$$

for any $t \in [0, 1 + p]$, $x \in [0, 1]$ and any $\delta > 0$.

Now, we give the rates of statistical convergence of these operators $\widetilde{S}_{n,p}(f; q; x)$ to the function $f \in C[0, 1 + p]$ by means of modulus of continuity.

Theorem 4.1. *Let $q = \{q_n\}$, $0 < q_n < 1$ be a sequence satisfying the condition (4). Then for any $f \in C[0, 1 + p]$ and $x \in [0, 1]$, we have*

$$|\widetilde{S}_{n,p}(f; q_n; x) - f(x)| \leq 2\omega(f; \delta_n(x)),$$

where

$$\delta_n(x) = \left[\frac{1}{[n]_{q_n}}(1 - \frac{[n]_{q_n} x}{[n + p]_{q_n}})\right]^{1/2}. \tag{9}$$

Proof. Using the linearity and positivity of these operators $\widetilde{S}_{n,p}(f; q; x)$, by Lemma 2.3(ii) and the inequality (8), for any $f \in C[0, 1 + p]$ and $x \in [0, 1]$, we get

$$\begin{aligned} |\widetilde{S}_{n,p}(f; q; x) - f(x)| &\leq \widetilde{S}_{n,p}(|f(t) - f(x)|; q; x) \\ &\leq (1 + \delta^{-2}\widetilde{S}_{n,p}((t - x)^2; q; x))\omega(f; \delta) \\ &\leq \left[1 + \delta^{-2} \frac{1}{[n]_q}(1 - \frac{[n]_q x}{[n + p]_q})\right] \omega(f; \delta). \end{aligned}$$

Taking $q = \{q_n\}$, $0 < q_n < 1$ be a sequence satisfying the condition (4) and choosing $\delta = \delta_n(x)$ as in (9), we have $|\widetilde{S}_{n,p}(f; q_n; x) - f(x)| \le 2\omega(f; \delta_n(x))$. □

From the condition (4), by simple calculation, we can get $st - \lim\limits_n \delta_n(x) = 0$, which can educe $st - \lim\limits_n \omega(f; \delta_n(x)) = 0$. This gives the pointwise rate of statistical convergence of these operators $\widetilde{S}_{n,p}(f; q; x)$ to the function $f \in C[0, 1+p]$.

In Theorem 4.1, replacing the condition (4) by the condition (3), similar to the proof of Theorem 4.1, we can give the rate of pointwise convergence of these operators $\widetilde{S}_{n,p}(f; q; x)$ to the function $f \in C[0, 1+p]$ by means of usual modulus of continuity as follows:

Proposition 4.2. *Let $q = \{q'_n\}$, $0 < q'_n < 1$ be a sequence satisfying the condition* (3)*. Then for any $f \in C[0, 1+p]$ and $x \in [0, 1]$, we have*

$$|\widetilde{S}_{n,p}(f; q'_n; x) - f(x)| \le 2\omega(f; \delta'_n(x)),$$

where

$$\delta'_n(x) = \left[\frac{1}{[n]_{q'_n}}(1 - \frac{[n]_{q'_n} x}{[n+p]_{q'_n}})\right]^{1/2}. \tag{10}$$

Proof. Using the linearity and positivity of these operators $\widetilde{S}_{n,p}(f; q; x)$, by Lemma 2.3(ii) and the inequality (8), for any $f \in C[0, 1+p]$ and $x \in [0, 1]$, we get

$$\begin{aligned} |\widetilde{S}_{n,p}(f; q; x) - f(x)| &\le \widetilde{S}_{n,p}(|f(t) - f(x)|; q; x) \\ &\le (1 + \delta^{-2}\widetilde{S}_{n,p}((t-x)^2; q; x))\omega(f; \delta) \\ &\le \left[1 + \delta^{-2}\frac{1}{[n]_q}(1 - \frac{[n]_q x}{[n+p]_q})\right]\omega(f; \delta). \end{aligned}$$

Taking $q = \{q'_n\}$, $0 < q'_n < 1$ be a sequence satisfying the condition (3) and choosing $\delta = \delta'_n(x)$ as in (10), we have $|\widetilde{S}_{n,p}(f; q'_n; x) - f(x)| \le 2\omega(f; \delta'_n(x))$. □

Remark 4.3. We compare the result in Theorem 4.1 with the result in Proposition 4.2. To outward seeming these two results are similar completely, however, their conditions are different. In Theorem 4.1, $q = \{q_n\}$, $0 < q_n < 1$ be a sequence satisfying the condition (4), and in Proposition 4.2, $q = \{q'_n\}$, $0 < q'_n < 1$ be a sequence satisfying the condition (3). Obviously, the condition (3) is stronger than the condition (4). Thus the result in Theorem 4.1 and the result in Proposition 4.2 are different essentially.

Corollary 4.4. *Let $q = \{q_n\}$, $0 < q_n < 1$ be a sequence satisfying the condition* (4)*. Then for any $f \in C[0, 1+p]$, we have*

$$\|\widetilde{S}_{n,p}(f; q_n; \cdot) - f\|_{C[0,1]} \le 2\omega(f; \eta_n),$$

where

$$\eta_n = \left(\frac{1}{[n]_{q_n}}\right)^{1/2}. \tag{11}$$

Proof. Since for any $x \in [0,1]$, we have $0 \le \frac{1}{[n]_{q_n}}(1 - \frac{[n]_{q_n} x}{[n+p]_{q_n}}) \le \frac{1}{[n]_{q_n}}$, thus, in view of the monotonicity of the usual modulus of continuity, we can get $\omega(f;\delta_n(x)) \le \omega(f;\eta_n)$, where $\delta_n(x)$ and η_n are given in (9) and (11) respectively. So, by Theorem 4.1, for any $x \in [0,1]$, we have

$$|\widetilde{S}_{n,p}(f;q_n;x) - f(x)| \le 2\omega(f;\delta_n(x)) \le 2\omega(f;\eta_n),$$

which implies the proof is complete. □

Theorem 4.5. *Let $q = \{q_n\}$, $0 < q_n < 1$ be a sequence satisfying the condition* (4)*. Then for any $f \in C^1[0,1+p]$ and $x \in [0,1]$, we have*

$$|\widetilde{S}_{n,p}(f;q_n;x) - f(x)| \le 2\delta_n(x)\omega(f';\delta_n(x)),$$

where $\delta_n(x)$ is given by (9)*.*

Proof. Let $f \in C^1[0,1+p]$. For any $t \in [0,1+p]$, $x \in [0,1]$, we have

$$f(t) - f(x) - f'(x)(t-x) = \int_x^t (f'(u) - f'(x))du,$$

so, for any $\delta > 0$, we get

$$\begin{aligned}|f(t) - f(x) - f'(x)(t-x)| &\le |\int_x^t |f'(u) - f'(x)|du| \\ &\le \omega(f';|t-x|)|t-x| \\ &\le \omega(f';\delta)(|t-x| + \delta^{-1}(t-x)^2).\end{aligned}$$

Thus, we have

$$\begin{aligned}&|\widetilde{S}_{n,p}(f(t) - f(x) - f'(x)(t-x);q;x)| \\ &\le \omega(f';\delta)(\widetilde{S}_{n,p}(|t-x|;q;x) + \delta^{-1}\widetilde{S}_{n,p}((t-x)^2;q;x)).\end{aligned}$$

Using Cauchy-Schwartz inequality, we obtain

$$\widetilde{S}_{n,p}(|t-x|;q;x) \le \sqrt{\widetilde{S}_{n,p}(1;q;x)}\sqrt{\widetilde{S}_{n,p}((t-x)^2;q;x)},$$

so, we have

$$\begin{aligned}&|\widetilde{S}_{n,p}(f(t) - f(x) - f'(x)(t-x);q;x)| \\ &\le \omega(f';\delta)(\sqrt{\widetilde{S}_{n,p}(1;q;x)} + \delta^{-1}\sqrt{\widetilde{S}_{n,p}((t-x)^2;q;x)})\sqrt{\widetilde{S}_{n,p}((t-x)^2;q;x)}.\end{aligned}$$

Thus, by Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned}&|\widetilde{S}_{n,p}(f;q;x) - f(x)| \\ &\le |f'(x)||\widetilde{S}_{n,p}(t-x;q;x)|\end{aligned}$$

$$+\,\omega(f';\delta)(1+\delta^{-1}\sqrt{\widetilde{S}_{n,p}((t-x)^2;q;x)})\sqrt{\widetilde{S}_{n,p}((t-x)^2;q;x)}$$
$$\le \omega(f';\delta)\left\{1+\delta^{-1}\left[\frac{1}{[n]_q}(1-\frac{[n]_qx}{[n+p]_q})\right]^{1/2}\right\}\left[\frac{1}{[n]_q}(1-\frac{[n]_qx}{[n+p]_q})\right]^{1/2}.$$

Taking $q=\{q_n\}$, $0<q_n<1$ be a sequence satisfying the condition (4) and choosing $\delta=\delta_n(x)$ as in (9), then from the above inequality we obtain the desired result. □

Corollary 4.6. *Let* $q=\{q_n\}$, $0<q_n<1$ *be a sequence satisfying the condition* (4). *Then for any* $f\in C^1[0,1+p]$, *we have*

$$\|\widetilde{S}_{n,p}(f;q_n;\cdot)-f\|_{C[0,1]}\le 2\eta_n\omega(f';\eta_n),$$

where η_n *is given by* (11).

Proof. Similar to the proof of Corollary 4.4, we can get the desired result. Here, the proof is omitted. □

Next, we give the rate of statistical convergence of the operators $\widetilde{S}_{n,p}(f;q;x)$ with the help of functions of the Lipschitz class.

Theorem 4.7. *Let* $0<\alpha\le 1$, $M>0$, $f\in Lip_M^\alpha$ *on* $[0,1+p]$, *also, let* $q=\{q_n\}$, $0<q_n<1$ *be a sequence satisfying the condition* (4). *Then for any* $x\in[0,1]$, *we have*

$$|\widetilde{S}_{n,p}(f;q_n;x)-f(x)|\le M\delta_n^\alpha(x),$$

where $\delta_n(x)$ *is given by* (9).

Proof. Let $0<\alpha\le 1$, $M>0$, $f\in Lip_M^\alpha$ on $[0,1+p]$. Then we obtain $f\in C[0,1+p]$, also, for any $t\in[0,1+p]$ and any $x\in[0,1]$, we have $|f(t)-f(x)|\le M|t-x|^\alpha$. Thus, using the linearity and positivity of the operator $\widetilde{S}_{n,p}(f;q;x)$, we obtain $|\widetilde{S}_{n,p}(f;q;x)-f(x)|\le \widetilde{S}_{n,p}(|f(t)-f(x)|;q;x)\le M\widetilde{S}_{n,p}(|t-x|^\alpha;q;x)$. Using the Hölder inequality with $m=\frac{2}{\alpha}$, $n=\frac{2}{2-\alpha}$, we get

$$|\widetilde{S}_{n,p}(f;q;x)-f(x)|\le M[\widetilde{S}_{n,p}((t-x)^2;q;x)]^{\alpha/2}.$$

So, by Lemma 2.3(ii), we have $|\widetilde{S}_{n,p}(f;q;x)-f(x)|\le M[\frac{1}{[n]_q}(1-\frac{[n]_qx}{[n+p]_q})]^{\alpha/2}$. Taking $q=\{q_n\}$, $0<q_n<1$ be a sequence satisfying the condition (4), the desired result follows immediately. □

5. A Voronovskaja type theorem

In this section, we give a Voronovskaja type theorem of the $\widetilde{S}_{n,p}(f;q;x)$.

Theorem 5.1. *Let* $x\in[0,1]$ *and* $q=\{q_n\}$, $0<q_n<1$ *be a sequence satisfying the condition* (3). *Then for any* $f\in C^2[0,1+p]$, *we have*

$$\lim_{n\to\infty}[n]_{q_n}(\widetilde{S}_{n,p}(f;q_n;x)-f(x))=\frac{x(1-x)}{2}f''(x).$$

Proof. Let $f \in C^2[0,1+p]$ and $x \in [0,1]$ be fixed. For any $t \in [0,1+p]$, by the Taylor formula, we have

$$f(t) - f(x) = f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + r(t,x)(t-x)^2,$$

where $r(t,x) \in C[0,1+p]$ and $\lim_{t\to x} r(t,x) = 0$. By Lemma 2.2, we get
(12)

$$\widetilde{S}_{n,p}(f;q_n;x) - f(x) = \frac{f''(x)}{2}\widetilde{S}_{n,p}((t-x)^2;q_n;x) + \widetilde{S}_{n,p}(r(t,x)(t-x)^2;q_n;x).$$

In view of $\lim_{t\to x} r(t,x) = 0$, for any $\varepsilon > 0$, there exists a constant $\delta > 0$, when $t \in U_x(\delta) = \{t \mid t \in [0,1+p] \text{ and } |t-x| < \delta\}$, we have $|r(t,x)| < \epsilon$.

Denoting

$$\lambda_\delta(t,x) = \begin{cases} 1, & |t-x| \geq \delta, \\ 0, & |t-x| < \delta, \end{cases}$$

then $|r(t,x)(t-x)^2| \leq \varepsilon(t-x)^2 + \lambda_\delta(t,x)|r(t,x)|(t-x)^2$, $|\widetilde{S}_{n,p}(r(t,x)(t-x)^2;q_n;x)| \leq \varepsilon\widetilde{S}_{n,p}((t-x)^2;q_n;x) + \widetilde{S}_{n,p}(\lambda_\delta(t,x)|r(t,x)|(t-x)^2;q_n;x)$.

Since $[0,1+p]\backslash U_x(\delta)$ is compact, also, $r(t,x)$ is bounded on $[0,1+p]$, so, there exists a constant $L > 0$, for any $t \in [0,1+p]$, we obtain $\lambda_\delta(t,x)|r(t,x)|(t-x)^2 \leq L(t-x)^4$, hence

$$|\widetilde{S}_{n,p}(r(t,x)(t-x)^2;q_n;x)| \leq \varepsilon\widetilde{S}_{n,p}((t-x)^2;q_n;x) + L\widetilde{S}_{n,p}((t-x)^4;q_n;x).$$

Note that $\varepsilon > 0$ is arbitrary, by Lemma 2.4, we obtain

$$\lim_{n\to\infty} [n]_{q_n} |\widetilde{S}_{n,p}(r(t,x)(t-x)^2;q_n;x)| = 0,$$

so

$$\lim_{n\to\infty} [n]_{q_n} \widetilde{S}_{n,p}(r(t,x)(t-x)^2;q_n;x) = 0. \tag{13}$$

By equalities (12), (13) and Lemma 2.4, we can obtain the desired result. □

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