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BERTRAND CURVES IN NON-FLAT 3-DIMENSIONAL (RIEMANNIAN OR LORENTZIAN) SPACE FORMS

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Abstract. Let $\mathbb{M}_q^3(c)$ denote the 3-dimensional space form of index $q = 0, 1$, and constant curvature $c \neq 0$. A curve α immersed in $\mathbb{M}_q^3(c)$ is said to be a Bertrand curve if there exists another curve β and a one-to-one correspondence between α and β such that both curves have common principal normal geodesics at corresponding points. We obtain characterizations for both the cases of non-null curves and null curves. For non-null curves our theorem formally agrees with the classical one: non-null Bertrand curves in $\mathbb{M}_q^3(c)$ correspond with curves for which there exist two constants $\lambda \neq 0$ and μ such that $\lambda\kappa + \mu\tau = 1$, where κ and τ stand for the curvature and torsion of the curve. As a consequence, non-null helices in $\mathbb{M}_q^3(c)$ are the only twisted curves in $\mathbb{M}_q^3(c)$ having infinite non-null Bertrand conjugate curves. In the case of null curves in the 3-dimensional Lorentzian space forms, we show that a null curve is a Bertrand curve if and only if it has non-zero constant second Frenet curvature. In the particular case where null curves are parametrized by the pseudo-arc length parameter, null helices are the only null Bertrand curves.

1. Introduction

Saint-Venant proposed [22], in the middle of the 19th century, the question whether upon the ruled surface generated by the principal normals of a curve in the 3-dimensional Euclidean space $\mathbb{R}^3$, a second curve can exist having the same principal normals. This question was answered by Bertrand [5], who showed that a necessary and sufficient condition is that a linear relationship with constant coefficients shall exist between the curvature and torsion of the

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given original curve. Pair of curves of this kind have been called Conjugate Bertrand curves or, more commonly, Bertrand curves.

The study of this kind of curves has been extended to many other ambient spaces. Pears [21] studied this problem for curves in the n-dimensional Euclidean space $\mathbb{R}^n$, $n > 3$, and showed that a Bertrand curve in $\mathbb{R}^n$ must belong to a three-dimensional subspace $\mathbb{R}^3 \subset \mathbb{R}^n$ (see also [1, p. 176] and [18]). Many authors have studied Bertrand curves in other ambient spaces ([8], [10], [11], [13], [16], [20], [23]).

Let $\mathbb{M}^3_q(c) \subset \mathbb{R}^4_v$ denote the 3-dimensional space form of index $q = 0, 1$, and constant curvature $c \neq 0$. A curve α immersed in $\mathbb{M}^3_q(c)$ is said to be a Bertrand curve if there exists another curve β and a one-to-one correspondence between α and β such that both curves have common principal normal geodesics at corresponding points (see Section 2 for details; Subsection 2.1 for non-null curves and Subsection 2.2 for null curves). The curves α and β are said to be a pair of Bertrand curves in $\mathbb{M}^3_q(c)$.

The first properties about non-null Bertrand curves are presented in Section 3:

a) the 'distance' between corresponding points is constant;

b) the angle between the tangent vectors at corresponding points (considered as vectors in $\mathbb{R}^4_v$) is constant;

c) the angle between the binormal vectors at corresponding points (considered as vectors in $\mathbb{R}^4_v$) is constant.

We obtain some relationships among curvatures and torsions of a pair of non-null Bertrand curves, and show that every non-null plane curve in $\mathbb{M}^3_q(c)$ is a Bertrand curve with infinite Bertrand conjugate plane curves. Here, a plane curve means a curve lying in a totally geodesic 2-dimensional surface of $\mathbb{M}^3_q(c)$. Our main theorem for non-null curves is a result which formally agrees with the classical one: non-null Bertrand curves in $\mathbb{M}^3_q(c)$ correspond with curves for which there exist two constants $\lambda \neq 0$ and μ such that $\lambda\kappa + \mu\tau = 1$, where κ and τ stand for the curvature and torsion of the curve. We conclude this section with a characterization of non-null helices as the only twisted non-null curves in $\mathbb{M}^3_q(c)$ having infinite non-null Bertrand conjugate curves. After that we present some examples.

Last section is devoted to the study of null Bertrand curves. As in the non-null case, the first result is that the 'distance' between corresponding points is constant. The key result in the null case states that Frenet curvatures κ_1^α, κ_2^α, κ_1^β and κ_2^β of a pair of null Bertrand curves α and β must satisfy the following relations:

(i) $\kappa_1^\alpha\kappa_1^\beta > 0$ is constant;

(ii) $\kappa_2^\alpha = \kappa_2^\beta$ is constant.

After that we show that a null curve is a Bertrand curve if and only if it has non-zero constant second Frenet curvature. In the particular case where null curves are parametrized by the pseudo-arc length parameter, we show that null helices are the only null Bertrand curves.

2. Setup

Let $\mathbb{R}^{n+1}_v$ denote the $(n+1)$-dimensional pseudo-Euclidean space of index $v \geq 0$ with metric tensor given by

$$\langle , \rangle = -\sum_{i=1}^{v} \mathrm{d}x_i^2 \otimes \mathrm{d}x_i^2 + \sum_{j=v+1}^{n+1} \mathrm{d}x_j^2 \otimes \mathrm{d}x_j^2,$$

where $(x_1, \ldots, x_{n+1})$ stands for the usual rectangular coordinates in $\mathbb{R}^{n+1}$. The pseudo-Euclidean hypersphere of index $q \geq 0$ and curvature $c > 0$ is defined by

$$\mathbb{S}^n_q(c) = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_q \mid \langle x, x\rangle = \frac{1}{c^2}\},$$

and the pseudo-Euclidean hyperbolic space of index $q \geq 0$ and curvature $c < 0$ is defined by

$$\mathbb{H}^n_q(c) = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_{q+1} \mid \langle x, x\rangle = -\frac{1}{c^2}\}.$$

Without loss of generality we can assume that the constant curvature c is equal to ± 1. In order to simplify our notation and computations, we will denote by $\mathbb{M}^n_q(c)$ the pseudo-Euclidean hypersphere $\mathbb{S}^n_q(1)$ or the pseudo-Euclidean hyperbolic space $\mathbb{H}^n_q(-1)$ according to $c = 1$ or $c = -1$, respectively. We will use $\mathbb{R}^{n+1}_v$ to denote the corresponding pseudo-Euclidean space where $\mathbb{M}^n_q(c)$ lives, so that $v = q$ if $c = 1$ and $v = q + 1$ if $c = -1$.

Let us recall the usual definition of wedge product (or cross product) in $\mathbb{R}^{n+1}_v$. Given n vectors $X_1, X_2, \ldots, X_n$ in $\mathbb{R}^{n+1}_v$, we define its wedge product $X_1 \times \cdots \times X_n$ as the unique vector in $\mathbb{R}^{n+1}_v$ such that

$$\langle X_1 \times \cdots \times X_n, Y\rangle = \det(X_1, \ldots, X_n, Y) \quad \text{for every } Y \in \mathbb{R}^{n+1}_v. \tag{2.1}$$

The wedge product $\times$ in $\mathbb{R}^{n+1}_v$ induces another wedge product $\wedge$ in $\mathbb{M}^n_q(c)$ as follows. Given a point $p \in \mathbb{M}^n_q(c)$ and $n-1$ vectors $X_1, X_2, \ldots, X_{n-1}$ in $T_p\mathbb{M}^n_q(c) \subset \mathbb{R}^{n+1}_v$, we define its wedge product $X_1 \wedge \cdots \wedge X_{n-1}$ as the vector in $T_p\mathbb{M}^n_q(c)$ given by

$$X_1 \wedge \cdots \wedge X_{n-1} = p \times X_1 \times \cdots \times X_{n-1}.$$

It is easy to see that the usual orientation in $\mathbb{R}^{n+1}_v$ induces an orientation in $\mathbb{M}^n_q(c)$ as follows: a basis $\{X_1, X_2, \ldots, X_n\}$ of $T_p\mathbb{M}^n_q(c)$ is said to be positively oriented if $\{X_1, X_2, \ldots, X_n, p\}$ is a basis of $\mathbb{R}^{n+1}_v$ positively oriented.

Many features of inner product spaces have analogues in the pseudo-Euclidean case. For example, in the Euclidean space $\mathbb{R}^n$ the Schwarz inequality permits the definition of the Euclidean angle θ between two vectors X and Y as the unique number $0 \leq \theta \leq \pi$ such that $\langle X, Y\rangle = |X|\,|Y|\cos\theta$. In the Lorentzian space $\mathbb{L}^n = \mathbb{R}^n_1$ we have the following. Let us consider two non-null vectors $X, Y \in \mathbb{R}^n_1$ such that they span a timelike plane $\mathbb{R}^2_1$; in this plane we can consider an orthonormal basis $\{e_1, e_2\}$, with $\langle e_1, e_1\rangle = -1$ and $\langle e_2, e_2\rangle = 1$. Vectors X, Y can be written in this basis as $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$.

Definition 1 ([6, 7, 19])**.** Let us consider two non-null vectors $X, Y \in \mathbb{R}^n_1$.

a) Let us assume that X and Y are spacelike vectors, then
   - if they span a spacelike plane, there is a unique number $0 \le \theta \le \pi$ such that $\langle X, Y\rangle = |X|\,|Y| \cos\theta$.
   - if they span a timelike plane, there is a unique number $\theta \ge 0$ such that $\langle X, Y\rangle = \varepsilon\,|X|\,|Y| \cosh\theta$, where $\varepsilon = +1$ or $\varepsilon = -1$ according to $\operatorname{sgn}(X_2) = \operatorname{sgn}(Y_2)$ or $\operatorname{sgn}(X_2) \ne \operatorname{sgn}(Y_2)$, respectively.

b) Let us assume that X and Y are timelike vectors, then there is a unique number $\theta \ge 0$ such that $\langle X, Y\rangle = \varepsilon\,|X|\,|Y| \cosh\theta$, where $\varepsilon = +1$ or $\varepsilon = -1$ according to X and Y have different time-orientation or the same time-orientation, respectively.

c) Let us assume that X is spacelike and Y is timelike, then there is a unique number $\theta \ge 0$ such that $\langle X, Y\rangle = \varepsilon\,|X|\,|Y| \sinh\theta$, where $\varepsilon = +1$ or $\varepsilon = -1$ according to $\operatorname{sgn}(X_2) = \operatorname{sgn}(Y_1)$ or $\operatorname{sgn}(X_2) \ne \operatorname{sgn}(Y_1)$, respectively.

Given two non-null vectors $X, Y \in \mathbb{R}^n_1$, the corresponding number θ given above will be called simply the *angle between X and Y*.

2.1. The Frenet apparatus of a non-null curve in $\mathbb{M}^3_q(c)$

Let $\alpha = \alpha(s) : I \subset \mathbb{R} \to \mathbb{M}^3_q(c) \subset \mathbb{R}^4_v$, $q = 0, 1$, be a non-null curve immersed in the 3-dimensional space $\mathbb{M}^3_q(c)$ and assume without loss of generality that α is parametrized by the arclength parameter. If α is not a geodesic and ∇^0 stands for the Levi-Civita connection of $\mathbb{R}^4_v$, then there exists the Frenet frame (positively oriented) $\{T, N, B\}$ along α in $\mathbb{M}^3_q(c)$ such that

$$\begin{aligned}\nabla^0_T T &= -\varepsilon_1 c\alpha + \varepsilon_2 \kappa N,\\ \nabla^0_T N &= -\varepsilon_1 \kappa T + \varepsilon_3 \tau B,\\ \nabla^0_T B &= -\varepsilon_2 \tau N,\end{aligned}$$

where κ and τ denote the curvature and torsion of α, respectively, and $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ stand for the causal characters of $\{T, N, B\}$.

For any point $\alpha(s)$ in the curve α, the *principal normal geodesic* in $\mathbb{M}^3_q(c)$ starting at $\alpha(s)$ is defined as the geodesic curve

$$\gamma^\alpha_s(t) = \exp_{\alpha(s)}(tN(s)) = f(t)\alpha(s) + g(t)N(s), \quad t \in \mathbb{R}, \tag{2.2}$$

where the functions f and g are given by $f(t) = \cos t$ and $g(t) = \sin t$ if $\varepsilon_2 c = 1$, whereas $f(t) = \cosh t$ and $g(t) = \sinh t$ if $\varepsilon_2 c = -1$.

2.2. The Frenet apparatus of a null curve in $\mathbb{M}^3_1(c)$

Let $\alpha : I \subset \mathbb{R} \to \mathbb{M}^3_1(c)$ be a null curve immersed in $\mathbb{M}^3_1(c)$. If α is not a null geodesic then α admits a Frenet frame $\{L = \alpha', W, N\}$ along α, where the metric is given by

$$\langle L, L\rangle = \langle N, N\rangle = 0, \quad \langle L, N\rangle = \delta, \quad \delta = \pm 1, \tag{2.3}$$

$$\langle W, L\rangle = \langle W, N\rangle = 0, \quad \langle W, W\rangle = 1, \tag{2.4}$$

and the constant δ is chosen in such a way that $\{L, W, N\}$ is positively oriented; without loss of generality we can assume $\delta = -1$. The Frenet equations are the following (see [9]):

$$\begin{aligned} \nabla^0_L L &= \kappa_1 W, \\ \nabla^0_L W &= -\kappa_2 L + \kappa_1 N, \\ \nabla^0_L N &= -\kappa_2 W + c\alpha, \end{aligned} \tag{2.5}$$

where the functions κ_1 and κ_2 are called the curvatures functions of the null curve α with respect to the Frenet frame $\{L, W, N\}$. The null curve α is said to be a *null helix* in $\mathbb{M}^3_1(c)$ if κ_1 and κ_2 are constant.

For a null curve which is not a geodesic we can choose a special parameter, the pseudo-arc parameter of the null curve α, characterized by $\langle \nabla_{\alpha'}\alpha', \nabla_{\alpha'}\alpha'\rangle = 1$. In this case, equations (2.5) reduce to

$$\begin{aligned} \nabla^0_L L &= W, \\ \nabla^0_L W &= -\kappa L + N, \\ \nabla^0_L N &= -\kappa W + c\alpha. \end{aligned} \tag{2.6}$$

These equations are called the Cartan equations of the null curve α and the function κ is called the Cartan curvature (see [12]). The fundamental theorem for null curves tell us that κ determines completely the null curve up to Lorentzian transformations (see [12]). Even more, given a function κ we can always construct a null curve, parametrized by the pseudo-arc length parameter, whose curvature function is precisely κ. A non-geodesic null curve parametrized by the pseudo-arc length parameter and admitting a Cartan frame $\{L, W, N\}$ as above is called a Cartan curve.

For any point $\alpha(s)$ in the null curve α, the principal normal geodesic in $\mathbb{M}^3_1(c)$ starting at $\alpha(s)$ is defined as the geodesic curve

$$\gamma^\alpha_s(t) = \exp_{\alpha(s)}(tW(s)) = f(t)\alpha(s) + g(t)W(s), \quad t \in I \subset \mathbb{R}, \tag{2.7}$$

where the functions f and g are given by $f(t) = \cos t$ and $g(t) = \sin t$ in the De Sitter space $\mathbb{S}^3_1(1)$, whereas $f(t) = \cosh t$ and $g(t) = \sinh t$ in the anti-De Sitter space $\mathbb{H}^3_1(-1)$.

3. Non-null Bertrand curves

Definition 2. A non-null curve α with non-zero curvature is said to be a *Bertrand curve* if there exists another immersed non-null curve $\beta = \beta(\sigma) : J \subset \mathbb{R} \to \mathbb{M}^3_q(c)$, $\beta \neq \pm\alpha$, and a one-to-one correspondence between α and β (i.e. a map $s \in I \to \sigma(s) \in J$), such that both curves have common principal normal geodesics at corresponding points. We will said that β is a Bertrand mate (or Bertrand conjugate) of α; the curves α and β are called a pair of non-null Bertrand curves.

Let $\alpha(s)$ and $\beta(\sigma)$ be a pair of Bertrand curves, then there exists a differentiable function $a(s)$ such that

$$\beta(\sigma(s)) = f(a(s))\alpha(s) + g(a(s))N_\alpha(s), \tag{3.8}$$

where $\{T_\alpha, N_\alpha, B_\alpha\}$ denotes the Frenet frame along α and $\beta(\sigma(s))$ is the point in β corresponding to $\alpha(s)$.

Proposition 1. *Let α and β be a pair of non-null Bertrand curves in $\mathbb{M}^3_q(c)$. Then the following properties hold*:

a) *The function $a(s)$ is constant.*
b) *The angle between the tangent vectors at corresponding points* (*considered as vectors in $\mathbb{R}^4_v$*) *is constant.*
c) *The angle between the binormal vectors at corresponding points* (*considered as vectors in $\mathbb{R}^4_v$*) *is constant.*

Proof. a) Since α and β have common principal normal geodesics at corresponding points, we have

$$\left.\frac{d}{dt}\right|_{t=a(s)} \gamma^\alpha_s(t) = \varepsilon N_\beta(\sigma(s)), \quad \varepsilon = \pm 1,$$

and then, since $f' = -\varepsilon_2 c g$ and $g' = f$, we obtain

$$N_\beta(\sigma(s)) = -\varepsilon\varepsilon_2 c g(a(s))\alpha(s) + \varepsilon f(a(s))N_\alpha(s), \tag{3.9}$$

where $\{T_\beta, N_\beta, B_\beta\}$ denotes the Frenet frame along β, with causal characters given by $\{\delta_1, \delta_2, \delta_3\}$. On the other hand, the tangent vector to β is given by

$$\begin{aligned}\frac{d}{ds}\beta(\sigma) = {}& a'(s)f'(a(s))\alpha(s) + (f(a(s)) - \varepsilon_1\kappa_\alpha(s)g(a(s)))T_\alpha(s) \\ &+ a'(s)g'(a(s))N_\alpha(s) + \varepsilon_3\tau_\alpha(s)g(a(s))B_\alpha(s).\end{aligned}$$

But $\frac{d}{ds}\beta(\sigma) = \sigma'(s)T_\beta(\sigma(s))$ and thus we have

$$0 = \left\langle \frac{d}{ds}\beta(\sigma), N_\beta(\sigma) \right\rangle = \varepsilon a'(s)(\varepsilon_2 f(a)^2 + c g(a)^2) = \varepsilon\varepsilon_2 a'(s),$$

and the proof finishes.

b) A straightforward computation shows that

$$\begin{aligned}\frac{d}{ds}\langle T_\alpha(s), T_\beta(\sigma(s))\rangle = {}& \langle -\varepsilon_1 c\alpha(s) + \varepsilon_2\kappa_\alpha(s)N_\alpha(s), T_\beta(\sigma(s))\rangle \\ &+ \sigma'(s)\langle T_\alpha(s), -\delta_1 c\beta(\sigma(s)) + \delta_2\kappa_\beta(\sigma(s))N_\beta(\sigma(s))\rangle.\end{aligned} \tag{3.10}$$

On the other hand, it is easy to get the following formula

$$T_\beta(\sigma(s)) = \frac{1}{\sigma'(s)}\left((f(a) - \varepsilon_1\kappa_\alpha(s)g(a))T_\alpha(s) + \varepsilon_3\tau_\alpha(s)g(a)\ B_\alpha(s)\right), \tag{3.11}$$

that jointly with (3.8), (3.9) and (3.10) yields

$$\frac{d}{ds}\langle T_\alpha(s), T_\beta(\sigma(s))\rangle = 0,$$

showing the claim.

c) Let θ denote the constant angle between $T_\alpha(s)$ and $T_\beta(\sigma(s))$, then it is not difficult to see that we can write

$$T_\beta(\sigma(s)) = \varphi(\theta)\ T_\alpha(s) + \eta(\theta)\ B_\alpha(s), \tag{3.12}$$

where the functions φ and η satisfy the following condition:

$$\delta_1 = \varepsilon_1\varphi^2 + \varepsilon_3\eta^2. \tag{3.13}$$

By using the wedge product in $\mathbb{R}^4_v$ we can compute the binormal vector B_α of the curve α as follows

$$B_\alpha(s) = -\varepsilon_3\ \alpha(s)\times T_\alpha(s)\times N_\alpha(s).$$

From this formula, it is not difficult to get

$$B_\beta(\sigma(s)) = \varepsilon\delta_3(-\varepsilon_1\eta(\theta)\ T_\alpha(s) + \varepsilon_3\varphi(\theta)\ B_\alpha(s)), \tag{3.14}$$

and finally we deduce

$$\langle B_\alpha(s), B_\beta(\sigma(s))\rangle = \varepsilon\delta_3\varphi(\theta) = \text{constant}. \qquad \square$$

Claims b) and c) of Proposition 1 admit an alternative statement. To do that, let $P_a^0(\gamma_s^\alpha)$ denote the parallel transport along the geodesic $\gamma_s^\alpha(t)$ between the points $\gamma_s^\alpha(a) = \beta(\sigma(s))$ and $\gamma_s^\alpha(0) = \alpha(s)$. Then for every differentiable vector field $Y \in \mathfrak{X}(\beta)$ along β we can define a differentiable vector field $X \in \mathfrak{X}(\alpha)$ along α by the equation

$$X(s) = P_a^0(\gamma_s^\alpha)(Y(\sigma(s))).$$

In short, we will write $X = PY$.

An interesting property of the parallel transport is that if two non-degenerate hypersurfaces $S_1 \subset \mathbb{R}^4_v$ and $S_2 \subset \mathbb{R}^4_v$ are tangent along a parametrized non-null curve α and v_0 is a vector of $T_{\alpha(s_0)}S_1 = T_{\alpha(s_0)}S_2$, then $V(s)$ is the parallel transport of v_0 along α relative to the hypersurface S_1 if and only if $V(s)$ is the parallel transport of v_0 along α relative to the hypersurface S_2. Indeed, the covariant derivative DV/ds of V is the same for both hypersurfaces.

Using this property we can show that the parallel transport along a geodesic $\gamma \subset \mathbb{M}^3_q(c)$ of a vector orthogonal to γ' is a constant vector field. In fact, let $\gamma = \gamma(s)$ be a geodesic of $\mathbb{M}^3_q(c)$, then there exists a 2-plane $\Pi \subset \mathbb{R}^4_v$ passing through the origin such that $\gamma \subset \mathbb{M}^3_q(c) \cap \Pi$. Let v_0 be a vector tangent to $\mathbb{M}^3_q(c)$ at some point $p = \gamma(s_0)$, and assume that v_0 is orthogonal to $\gamma'(s_0)$. Consider the hypercylinder $C = \gamma \times \Pi^\perp$ tangent to $\mathbb{M}^3_q(c)$ along γ, then the constant vector field $V(s) = v_0$ is the parallel transport of v_0 along γ relative to the hypercylinder. But the above property yields that $V(s)$ is also the parallel transport of v_0 along γ relative to the 3-space $\mathbb{M}^3_q(c)$.

Therefore, we have the following result.

Proposition 2. *Let α and β be a pair of non-null Bertrand curves in $\mathbb{M}^3_q(c)$. Then*

a) *the angle between T_α and PT_β is constant.*
b) *the angle between B_α and PB_β is constant.*

Proof. Observe that $(PT_\beta)(s) = P^0_a(\gamma^\alpha_s)(T_\beta(\sigma(s))) = T_\beta(\sigma(s))$ and $(PB_\beta)(s) = B_\beta(\sigma(s))$, so that the result is a direct consequence of Claims b) and c) of Proposition 1. □

The following theorem is an extension of a result obtained by Lai [17] for Bertrand curves in the 3-dimensional Euclidean space.

Theorem 3. *Let α and β be a pair of non-null Bertrand curves in $\mathbb{M}^3_q(c)$. Then there exist two constants a and θ such that the following relations hold:*

a) $(f(a) - \varepsilon_1 g(a)\kappa_\alpha)\eta(\theta) = \varepsilon_3 g(a)\varphi(\theta)\tau_\alpha,$
b) $(f(a) + \varepsilon\delta_1 g(a)\kappa_\beta)\eta(\theta) = \varepsilon_3 g(a)\varphi(\theta)\tau_\beta,$
c) $(f(a) - \varepsilon_1 g(a)\kappa_\alpha)(f(a) + \varepsilon\delta_1 g(a)\kappa_\beta) = \varepsilon_1\delta_1\varphi(\theta)^2,$
d) $g(a)^2\tau_\alpha\tau_\beta = \varepsilon_1\delta_1\eta(\theta)^2,$

where κ_α, τ_α, κ_β and τ_β denote the curvature and torsion of α and β, respectively.

Proof. a) Taking covariant derivative in (3.8) and using (3.12) we have

$$\frac{d}{ds}\beta(\sigma(s)) = \sigma'(s)\varphi(\theta)T_\alpha(s) + \sigma'(s)\eta(\theta)B_\alpha(s).$$

On the other hand, by using the Frenet equations we get

$$\frac{d}{ds}\beta(\sigma(s)) = (f(a) - \varepsilon_1 g(a)\kappa_s(s))T_\alpha(s) + \varepsilon_3 g(a)\tau_\alpha(s)B_\alpha(s),$$

where we have used that $a(s) = a$ is constant. Last two equations leads to

$$\sigma'(s)\varphi(\theta) = f(a) - \varepsilon_1 g(a)\kappa_\alpha(s), \tag{3.15}$$

$$\sigma'(s)\eta(\theta) = \varepsilon_3 g(a)\tau_\alpha(s), \tag{3.16}$$

from which we deduce a).

b) Now we need to write the Frenet frame of α in terms of the Frenet frame of β:

$$\begin{aligned}
\alpha(s(\sigma)) &= f(a)\beta(\sigma) - \varepsilon g(a)N_\beta(\sigma),\\
T_\alpha(s(\sigma)) &= \varepsilon_1\delta_1\varphi(\theta)T_\beta(\sigma) - \varepsilon\eta(\theta)B_\beta(\sigma),\\
N_\alpha(s(\sigma)) &= \varepsilon_2 c g(a)\beta(\sigma) + \varepsilon f(a)N_\beta(\sigma),\\
B_\alpha(s(\sigma)) &= \delta_1\varepsilon_3\eta(\theta)T_\beta(\sigma) + \varepsilon\varphi(\theta)B_\beta(\sigma).
\end{aligned}$$

Reasoning as in case a) we deduce

$$\varepsilon_1\delta_1 s'(\sigma)\varphi(\theta) = f(a) + \varepsilon\delta_1 g(a)\kappa_\beta(\sigma), \tag{3.17}$$

$$\varepsilon_1\delta_1 s'(\sigma)\eta(\theta) = \varepsilon_3 g(a)\tau_\beta(\sigma), \tag{3.18}$$

from which we deduce b).

c) It is a consequence of Eqs. (3.15) and (3.17).

d) It is a consequence of Eqs. (3.16) and (3.18). □

If α and β are non-null Bertrand curves in $\mathbb{M}^3_q(c)$, part d) of above theorem implies that the product of their torsions at corresponding points is constant and non-negative (or non-positive), according to the curves have the same causal character (or different causal character, respectively). This is a generalization of the classical Schell's theorem for curves in $\mathbb{R}^3$.

A non-null curve α immersed in $\mathbb{M}^3_q(c)$ is said to be a *plane curve* if it lies in a totally geodesic surface $\mathbb{M}^2 \subset \mathbb{M}^3_q(c)$. As a consequence, its torsion is zero at all points. A twisted curve α in $\mathbb{M}^3_q(c)$ (i.e., a curve with torsion $\tau \neq 0$) is said to be a *helix* if its curvature and torsion are non-zero constants. More generally, a curve $\alpha = \alpha(s)$ in $\mathbb{M}^3_q(c)$ is said to be a *general helix* if there exists a Killing vector field $V(s)$ with constant length along α and such that the angle between V and α' is a non-zero constant along α. The vector field V is called an axis of the general helix α. Observe that plane curves and helices are obvious examples of general helices.

Proposition 4 (Bertrand plane curves)**.** a) *Every non-null plane curve in* $\mathbb{M}^3_q(c)$ *is a Bertrand curve and it has infinite Bertrand conjugate plane curves.*

b) *If a non-null Bertrand curve* α *in* $\mathbb{M}^3_q(c)$ *has a Bertrand conjugate* β *which is a plane curve, then* α *is a plane curve on the same totally geodesic surface* $\mathbb{M}^2$*.*

Proof. a) Let α be a plane curve in $\mathbb{M}^3_q(c)$, and for each real number $a \in (-\varepsilon, \varepsilon)$ let β_a be the curve in $\mathbb{M}^3_q$ defined by

$$\beta_a(s) = f(a)\alpha(s) + g(a)N_\alpha(s). \tag{3.19}$$

We will see that β_a is a Bertrand conjugate for all $a \in (-\varepsilon, \varepsilon)$. Taking covariant derivative in (3.19), and using the Frenet equations, we can assume without loss of generality that

$$T_{\beta_a}(\sigma(s)) = T_\alpha(s), \tag{3.20}$$

$$\sigma'(s) = f(a) - \varepsilon_1 g(a)\kappa_\alpha(s), \tag{3.21}$$

where $\sigma = \sigma(s)$ denotes the arc-length parameter of β_a. Taking again covariant derivative in (3.20) we easily get

$$N_{\beta_a}(\sigma(s)) = -\varepsilon_2 c g(a)\alpha(s) + f(a)N_\alpha(s), \tag{3.22}$$

$$\kappa_{\beta_a}(\sigma(s)) = \frac{\varepsilon_1 c g(a) + \varepsilon_2 f(a)\kappa_\alpha(s)}{f(a) - \varepsilon_1 g(a)\kappa_\alpha(s)}, \tag{3.23}$$

and then the principal normal geodesic starting at a point $\beta_a(\sigma_0)$, $\sigma_0 = \sigma(s_0)$, is given by

$$\gamma(t) = f(t)\beta_a(\sigma_0) + g(t)N_{\beta_a}(\sigma_0) = f(t+a)\alpha(s_0) + g(t+a)N_\alpha(s_0),$$

which is nothing but a reparametrization of the principal normal geodesic starting at $\alpha(s_0)$.

Finally, by taking covariant derivative in (3.22), and using the Frenet equations, we have

$$\sigma'(s)\frac{d}{d\sigma}N_{\beta_a}(\sigma(s)) = -(\varepsilon_2 c g(a) + \varepsilon_1 f(a)\kappa_\alpha(s))T_\alpha(s),$$

that jointly with (3.20) yields $\tau_{\beta_a} = 0$, that is, β_a is also a plane curve in $\mathbb{M}^3_q(c)$.

b) Since $\tau_\beta = 0$, then from Theorem 3(d) we get $\eta(\theta) = 0$ (and thus $\varphi(\theta)^2 = 1$). By using Theorem 3(a) we deduce that $g(a)\tau_\alpha = 0$. If $g(a) = 0$, then $f(a)^2 = 1$ and $\alpha = \pm\beta$, and so it is a plane curve on the same totally geodesic surface; otherwise, $\tau_\alpha = 0$ and we reach the same conclusion. □

In [2], Barros shows the Lancret theorem in the 3-sphere: A curve α in $\mathbb{S}^3$ is a Lancret curve if and only if either (1) $\tau_\alpha \equiv 0$ and α is a curve in some unit 2-dimensional sphere $\mathbb{S}^2(1)$, or (2) there exist a constant $b \neq 0$ such that $\tau_\alpha = b\kappa_\alpha \pm 1$. Barros' result can be extended to curves immersed into a 3-dimensional ambient space endowed with an indefinite metric; in [4] the authors obtain a characterization of (non-null or null) Lancret curves in 3-dimensional Lorentzian space forms ($\mathbb{L}^3$, $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$). Now we extend this result to non-null Bertrand curves in $\mathbb{M}^3_q(c)$.

Theorem 5. *A non-null twisted curve α in $\mathbb{M}^3_q(c)$ is a Bertrand curve if and only if there exist two constants $\lambda \neq 0$ and μ such that $\lambda\kappa_\alpha + \mu\tau_\alpha = 1$.*

Proof. Let α be a non-null Bertrand curve. If α is not a plane curve ($\tau_\alpha \neq 0$), then from Theorem 3(a) we have that $\lambda\kappa_\alpha + \mu\tau_\alpha = 1$, for constants $\lambda = \varepsilon_1\frac{g(a)}{f(a)}$ and $\mu = \varepsilon_3\frac{g(a)\varphi(\theta)}{f(a)\eta(\theta)}$.

Now, let us suppose that $\lambda\kappa_\alpha + \mu\tau_\alpha = 1$ for certain constants $\lambda \neq 0$ and μ. Let β be the curve in $\mathbb{M}^3_q(c)$ defined by

$$\beta(s) = f(a)\alpha(s) + g(a)N_\alpha(s), \tag{3.24}$$

where a is the number such that $\lambda f(a) - \varepsilon_1 g(a) = 0$. We will see that $\beta(\sigma)$ is a Bertrand conjugate, where $\sigma = \sigma(s)$ denotes the arc-length parameter of β. Taking covariant derivative in (3.24) and using the Frenet equations we obtain

$$N_\beta(\sigma(s)) = -\varepsilon\varepsilon_2 c g(a)\alpha(s) + \varepsilon f(a)N_\alpha(s), \quad \varepsilon = \pm 1. \tag{3.25}$$

Then the principal normal geodesic starting at a point $\beta(\sigma_0)$, $\sigma_0 = \sigma(s_0)$, is given by

$$\gamma(t) = f(t)\beta(\sigma_0) + g(t)N_\beta(\sigma_0) = f(t+\varepsilon a)\alpha(s_0) + g(t+\varepsilon a)N_\alpha(s_0),$$

which is nothing but a reparametrization of the principal normal geodesic starting at $\alpha(s_0)$. That concludes the proof. □

Proposition 6. *Let α be a twisted non-null curve in $\mathbb{M}^3_q(c)$. Then the following conditions are equivalent*:

a) *α is a helix.*
b) *α has infinite Bertrand conjugate curves.*
c) *α has two Bertrand conjugate curves.*

Proof. a) ⇒ b) Let us assume that κ_α and τ_α are non-zero constants. Then it is very easy to see that there are infinite pairs of constants (λ, μ) such that $\lambda\kappa_\alpha + \mu\tau_\alpha = 1$; but for each different linear relationship we can construct a different Bertrand conjugate curve, which is also a helix.

b) ⇒ c) Nothing to prove.

c) ⇒ a) If α has two Bertrand conjugate curves β_1 and β_2, then we can find four constants $a_1 \neq 0$, $a_2 \neq 0$, θ_1 and θ_2 such that

$$\varepsilon_1 \frac{g(a_1)}{f(a_1)}\kappa_\alpha(s) + \varepsilon_3 \frac{g(a_1)\varphi(\theta_1)}{f(a_1)\eta(\theta_1)}\tau_\alpha(s) = 1,$$
$$\varepsilon_1 \frac{g(a_2)}{f(a_2)}\kappa_\alpha(s) + \varepsilon_3 \frac{g(a_2)\varphi(\theta_2)}{f(a_2)\eta(\theta_2)}\tau_\alpha(s) = 1,$$

where $a_1 \neq a_2$ because of β_1 and β_2 are two different Bertrand conjugate curves. By taking covariant derivative in these equations we obtain

$$\varepsilon_1 \kappa'_\alpha(s) + \varepsilon_3 \frac{\varphi(\theta_1)}{\eta(\theta_1)}\tau'_\alpha(s) = 0,$$
$$\varepsilon_1 \kappa'_\alpha(s) + \varepsilon_3 \frac{\varphi(\theta_2)}{\eta(\theta_2)}\tau'_\alpha(s) = 0.$$

Therefore, $\kappa'_\alpha(s) = \tau'_\alpha(s) = 0$, that is, α has both constant curvature and constant torsion. That concludes the proof. □

3.1. Some examples

We have seen that plane curves and helices are Bertrand curves; in fact, we have shown that these curves are the only ones with infinite Bertrand conjugate curves (see Propositions 4 and 6). On the other hand, from Theorem 5 we deduce that any curve with constant curvature is also a Bertrand curve.

Example 1 (Euler spirals or Clothoids)**.** We present a 4-parametric family of Bertrand curves with non-constant curvature and non-constant torsion (see [15] and references therein). A curve $\alpha = \alpha(s)$ in $\mathbb{M}^3_q(c)$ is said to be an Euler spiral (or Clothoid or Cornu spiral) if both its curvature and torsion evolve linearly along the curve. Thus, there exist constants κ_0, τ_0, γ, δ such that

$$\kappa(s) = \kappa_0 + \gamma s, \quad \tau(s) = \tau_0 + \delta s.$$

It is easy to see that an Euler spiral is a Bertrand curve provided that $\kappa_0\delta - \tau_0\gamma \neq 0$.

Example 2 (n-Clothoids)**.** As a generalization of classical Clothoid, we define the n-Clothoid (or Clothoid of degree n) in $\mathbb{M}^3_q(c)$ as the curve whose curvature and torsion are given by

$$\kappa(s) = \kappa_0 + \gamma s^n, \quad \tau(s) = \tau_0 + \delta s^n,$$

where κ_0, τ_0, γ and δ are constants (see [14]). As before, it is very easy to see that n-Clothoids are Bertrand curves provided that $\kappa_0\delta - \tau_0\gamma \neq 0$.

Example 3 (Generalized conical helices)**.** A twisted curve $\alpha = \alpha(s)$ in $\mathbb{M}^3_q(c)$ with non constant curvatures is said to be a *conical helix* if both the curvature radius $1/\kappa$ and the torsion radius $1/\tau$ evolve linearly along the curve. Thus, there exist constants $r_0, r_1, \gamma \neq 0$ and $\delta \neq 0$, such that

$$\kappa(s) = \frac{\gamma}{s + r_0}, \quad \tau(s) = \frac{\delta}{s + r_1}.$$

In the particular case where r_0 and r_1 vanishes, the curve is called an standard conical helix. Now we are going to slightly generalize that definition. A twisted curve $\alpha = \alpha(s)$ in $\mathbb{M}^3_q(c)$ is said to be a *generalized conical helix* if there exist constants $r_0, r_1, \gamma \neq 0, \gamma_0, \delta \neq 0$ and δ_0, such that

$$\kappa(s) = \frac{\gamma}{s + r_0} + \gamma_0, \quad \tau(s) = \frac{\delta}{s + r_1} + \delta_0.$$

Then it is easy to see that α is a Bertrand curve provided that $r_0 = r_1$ and $\delta\gamma_0 - \gamma\delta_0 \neq 0$.

Example 4 (General helices)**.** We know that a twisted curve α in $\mathbb{S}^3$ is a general helix if there exists a constant b such that $\tau = b\kappa \pm 1$ (see [2]), then it is a Bertrand curve provided that $b \neq 0$. A nice construction of this family of curves is given in [2] by using the usual Hopf map $\pi : \mathbb{S}^3 \to \mathbb{S}^2(4)$. In the Lorentzian case, the problem has been studied and solved in [4]; in this case, the general helix is said to be degenerate or non-degenerate according to its axis is a null or non-null vector, respectively. As in the spherical case, a twisted curve α in $\mathbb{H}^3_1$ is a general helix if there exists a constant b such that $\tau = b\kappa \pm 1$. The general helix is degenerate if and only if $b = \pm 1$ and its normal vector is spacelike. By using the semi-Riemannian Hopf submersions $\pi_s : \mathbb{H}^3_1 \to \mathbb{H}^2_s(-4)$, $s = 0, 1$, the authors prove that a non-null curve in $\mathbb{H}^3_1$ is a non-degenerate general helix if and only if it is a geodesic in some Hopf cylinder. The definition and basic properties of pseudo-Riemannian submersions π_s are given in [3]. In what follows, we present in a unified way the construction of general helices given in [2] and [4].

Let $\mathbb{M}^2_s(4c)$, $s = 0, 1$, denote the 2-dimensional sphere $\mathbb{S}^2(4)$ or the 2-dimensional hyperbolic plane $\mathbb{H}^2_s(-4)$, $s = 0, 1$, according to $c > 0$ or $c < 0$, respectively. Let us consider $\pi_{qs} : \mathbb{M}^3_q(c) \to \mathbb{M}^2_s(4c)$ the usual Hopf map, which is a Riemannian or semi-Riemannian submersion. Therefore, for any point $p \in \mathbb{M}^3_q(c)$ the tangent space $T_p\mathbb{M}^3_q(c)$ splits into the horizontal plane (which is isometric to $T_{\pi_{qs}(p)}\mathbb{M}^2_s(4c)$) and the vertical line (which is the tangent line to the fiber through p). Let $\beta : I \to \mathbb{M}^2_s(4c)$ be a unit speed curve in $\mathbb{M}^2_s(4c)$ and consider a horizontal lift $\bar{\beta}$. The Hopf cylinder over β is defined as the total lift $M_\beta = \pi^{-1}_{qs}(\beta)$. It is not difficult to see that M_β is a flat surface in $\mathbb{M}^3_q(c)$ and can be parametrized by $X : I \times \mathbb{R} \to \mathbb{M}^3_q(c)$ defined by

$$X(t, z) = f(z)\ \bar{\beta}(t) + g(z)\ V(t),$$

where V is a unit vertical vector field, and functions f and g are given by $f(z) = \cos z$ and $g(z) = \sin z$ when $s = 0$. In the case $s = 1$, the functions are given by $f(z) = \cosh z$ and $g(z) = \sinh z$.

Notice that t-curves are the horizontal lifts of β while z-curves correspond to the fibers. Both families of curves are arclength parametrized and mutually orthogonal; furthermore, they are geodesics in M_β. Even more, every geodesic in M_β can be obtained as the image by X of a straight line in the (t, z)-plane. Thus, if $\alpha = \alpha(s)$ is a geodesic in M_β, then there exist constants a_i, b_i such that

$$\alpha(s) = X(a_1 s+a_2, b_1 s+b_2) = \cos(b_1 s+b_2)\ \bar{\beta}(a_1 s+a_2)+\sin(b_1 s+b_2)\ V(a_1 s+a_2).$$

It is shown in [2] and [4] that each general helix lying fully in $\mathbb{S}^3$ or non-degenerate a general helix lying fully in $\mathbb{H}^3_1$ can be regarded as a geodesic in a certain Hopf cylinder $M_\beta \subset \mathbb{M}^3_q(c)$ over a curve $\beta \subset \mathbb{M}^2_s(4c)$. This provides a beautiful method to construct a family of Bertrand curves in $\mathbb{M}^3_q(c)$.

A final remark. If $\alpha(s)$ is a general helix in $\mathbb{S}^3$ or a non-degenerate general helix in $\mathbb{H}^3_1$ with Frenet frame $\{T_\alpha, N_\alpha, B_\alpha\}$, then an axis is given by ([2], [4]):

$$f(\theta)T_\alpha(s) + g(\theta)B_\alpha(s),$$

for a certain constant θ, where f and g are the functions defined above. Since $\alpha(s)$ is a Bertand curve, there exists a conjugate $\beta(\sigma)$, so that equation (3.12) holds. Then we deduce that the unit tangent vector T_β is parallel (in $\mathbb{R}^4_v$) to the axis of the general helix α.

Example 5 (Degenerate general helices)**.** We know that a non-null curve in $\mathbb{H}^3_1$ is a degenerate general helix if and only if $\tau = \pm\kappa \pm 1$ and its normal vector is spacelike, [4]. It is also shown in [4] that degenerate general helices in $\mathbb{H}^3_1$ can be obtained as geodesics in some flat B-scroll over a null curve as follows. Let β be a curve in $\mathbb{H}^3_1$ with curvature κ and torsion τ satisfying that $\tau = \kappa + \varepsilon_1$ ($\varepsilon_1 = \langle \beta', \beta' \rangle$) and the normal vector N_β of β is spacelike (the other cases are similar). We define the null curve α in $\mathbb{H}^3_1$ by the equation

$$\alpha(s) = \beta(s) - \frac{1}{2}s(T_\beta(s) - B_\beta(s)).$$

It is not difficult to see that the Cartan frame $\{A, B, C\}$ along α is given by (see [12])

$$\begin{aligned} A(s) &= -\frac{\varepsilon_1}{2}s\beta(s) + \frac{1}{2}(T_\beta(s) + B_\beta(s)) + \frac{\varepsilon_1}{2}sN_\beta(s), \\ B(s) &= -\varepsilon_1(T_\beta(s) - B_\beta(s)), \end{aligned}$$

$$C(s) = -\frac{1}{2}s(T_\beta(s) - B_\beta(s)) + N_\beta(s).$$

Let $S_{\alpha,B}$ be the flat B-scroll in $\mathbb{H}^3_1$ parametrized by $X(s,t) = \alpha(s) + tB(s)$. Then it is clear that $\beta(s) = X(s, -\frac{\varepsilon_1}{2}s)$ and so β is a geodesic of that B-scroll.

4. Null Bertrand curves

Definition 3. A null curve α with non-zero first curvature is said to be a *null Bertrand curve* if there exists another null curve $\beta = \beta(\sigma) : J \subset \mathbb{R} \to \mathbb{M}^3_1(c)$, $\beta \neq \pm\alpha$, and a one-to-one correspondence between α and β (i.e., a map $s \in I \to \sigma(s) \in J$), such that both curves have common principal normal geodesics at corresponding points. We will said that β is a null Bertrand mate (or null Bertrand conjugate) of α; the curves α and β are called a pair of null Bertrand curves.

Let $\alpha(s)$ and $\beta(\sigma)$ be a pair of null Bertrand curves, then there exists a differentiable function $a(s)$ such that

$$\beta(\sigma(s)) = \gamma^\alpha_s(a(s)) = f(a(s))\alpha(s) + g(a(s))W_\alpha(s), \tag{4.26}$$

where $\{L_\alpha, W_\alpha, N_\alpha\}$ denotes the Frenet frame along α and $\beta(\sigma(s))$ is the point in β corresponding to $\alpha(s)$.

Proposition 7. *Let α and β be a pair of null Bertrand curves in $\mathbb{M}^3_1(c)$. Then the function $a(s)$ is a non-zero constant.*

Proof. Since α and β have common principal normal geodesics at corresponding points, we have

$$\left.\frac{d}{dt}\right|_{t=a(s)} \gamma^\alpha_s(t) = \varepsilon W_\beta(\sigma(s)), \quad \varepsilon = \pm 1.$$

By using that $f' = -cg$ and $g' = f$, we obtain

$$W_\beta(\sigma(s)) = -\varepsilon c g(a(s))\alpha(s) + \varepsilon f(a(s))W_\alpha(s), \tag{4.27}$$

where $\{L_\beta, W_\beta, N_\beta\}$ denotes the Frenet frame along $\beta(\sigma)$. On the other hand, the tangent vector to β is given by

$$\begin{aligned}\frac{d}{ds}\beta(\sigma(s)) &= a'(s)f'(a(s))\alpha(s) + (f(a(s)) - g(a(s))\kappa^\alpha_2(s))L_\alpha(s)\\ &\quad + a'(s)g'(a(s))W_\alpha(s) + g(a(s))\kappa^\alpha_1(s)N_\alpha(s).\end{aligned} \tag{4.28}$$

But $\frac{d}{ds}\beta(\sigma) = \sigma'(s)L_\beta(\sigma(s))$ and then, from (4.27) and (4.28), we have

$$0 = \left\langle \frac{d}{ds}\beta(\sigma), W_\beta(\sigma) \right\rangle = \varepsilon a'(s)(f(a)^2 + cg(a)^2) = \varepsilon a'(s),$$

and the proof finishes. □

In the following result we characterize null Bertrand curves in $\mathbb{M}^3_1(c)$.

Proposition 8. *Let $\alpha = \alpha(s)$ and $\beta = \beta(\sigma)$ be a pair of null Bertrand curves in $\mathbb{M}_1^3(c)$. Then there exist two constants a and b such that the following relations hold*:

(i) $\kappa_1^\alpha(s)\kappa_1^\beta(\sigma(s)) = \left(\dfrac{b}{g(a)}\right)^2$.

(ii) $\kappa_2^\alpha = \kappa_2^\beta = \dfrac{f(a)}{g(a)}$.

Proof. Since a is constant, from (4.28) we deduce

$$\frac{d}{ds}\beta(\sigma(s)) = (f(a) - g(a)\kappa_2^\alpha(s))L_\alpha(s) + g(a)\kappa_1^\alpha(s)N_\alpha(s), \tag{4.29}$$

and then

$$0 = \left\langle \frac{d}{ds}\beta(\sigma(s)), \frac{d}{ds}\beta(\sigma(s))\right\rangle = -(f(a) - g(a)\kappa_2^\alpha(s))g(a)\kappa_1^\alpha(s).$$

Since $g(a)\kappa_1^\alpha$ is non-zero, we deduce that $\kappa_2^\alpha(s) = \frac{f(a)}{g(a)}$ is constant. From here and (4.29) we get $\sigma'(s)L_\beta(\sigma(s)) = g(a)\kappa_1^\alpha(s)N_\alpha(s)$, hence there exists a non-zero function $b(s)$ such that

$$L_\beta(\sigma(s)) = b(s)N_\alpha(s), \quad b(s) = \frac{g(a)\kappa_1^\alpha(s)}{\sigma'(s)}, \tag{4.30}$$

and then

$$N_\beta(\sigma(s)) = \frac{1}{b(s)}L_\alpha(s). \tag{4.31}$$

Since $\{L_\alpha, W_\alpha, N_\alpha\}$ and $\{L_\beta, W_\beta, N_\beta\}$ are positively oriented, from (4.27), (4.30) and (4.31) we deduce $\varepsilon = -1$. By taking derivative in (4.27) we obtain

$$\frac{d}{ds}W_\beta(\sigma(s)) = (cg(a) + f(a)\kappa_2^\alpha(s))L_\alpha(s) - f(a)\kappa_1^\alpha(s)N_\alpha(s), \tag{4.32}$$

but we also have

$$\begin{aligned}\frac{d}{ds}W_\beta(\sigma(s)) &= \sigma'(s)\frac{d}{d\sigma}W_\beta(\sigma(s)) \\ &= -\sigma'(s)\kappa_2^\beta(\sigma(s))L_\beta(\sigma(s)) + \sigma'(s)\kappa_1^\beta(\sigma(s))N_\beta(\sigma(s)).\end{aligned} \tag{4.33}$$

Combining equations (4.30) to (4.33) we deduce

$$\sigma'(s)\kappa_1^\beta(\sigma(s)) = (cg(a) + f(a)\kappa_2^\alpha(s))b(s), \tag{4.34}$$

$$\sigma'(s)\kappa_2^\beta(\sigma(s))b(s) = f(a)\kappa_1^\alpha(s). \tag{4.35}$$

Since $b(s)\sigma'(s) = g(a)\kappa_1^\alpha(s)$, from (4.35) we deduce

$$\kappa_2^\beta(\sigma(s)) = \frac{f(a)}{g(a)} = \kappa_2^\alpha(s).$$

Now we will prove that $b(s)$ is constant. By taking derivative in (4.30) we get

$$\frac{d}{ds}L_\beta(\sigma(s)) = b'(s)N_\alpha(s) - b(s)\kappa_2^\alpha(s)W_\alpha(s) + cb(s)\alpha(s),$$

but we also have

$$\frac{d}{ds}L_\beta(\sigma(s)) = \sigma'(s)\frac{d}{d\sigma}L_\beta(\sigma(s)) = \sigma'(s)\kappa_1^\beta(\sigma(s))W_\beta(\sigma(s)).$$

Last two equations jointly with (4.27) imply $b'(s) = 0$, showing that b is constant. Now we use (4.34) and (4.35) to get (i). □

Proposition 9. *Let α be a null curve in $\mathbb{M}_1^3(c)$. Then α is a null Bertrand curve if and only if it has non-zero constant second curvature.*

Proof. From Proposition 8(ii), we only need to prove the converse part. Let us suppose that α has non-zero constant second curvature κ_2^α. Let a be the non-zero constant such that $f(a) - \kappa_2^\alpha g(a) = 0$ and take any non-zero constant b. Let us define σ by the equation

$$\sigma(s) = \frac{g(a)}{b}\int \kappa_1^\alpha(u)du.$$

Observe that σ is defined to fulfill the second relation in (4.30).

Let $\beta = \beta(\sigma)$ be the null curve defined by

$$\beta(\sigma(s)) = f(a)\alpha(s) + g(a)W_\alpha(s). \tag{4.36}$$

We are going to prove that β is a Bertrand conjugate. Taking covariant derivative in (4.36) and using Frenet equations we obtain

$$L_\beta(\sigma(s)) = bN_\alpha(s).$$

Taking again covariant derivative here we get

$$\frac{g(a)}{b}\kappa_1^\alpha(s)\kappa_1^\beta(\sigma(s))W_\beta(\sigma(s)) = bc\alpha(s) - b\kappa_2^\alpha W_\alpha(s).$$

This equation implies

$$\kappa_1^\alpha(s)\kappa_1^\beta(\sigma(s)) = \varepsilon\frac{b^2}{g(a)^2}, \quad \varepsilon = \pm 1, \tag{4.37}$$

$$W_\beta(\sigma(s)) = \varepsilon(cg(a)\alpha(s) - f(a)W_\alpha(s)), \tag{4.38}$$

$$N_\beta(\sigma(s)) = \frac{1}{b}L_\alpha(s). \tag{4.39}$$

By using that Frenet frames are positively oriented we obtain $\varepsilon = 1$, then from (4.26) and (4.27) the principal normal geodesic starting at a point $\beta(\sigma_0)$, $\sigma_0 = \sigma(s_0)$, is given by

$$\gamma_{\sigma_0}^\beta(t) = f(t)\beta(\sigma_0) + g(t)W_\beta(\sigma_0) = f(t+a)\alpha(s_0) + g(t+a)W_\alpha(s_0) = \gamma_{s_0}^\alpha(t+a),$$

showing that null curves α and β have common principal normal geodesics. That concludes the proof. □

If we consider that our null curves are parametrized by the pseudo-arc length parameter, then the first curvature is always constant and equal to 1. Then we can state the following consequence.

Corollary 10. *Let α be a null curve in $\mathbb{M}_1^3(c)$ parametrized by the pseudo-arc length parameter with Cartan curvature κ. Then α is a null Bertrand curve if and only if it is a null helix.*

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