Bull. Korean Math. Soc. **50** (2013), No. 4, pp. 1099–1108
http://dx.doi.org/10.4134/BKMS.2013.50.4.1099

MANNHEIM CURVES IN 3-DIMENSIONAL SPACE FORMS

Jin Ho Choi, Tae Ho Kang, and Young Ho Kim

Abstract. We define the Mannheim curve in a 3-dimensional Riemannian manifold, which is a generalization of the Mannheim curve in Euclidean space. In particular, we study the Mannheim curves and their partner curves in 3-dimensional space forms.

1. Introduction

In a classical differential geometry, the properties of curves in a 3-dimensional Euclidean or Minkowski space are generally characterized by the algebraic equations concerning their curvature and torsion function. For example, the general helices and the Bertrand curves in a 3-dimensional Euclidean space $\mathbb{E}^3$ are characterized respectively by

$$a\kappa - \tau = 0 \quad \text{and} \quad a\kappa + b\tau = 1$$

for some constants $a \neq 0$ and b, where κ and τ are the curvature function and the torsion function under consideration.

On the other hand, it is well-known that the Mannheim curves in $\mathbb{E}^3$ are characterized by

$$\kappa = a(\kappa^2 + \tau^2)$$

for a constant $a \neq 0$. Moreover, H. Liu and F. Wang proved that Mannheim partner curves in $\mathbb{E}^3$ are characterized by

$$\kappa' = \frac{\kappa}{a}(1 + a^2\tau^2),$$

where $'$ denotes the differentiation with respect to the arc length parameter of a given curve (see, [2]).

Recently, the authors defined the Bertrand curves in a 3-dimensional Riemannian manifold and proved that those in 3-dimensional space forms are characterized by the same algebraic equation as that in $\mathbb{E}^3$ (see, [1]).

Received March 30, 2012; Revised October 25, 2012.

2010 *Mathematics Subject Classification.* Primary 53A04, 53A35.

Key words and phrases. Mannheim curves, Mannheim partner curves, space form.

The third author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0007184).

©2013 The Korean Mathematical Society

In this paper, we define the Mannheim curves in a 3-dimensional Riemannian manifold and give the algebraic characterizations of Mannheim curves and their partner curves in 3-dimensional space forms.

2. Preliminaries and definitions

Let $M := (M, \langle , \rangle)$ be a 3-dimensional Riemannian manifold and ∇ denotes the Levi-Civita connection of M. Let T_pM denotes the set of all tangent vectors to M at $p \in M$. For a vector v in T_pM, we define the *norm* of v by $||v|| := \sqrt{\langle v, v\rangle}$.

Let $\alpha : I \to M$ be a smooth curve in M. If $\| \alpha'(s) \| = 1$ for each $s \in I$, the curve α is called a *unit speed curve*. In this case, the parameter s can be assumed to be the *arc length parameter* of α.

A vector field X on M along α is said to be *parallel along* α if $\nabla_s X = 0$ for all s, where ∇_s denotes the covariant derivative along α. When a vector field X on M is parallel along α, a vector $X_{\alpha(s_1)}$ at $\alpha(s_1)$ is called the *parallel displacement* of a vector $X_{\alpha(s_0)}$ at $\alpha(s_0)$ along α.

A smooth curve $\alpha : I \to M$ is called a *geodesic* in M if its velocity vector field α' is parallel along α. A non-geodesic unit speed smooth curve α is called a *Frenet curve* on M if there exists an orthonormal frame $\{T = \alpha', N, B\}$ along α and smooth functions $\kappa_\alpha > 0$ and τ_α satisfying the following system of ordinary differential equations

$$\nabla_s \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{2.1}$$

The equation (2.1) is called the *Frenet formula* of the Frenet curve α. The functions κ_α, τ_α and the orthonormal frame $\{T, N, B\}$ are called the *curvature*, the *torsion* and the *Frenet frame* of α, respectively. It is well-known that the curvature and the torsion are invariant under the isometries of M. Three unit vector fields T, N and B consisting of the Frenet frame of α are called the *tangent*, *principal normal* and *binormal vector field*, respectively.

Denote the exponential map at $p \in M$ by $\exp_p$. Recall that the *exponential map* $\exp_p : T_pM \to M$ at $p \in M$ is defined by

$$\exp_p(v) = \gamma_v(1),$$

where $\gamma_v : [0, \infty) \to M$ is the constant speed geodesic starting from p with the initial velocity $\gamma_v(0) = v$.

Now, we define a Mannheim curve in a 3-dimensional Riemannian manifold M as follows:

Definition 2.1. Let $\alpha(s)$ be a Frenet curve in a 3-dimensional Riemannian manifold M and $\{T_\alpha, N_\alpha, B_\alpha\}$ the Frenet frame of α. Consider a surface X_{N_α} defined by

$$X_{N_\alpha}(s, t) = \exp_{\alpha(s)}\left(tN_\alpha(s)\right).$$

A Frenet curve $\beta = \beta(s)$ in M is called a *Mannheim partner curve* of α if the binormal vector field of β defined by $\beta(s) = X_{N_\alpha}(s, t(s))$ is the parallel displacement of $N_\alpha(s_0)$ or $-N_\alpha(s_0)$ for each $s = s_0$. When a Frenet curve α in M admits its Mannheim partner curve, we call α a *Mannheim curve* in M.

Definition 2.1 is trivially equivalent to the following:

Definition 2.2. For two different Frenet curves $\alpha(s)$ and $\beta(s)$ on M, β is a Mannheim partner curve of α if $\alpha(s) = X_{B_\beta}(s, t(s))$ and for each $s = s_0$, its principal normal vector N_α at $\alpha(s_0)$ is the parallel displacement of B_β or $-B_\beta$ at $\beta(s_0)$ along the ruling curve at $\beta(s_0)$. When a Frenet curve α in M admits its Mannheim partner curve, we call α a *Mannheim curve* on M.

Remark 2.1. Generally, a straight line ℓ in a 3-dimensional Euclidean space $\mathbb{E}^3$ can not define its Frenet frame. But, in the study of Bertrand curves and Mannheim curves, the straight line ℓ is regarded as a Frenet curve with arbitrary Frenet frame $\{T, N, B\}$ for a unit tangent vector T of ℓ. In this paper, we consider a geodesic in a 3-dimensional manifold M as a Frenet curve with a Frenet frame $\{T, N, B\}$ by choosing N and B properly, where T is the unit tangent vector field T of the geodesic.

3. Mannheim curves in 3-dimensional space forms

Let M be a 3-dimensional simply connected space form, i.e., $\mathbb{E}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$, and ∇ the Levi-Civita connection of M. Here, we consider M as a subspace of $\mathbb{E}^4$ with the induced metric from the natural inner product $\langle , \rangle$ of $\mathbb{E}^4$ (resp., $\mathbb{E}^4_1$) if $M = \mathbb{E}^3$ or $\mathbb{S}^3 := \{p \in \mathbb{E}^4 \mid \langle p, p\rangle = 1\}$ (resp., $\mathbb{H}^3 := \{p \in \mathbb{E}^4_1 \mid \langle p, p\rangle = -1\}$).

Let α be a smooth curve in M and $'$ denotes the ordinary differentiation with respect to the parameter of α in $\mathbb{E}^4$ or $\mathbb{E}^4_1$. Then, the Gauss formula of M along α is given by

$$X' = \nabla_s X - \epsilon\langle X, \alpha'\rangle\alpha \tag{3.1}$$

for any vector field X on M along α, where ∇_s denotes the covariant derivative of M along the curve α and $\epsilon = -1$, 0 or 1 if M is $\mathbb{H}^3$, $\mathbb{E}^3$ or $\mathbb{S}^3$, respectively.

The exponential map $\exp_p(tv)$ on M and the parallel transport $P^t(v)$ from p to $\exp_p(tv)$ along the geodesic γ_v are well-known as the following simple expressions (see, [1]):

$$\exp_p(tv) = f(t)p + g(t)v \tag{3.2}$$

and

$$P^t(v) = -\epsilon g(t)p + f(t)v, \tag{3.3}$$

where $p \in M$ and $v \in T_pM$ with $||v|| = 1$ are considered as vectors in $\mathbb{E}^4$ and the functions f and g are given by

$$\begin{cases} f(t) = 1 & g(t) = t, \quad \text{if } M = \mathbb{E}^3, \\ f(t) = \cos t & g(t) = \sin t, \quad \text{if } M = \mathbb{S}^3, \\ f(t) = \cosh t & g(t) = \sinh t, \quad \text{if } M = \mathbb{H}^3 \end{cases} \tag{3.4}$$

and $\epsilon = -1$, 0 or 1 if M is $\mathbb{H}^3$, $\mathbb{E}^3$ or $\mathbb{S}^3$, respectively.

Now, we will consider the characterizations with respect to the curvature and the torsion of Mannheim curves in 3-dimensional space forms.

In case of $M = \mathbb{E}^3$, the following proposition is well-known.

Proposition 3.1. *A Frenet curve α in $\mathbb{E}^3$ with the curvature κ and the torsion τ is a Mannheim curve if and only if it satisfies $\kappa = a(\kappa^2 + \tau^2)$ for a constant $a \neq 0$.*

Let M be a 3-dimensional non-flat space form, i.e., $\mathbb{S}^3$ or $\mathbb{H}^3$, and let $\alpha = \alpha(s)$ be a unit speed Frenet curve in M satisfying (2.1) and $\beta = \beta(u)$ a Mannheim partner curve of α, where u is the arc length parameter of β. Note that we can assume $du/ds > 0$. Then, without loss of generality, the curve $\beta(u)$ and its binormal vector field $B_\beta(u)$ can be expressed by

$$\begin{aligned} \beta(u) &= \exp_{\alpha(u)}(t(u)N_\alpha(u)) \\ &= f(t(u))\alpha(u) + g(t(u))N_\alpha(u) \end{aligned} \tag{3.5}$$

and $B_\beta(u) = \pm P^{t(u)}(N_\alpha(u))$, respectively, where the functions f and g are given in (3.4).

By differentiating β with respect u as a vector valued function in $\mathbb{E}^4$, we have

$$\begin{aligned} \beta'(u) &= \{f(t(u))\}'\alpha(u) + f(t(u))\frac{ds}{du}T_\alpha(u) \\ &\quad + \{g(t(u))\}'N_\alpha(u) + g(t(u))\frac{ds}{du}\nabla_s N_\alpha(u) \\ &= \{f(t(u))\}'\alpha(u) + \frac{ds}{du}\{f(t(u)) - \kappa_\alpha(u)g(t(u))\}T_\alpha(u) \\ &\quad + \{g(t(u))\}'N_\alpha(u) + \tau_\alpha(u)g(t(u))\frac{ds}{du}B_\alpha(u). \end{aligned} \tag{3.6}$$

Since $\langle \beta', \beta \rangle = \langle \beta', B_\beta \rangle = 0$ and $B_\beta = -\epsilon g(t)\alpha + f(t)N_\alpha$, (3.5) shows that β' is orthogonal to α and N_α in $\mathbb{E}^4$ or $\mathbb{E}^4_1$. Thus, from (3.6), we get

$$\{f(t(u))\}' = \{g(t(u))\}' = 0. \tag{3.7}$$

Since t is a non-zero smooth function, $t(u)$ is a non-zero constant, say $t(u) = \theta \neq 0$. Then, (3.5) and (3.6) are respectively reduced to

$$\beta(u) = f(\theta)\alpha(u) + g(\theta)N_\alpha(u) \tag{3.8}$$

and

$$\beta'(u) = \frac{ds}{du}\left[\{f(\theta) - \kappa_\alpha(u)g(\theta)\} T_\alpha(u) + \tau_\alpha(u)g(\theta)B_\alpha(u)\right], \tag{3.9}$$

from which,

$$du/ds = \sqrt{\{f(\theta) - \kappa_\alpha(u)g(\theta)\}^2 + \tau_\alpha(u)^2 g(\theta)^2}. \tag{3.10}$$

Note that if $M = \mathbb{S}^3$ and $\theta = \pi$ in (3.8), then $\beta = -\alpha$. Also, we can assume $g(\theta) \neq 0$.

For the sake of simplicity, we put

$$T_\beta(u) = a_1(u)T_\alpha(u) + a_2(u)B_\alpha(u), \tag{3.11}$$

where $a_1(u) = \frac{ds}{du}\{f(\theta) - \kappa_\alpha(u)g(\theta)\}$, $a_2(u) = \frac{ds}{du}\tau_\alpha(u)g(\theta)$.

Differentiating (3.11), we have

$$\begin{aligned}\nabla_u T_\beta(u) - \epsilon\beta(u) = &-\epsilon a_1(u)\frac{ds}{du}\,\alpha(u) + a_1'(u)T_\alpha(u)\\ &+ \frac{ds}{du}\left\{a_1(u)\kappa_\alpha(u) - a_2(u)\tau_\alpha(u)\right\} N_\alpha(u) + a_2'(u)B_\alpha(u),\end{aligned}$$

from which,

$$\begin{aligned}\nabla_u T_\beta(u) = {}&\epsilon\{f(\theta) - a_1(u)\frac{ds}{du}\}\,\alpha(u) + a_1'(u)T_\alpha(u)\\ &+ \left\{a_1(u)\kappa_\alpha(u)\frac{ds}{du} - a_2(u)\tau_\alpha(u)\frac{ds}{du} + \epsilon g(\theta)\right\} N_\alpha(u)\\ &+ a_2'(u)B_\alpha(u).\end{aligned} \tag{3.12}$$

Since $\langle N_\beta, \beta\rangle = \langle N_\beta, B_\beta\rangle = 0$ and $B_\beta = -\epsilon g(t)\alpha + f(t)N_\alpha$, (3.5) implies that N_β is orthogonal to α and N_α in $\mathbb{E}^4$ or $\mathbb{E}^4_1$. Thus, from (3.12), we have

$$\begin{cases} f(\theta) - a_1\frac{ds}{du} = 0,\\ a_1\kappa_\alpha\frac{ds}{du} - a_2\tau_\alpha\frac{ds}{du} + \epsilon g(\theta) = 0\end{cases} \tag{3.13}$$

and

$$\nabla_u T_\beta = a_1' T_\alpha + a_2' B_\alpha. \tag{3.14}$$

Then, we get

$$a_1 = f(\theta)\frac{du}{ds} \quad\text{and}\quad a_2 = \frac{1}{\tau_\alpha}\left(f(\theta)\kappa_\alpha + \epsilon g(\theta)\right)\frac{du}{ds}, \tag{3.15}$$

or equivalently,

$$\begin{aligned} f(\theta) &= \frac{f(\theta) - g(\theta)\kappa_\alpha}{(f(\theta) - g(\theta)\kappa_\alpha)^2 + g(\theta)^2{\tau_\alpha}^2},\\ f(\theta)\kappa_\alpha + \epsilon g(\theta) &= \frac{g(\theta){\tau_\alpha}^2}{(f(\theta) - g(\theta)\kappa_\alpha)^2 + g(\theta)^2{\tau_\alpha}^2}.\end{aligned} \tag{3.16}$$

It follows that

$$\frac{f(\theta)}{f(\theta)\kappa_\alpha + \epsilon g(\theta)} = \frac{f(\theta) - g(\theta)\kappa_\alpha}{g(\theta){\tau_\alpha}^2}, \tag{3.17}$$

or, equivalently,

$$\kappa_\alpha^2 + \tau_\alpha^2 = \lambda\kappa_\alpha + \epsilon, \tag{3.18}$$

where λ is a constant given by

$$\lambda = \frac{f(\theta)^2 - \epsilon g(\theta)^2}{f(\theta)g(\theta)}.$$

Note that $\lambda = \theta$ ($\epsilon = 0$), $2\cot 2\theta$ ($\epsilon = 1$) or $\lambda = 2\,\mathrm{coth}2\theta$, $|\lambda| > 2$ ($\epsilon = -1$).

Conversely, for a curve α in M satisfying (3.18) for some λ ($\lambda \neq 0$ if $\epsilon = 1$, $|\lambda| > 2$ if $\epsilon = -1$), we may choose $\theta \in \mathbb{R}$ such that $\lambda = \frac{f(\theta)^2 - \epsilon g(\theta)^2}{f(\theta)g(\theta)}$, and hence it satisfies (3.17). Equivalently, (3.17) can be rewritten as

$$f(\theta) - g(\theta)\kappa_\alpha = \frac{f(\theta)g(\theta)\tau_\alpha^2}{f(\theta)\kappa_\alpha + \epsilon g(\theta)}. \tag{3.19}$$

We define a curve β by (3.8). Since either $f(\theta) = \cos\theta$ and $g(\theta) = \sin\theta$, or $f(\theta) = \cosh\theta$ and $g(\theta) = \sinh\theta$, we can check from (3.19) that two equations of (3.16) are satisfied. Note that (3.16) derives the system of equations of (3.13). Thus, we get from (3.12) the principal normal vector field N_β of β given by

$$N_\beta = \frac{a_1'}{\sqrt{a_1'^2 + a_2'^2}} T_\alpha + \frac{a_2'}{\sqrt{a_1'^2 + a_2'^2}} B_\alpha. \tag{3.20}$$

Since $\{\beta, T_\beta, N_\beta, B_\beta\}$ forms an orthonormal frame in $\mathbb{E}^4$ or $\mathbb{E}^4_1$, (3.8), (3.11) and (3.20) yield the binormal vector field B_β as

$$B_\beta(u) = \pm\{-\epsilon g(\theta)\alpha(u) + f(\theta)N_\alpha(u)\} = \pm P^\theta(N_\alpha). \tag{3.21}$$

Therefore, β is a Mannheim partner curve of α, i.e., α is a Mannheim curve in M.

Note that if $f(\theta) = 1, g(\theta) = \theta$ and $\epsilon = 0$, (3.18) is replaced by

$$\kappa_\alpha = \frac{1}{\theta}(\kappa_\alpha^2 + \tau_\alpha^2).$$

Thus, we have an extended characterization of Mannheim curves on a 3-dimensional space form.

Theorem 3.2. *A Frenet curve α on a 3-dimensional space form M with the curvature κ and τ is a Mannheim curve if and only if it satisfies* (3.18) *for a constant λ ($\lambda \neq 0$ if $M = \mathbb{E}^3$ or $\mathbb{S}^3$, $|\lambda| > 2$ if $M = \mathbb{H}^3$) with* (3.4).

4. Mannheim partner curves in 3-dimensional space forms

In 2008, H. Liu and W. Fang defined the notion of Mannheim partner curve and they proved the following:

Proposition 4.1. *Let $\beta = \beta(s)$ be a Frenet curve in $\mathbb{E}^3$ with the arc length parameter s. Then β is the Mannheim partner curve of some Mannheim curve α in $\mathbb{E}^3$ if and only if the curvature κ and the torsion τ of β satisfy the following equation*:

$$\frac{d\tau}{ds} = \frac{\kappa}{\lambda}\left(1 + \lambda^2\tau^2\right)$$

for some nonzero constant λ.

Let M be a 3-dimensional non-flat space form, i.e., $\mathbb{S}^3$ or $\mathbb{H}^3$, and let $\alpha = \alpha(s)$ be a unit speed Frenet curve in M satisfying (2.1) and $\beta = \beta(u)$ a Mannheim partner curve of α, where u is the arc length parameter of β. Then, without loss of generality, the curve α and its principal normal vector field $N_\alpha(u)$ can be expressed by

$$\begin{aligned} \alpha(u) &= \exp_{\beta(u)}\left(t(u)B_\beta(u)\right) \\ &= f(t(u))\beta(u) + g(t(u))B_\beta(u) \end{aligned} \tag{4.1}$$

and $N_\alpha(u) = \pm P^{t(u)}(B_\beta(u))$, respectively, where the functions f and g are given in (3.4).

Differentiating α with respect to u, we have

$$\begin{aligned} \dot\alpha(u)\frac{ds}{du} &= \{f(t(u))\}'\,\beta(u) + f(t(u))\,T_\beta(u) \\ &\quad + \{g(t(u))\}'\,B_\beta(u) + g(t(u))\nabla_u B_\beta(u) \\ &= \{f(t(u))\}'\,\beta(u) + f(t(u))T_\beta(u) \\ &\quad - \tau_\beta(u)g(t(u))\,N_\beta(u) + \{g(t(u))\}'B_\beta(u), \end{aligned} \tag{4.2}$$

where $\dot\alpha$ denotes the differentiation of α with respect to s.

Since $\langle\dot\alpha, \alpha\rangle = \langle\dot\alpha, N_\alpha\rangle = 0$ and $N_\alpha = -\epsilon g(t)\beta + f(t)B_\beta$, (3.5) shows that $\dot\alpha$ is orthogonal to β and B_β in $\mathbb{E}^4$ or $\mathbb{E}^4_1$. Thus, from (4.2), we get

$$\{f(t(u))\}' = \{g(t(u))\}' = 0. \tag{4.3}$$

Since t is a non-zero smooth function, $t(u)$ is a non-zero constant, say $t(u) = \theta \neq 0$. Then, (4.1) and (4.2) are respectively reduced to

$$\alpha(u) = f(\theta)\beta(u) + g(\theta)B_\beta(u) \tag{4.4}$$

and

$$\dot\alpha(u)\frac{ds}{du} = \left[f(\theta)T_\beta(u) - g(\theta)\tau_\beta(u)N_\beta(u)\right], \tag{4.5}$$

from which,

$$ds/du = \sqrt{f(\theta)^2 + \tau_\beta(u)^2 g(\theta)^2}.$$

Hence, we get

$$T_\alpha(u) = b_1(u)T_\beta(u) + b_2(u)N_\beta(u), \tag{4.6}$$

where $b_1(u) = \frac{f(\theta)}{\sqrt{f(\theta)^2+\tau_\beta(u)^2 g(\theta)^2}}$, $b_2(u) = -\frac{\tau_\beta(u)g(\theta)}{\sqrt{f(\theta)^2+\tau_\beta(u)^2 g(\theta)^2}}$.

Differentiating (4.6) with respect to u gives

$$\begin{aligned}\frac{ds}{du}\left(\nabla_s T_\alpha(u) - \epsilon\alpha(u)\right) = &-\epsilon b_1(u)\,\beta(u) + \left\{b_1'(u) - b_2(u)\kappa_\beta(u)\right\}T_\beta(u)\\ &+ \left\{b_1(u)\kappa_\beta(u) + b_2'(u)\right\}N_\beta(u) + b_2(u)\tau_\beta(u)B_\beta(u),\end{aligned}$$

from which,

$$\begin{aligned}\frac{ds}{du}\nabla_s T_\alpha(u) = \epsilon\{f(\theta)\frac{ds}{du} &- b_1(u)\}\,\beta(u) + \left\{b_1'(u) - b_2(u)\kappa_\beta(u)\right\}T_\beta(u)\\ &+ \left\{b_1(u)\kappa_\beta(u) + b_2'(u)\right\}N_\beta(u)\\ &+ \left\{b_2(u)\tau_\beta(u) + \epsilon g(\theta)\frac{ds}{du}\right\}B_\beta(u).\end{aligned} \tag{4.7}$$

Since $\nabla_s T_\alpha$ is proportional to $N_\alpha = -\epsilon g(\theta)\beta + f(\theta)B_\beta$, (4.7) implies that

$$\begin{cases} b_1'(u) - \kappa_\beta(u)b_2(u) = 0,\\ b_2'(u) + \kappa_\beta(u)b_1(u) = 0\end{cases} \tag{4.8}$$

and

$$\nabla_s T_\alpha = \epsilon\{f(\theta) - b_1\frac{du}{ds}\}\,\beta + \left\{b_2\frac{du}{ds}\tau_\beta + \epsilon g(\theta)\right\}B_\beta. \tag{4.9}$$

By a direct computation, we have

$$b_1' - \kappa_\beta b_2 = \frac{g(\theta)\tau_\beta\left\{\kappa_\beta\left(f(\theta)^2 + g(\theta)^2\tau_\beta{}^2\right) - f(\theta)g(\theta)\tau_\beta{}'\right\}}{\sqrt{\left[f(\theta)^2 + g(\theta)^2\tau_\beta{}^2\right]^3}} \tag{4.10}$$

and

$$b_2'(u) + \kappa_\beta(u)b_1(u) = \frac{f(\theta)\left\{\kappa_\beta\left(f(\theta)^2 + g(\theta)^2\tau_\beta{}^2\right) - f(\theta)g(\theta)\tau_\beta{}'\right\}}{\sqrt{\left[f(\theta)^2 + g(\theta)^2\tau_\beta{}^2\right]^3}}. \tag{4.11}$$

Thus, equation (4.8) yields

$$\tau_\beta{}' = \frac{\kappa_\beta}{\bar\lambda}\left\{1 + \bar\lambda^2\tau_\beta{}^2\right\}, \tag{4.12}$$

where $\bar\lambda$ is a constant defined by $\bar\lambda = \frac{g(\theta)}{f(\theta)}$. Note that $\bar\lambda = \theta$ $(\epsilon = 0)$, $\tan\theta$ $(\epsilon = 1)$ or $\bar\lambda = \tanh\theta$, $|\bar\lambda| < 1$ $(\epsilon = -1)$.

Conversely, for a curve β in M satisfying (4.12) for some $\bar\lambda$ ($\bar\lambda \neq 0$ if $\epsilon = 1$, $|\bar\lambda| < 1$ if $\epsilon = -1$), we may choose $\theta \in \mathbb{R}$ such that $\bar\lambda = \frac{g(\theta)}{f(\theta)}$. We define a curve α by (4.4). Then, from (4.10) and (4.11), (4.7) implies (4.9).

When $M = \mathbb{S}^3$, (4.9) becomes

$$\nabla_s T_\alpha = \frac{\sin\theta\cos\theta(\tau_\beta{}^2 - 1)}{\cos^2\theta + \sin^2\theta\tau_\beta{}^2}\left\{\sin\theta\,\beta - \cos\theta\, B_\beta\right\}, \tag{4.13}$$

from which,

$$||\nabla_s T_\alpha|| = \frac{|\sin 2\theta(\tau_\beta{}^2 - 1)|}{2(\cos\theta^2 + \sin\theta^2\tau_\beta{}^2)}. \tag{4.14}$$

This means that if $\sin 2\theta(\tau_\beta{}^2 - 1) > 0$, β is a Mannheim partner curve of α. In fact, for a function $\tau_\beta \neq \pm 1$, we can choose locally a constant θ satisfying $\sin 2\theta(\tau_\beta{}^2 - 1) > 0$. Thus, the principal normal vector field N_α of α is given by

$$N_\alpha(u) = -\sin\theta\beta(u) + \cos\theta B_\beta(u).$$

By the similar method, when $M = \mathbb{H}^3$, β is a Mannheim partner curve of α.

Consequently, we have a characterization of Mannheim partner curves in a 3-dimensional space form.

Theorem 4.2. *Let $\beta = \beta(s)$ be a Frenet curve in a 3-dimensional space form M with the arc parameter s. Then, β is the Mannheim partner curve of some Mannheim curve α in M if and only if the curvature κ and the torsion τ of β satisfy* (4.12) *for a constant $\bar\lambda$ ($\bar\lambda \neq 0$ if $M = \mathbb{E}^3$ or $\mathbb{S}^3$, $|\bar\lambda| < 1$ if $M = \mathbb{H}^3$) with* (3.4).

A helix in a 3-dimensional manifold M is defined by a curve whose curvature and torison are constants. In [2], Huili Liu and Fan Wang stated that the Mannheim partner curve of a helix in $\mathbb{E}^3$ is a straight line. Motivated by this result, we give the following example:

Example. Let M be a 3-dimensional non-flat space form, i.e., $M = \mathbb{S}^3$ or $M = \mathbb{H}^3$, and $\alpha = \alpha(s)$ a helix in M parametrized by the arc length with the curvature κ_0 and the torsion τ_0.

By applying κ_0 and τ_0 to (3.18), we have $\lambda = \frac{\kappa_0^2 + \tau_0^2 - 1}{\kappa_0}$ and $\lambda = \frac{\kappa_0^2 + \tau_0^2 + 1}{\kappa_0}$ according to $M = \mathbb{S}^3$ and $M = \mathbb{H}^3$, respectively. This means that α is a Mannheim curve in M. Moreover, the Mannheim partner curve β of α in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$) is given by $\beta(u) = \cos\theta\alpha(u) + \sin\theta N_\alpha(u)$ (resp. $\beta(u) = \cosh\theta\alpha(u) + \sinh\theta N_\alpha(u)$) with

$$\begin{aligned} u &= \sqrt{(\cos\theta - \kappa_0\sin\theta)^2 + \tau_0^2\sin^2\theta}\, s \text{ (resp. } \sqrt{(\cosh\theta - \kappa_0\sinh\theta)^2 + \tau_0^2\sinh^2\theta}\, s), \\ \theta &= \frac{1}{2}\tan^{-1}\left(\frac{2\kappa_0}{\kappa_0^2 + \tau_0^2 - 1}\right) \text{ (resp. } \frac{1}{2}\tanh^{-1}\left(\frac{2\kappa_0}{\kappa_0^2 + \tau_0^2 + 1}\right)), \end{aligned}$$

where N_α is the principal normal vector field of α and u is the arc length parameter of β.

On the other hand, equations (3.10), (3.14) and (3.15) lead to $\nabla_u T_\beta = 0$, from which, β is a geodesic in M.

Consequently, we have the following proposition:

Proposition 4.3. *A helix in a 3-dimensional simply connected space form M is a Mannheim curve. Moreover, the Mannheim partner curve of a helix in M is a geodesic.*

References

[1] J. H. Choi, T. H. Kang, and Y. H. Kim, *Bertrand curves in 3-dimensional space forms*, to appear in Applied Mathematics and Computation.

[2] H. Liu and F. Wang, *Mannheim partner curves in 3-space*, J. Geom. **88** (2008), no. 1-2, 120–126.

Jin Ho Choi
Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea
E-mail address: `jino7142@knu.ac.kr`

Tae Ho Kang
Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea
E-mail address: `thkang@ulsan.ac.kr`

Young Ho Kim
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail address: `yhkim@knu.ac.kr`