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MELTING OF THE EUCLIDEAN METRIC TO NEGATIVE SCALAR CURVATURE

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Abstract. We find a C^∞-continuous path of Riemannian metrics g_t on $\mathbb{R}^k$, $k \geq 3$, for $0 \leq t \leq \varepsilon$ for some number $\varepsilon > 0$ with the following property: g_0 is the Euclidean metric on $\mathbb{R}^k$, the scalar curvatures of g_t are strictly decreasing in t in the open unit ball and g_t is isometric to the Euclidean metric in the complement of the ball. Furthermore we extend the discussion to the Fubini-Study metric in a similar way.

1. Introduction

In a remarkable paper [11], Lohkamp has made the following conjecture in Riemannian geometry.

Conjecture. Let (M^k, g_0), $k \geq 3$, be a manifold and $B \subset M$ a ball. Then there is a C^∞-continuous path of Riemannian metrics g_t, $0 \leq t \leq \varepsilon$, on M with

(i) Ricci curvature of g_t is strictly decreasing in t on B.

(ii) $g_t \equiv g_0$ on $M \backslash B$.

If such a path g_t exists, we call it a *Ricci-curvature melting* of g_0 on B. This conjecture, if true, would certainly imply a *scalar-curvature melting*, meaning a path g_t as above but with scalar curvature replacing the Ricci curvature in the condition (i). We note that common metric-surgery arguments do not seem to yield a scalar-curvature melting. If one considers the scalar curvatures $s(g_t)$ for a scalar-curvature melting g_t, then $\frac{ds(g_t)}{dt}|_{t=0} \leq 0$ on B. In this way, the scalar-curvature melting is related to the deformation theory of the scalar curvature functional [4, Chapter 4]. A remarkable approach is the theory of local scalar curvature deformation of J. Corvino [6, Theorem 4]. He considered the formal adjoint L_g^* of the linearization L_g of the scalar curvature functional on the space of Riemannian metrics restricted to a domain. According to his work, a scalar-curvature melting of g seems to exist when L_g^* is injective. Years

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later, this injectivity condition of L_g^* on domains was shown to be a generic one by Beig, Chruściel and Schoen (see Theorem 6.1 and Theorem 7.4 in [3]). Now the question is how to melt a Riemannian metric which does not satisfy this condition.

In this context, Euclidean metrics arise importantly because they are outstanding ones, not satisfying this condition. In a recent paper [8], we explained the scalar-curvature melting of Euclidean metric in 3 dimension. The purpose of this article is to complete the scalar-curvature melting of Euclidean metrics in any dimension ≥ 3 and then extend the discussion to the Fubini-Study metric in a similar way.

We shall first construct a family of Riemannian metrics on $\mathbb{R}^k$, $k \geq 3$ which have negative scalar curvatures on a pre-compact (open) set and are Euclidean away from it. In even dimension we already have such a family of metrics [7]. In odd dimension, we use the coordinates $(r_1, \theta_1, \dots, r_n, \theta_n, z)$ on $\mathbb{R}^{2n+1}$ where (r_i, θ_i) are the polar coordinates on the i-th direct summand of $\mathbb{R}^{2n+1} := \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R}$ and z is the coordinate for the last summand $\mathbb{R}$. We express the Euclidean metric as $g_0 = \sum_{i=1}^n (dr_i^2 + r_i^2 d\theta_i^2) + dz^2$. We deform it to $g = \sum_{i=1}^n (f_i^2 dr_i^2 + \frac{r_i^2}{f_i^2} d\theta_i^2) + dz^2$ and choose smooth functions f_i so that g has negative scalar curvature on a pre-compact set near origin and is Euclidean away from it.

Then by conformal change of g (also for the even dimensional metrics mentioned above), we spread the negativity inside the pre-compact set over to a larger ball. In the process, we found a natural choice of parameter t to get g_t. In this way we get a scalar-curvature melting:

Theorem 1.1. *There exists a C^∞-continuous path of Riemannian metrics g_t on $\mathbb{R}^k$, $k \geq 3$ which exists for $0 \leq t \leq \varepsilon$ for some number ε with the following property: g_0 is the Euclidean metric on $\mathbb{R}^k$, $s(g_{\tilde{t}}) < s(g_t)$ for $0 \leq t < \tilde{t} \leq \varepsilon$ in the open unit ball and g_t is the Euclidean metric in the complement of the ball.*

In Section 2, we construct Riemannian metrics on $\mathbb{R}^{2n+1}$ that have negative scalar curvatures on a pre-compact set and are Euclidean away from it. In Section 3, we demonstrate a C^∞-continuous path of metrics g_t such that the scalar curvature $s(g_t)$ is monotonically decreasing in t. In Section 4, by a conformal deformation we get a genuine scalar-curvature melting on the unit ball in $\mathbb{R}^{2n+1}$. We also observe that similar argument works for even dimensions. In Section 5 we discuss the Fubini-Study metric in a similar way.

2. Construction of the metric

We will deform the Euclidean metric $g_0 = \sum_{i=1}^n (dr_i^2 + r_i^2 d\theta_i^2) + dz^2$ on $\mathbb{R}^{2n+1} = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R}$ to a metric of the form

$$\tilde{g} = \sum_{i=1}^n (f_i^2 dr_i^2 + \frac{r_i^2}{f_i^2} d\theta_i^2) + dz^2, \tag{1}$$

where f_i's are smooth positive functions on $\mathbb{R}^{2n+1}$ depending only on the variables $r_1, \dots, r_n, z$. $\tilde{g}$ is a metric on $\mathbb{R}^{2n+1} \backslash \{(r_1, \theta_1, \dots, r_n, \theta_n, z) \mid r_i = 0 \text{ for some } i\}$. Below we shall choose f_i so that $\tilde{g}$ is smooth on $\mathbb{R}^{2n+1}$. Let $e_{2i-1} = \frac{1}{f_i}\frac{\partial}{\partial r_i}$, $e_{2i} = \frac{f_i}{r_i}\frac{\partial}{\partial \theta_i}$, $i = 1, 2, \dots, n$, $e_{2n+1} = \frac{\partial}{\partial z}$.

Let ω_i be the dual co-frame fields of e_i: $\omega_{2i-1} = f_i dr_i$, $\omega_{2i} = \frac{r_i}{f_i} d\theta_i$, $\omega_{2n+1} = dz$. We compute the connection 1-forms ω_{ij} with respect to ω_i: $d\omega_i = \sum_{j=1}^{2n+1} \omega_{ij} \wedge \omega_j$, with $\omega_{ij} = -\omega_{ji}$; one may compute

$$2a_{ijk} = \langle d\omega_k, \omega_i \wedge \omega_j \rangle_g - \langle d\omega_i, \omega_j \wedge \omega_k \rangle_g - \langle d\omega_j, \omega_k \wedge \omega_i \rangle_g,$$

where $\omega_{ij} = \sum_{k=1} a_{ijk}\omega_k$. We get

$$d\omega_{2n+1} = 0,$$

$$d\omega_{2i-1} = \frac{f_{i,2n+1}}{f_i}\omega_{2n+1} \wedge \omega_{2i-1} + \sum_{j=1}^{n} \frac{f_{i,j}}{f_i f_j}\omega_{2j-1} \wedge \omega_{2i-1} \text{ and}$$

$$d\omega_{2i} = -\frac{f_{i,2n+1}}{f_i}\omega_{2n+1} \wedge \omega_{2i} + \sum_{j=1}^{n} \frac{\delta_{ij} f_i - r_i f_{i,j}}{r_i f_i f_j}\omega_{2j-1} \wedge \omega_{2i}$$

$$\text{for } i = 1, 2, \dots, n.$$

Here we write $f_{i,j} = \frac{\partial f_i}{\partial r_j}$, $f_{i,jk} = \frac{\partial^2 f_i}{\partial r_k \partial r_j}$. Then we can get $\omega_{2i-1\ 2j-1} = -\frac{f_{i,j}}{f_j f_i}\omega_{2i-1} + \frac{f_{j,i}}{f_j f_i}\omega_{2j-1}$, $\omega_{2i-1\ 2j} = \frac{\delta_{ji} f_j - r_j f_{j,i}}{r_j f_i f_j}\omega_{2j}$ and $\omega_{2i\ 2j} = 0$ for $i, j = 1, 2, \dots, n$ and $\omega_{2n+1\ 2i-1} = \frac{f_{i,2n+1}}{f_i}\omega_{2i-1}$, $\omega_{2n+1\ 2i} = -\frac{f_{i,2n+1}}{f_i}\omega_{2i}$. We use the formula $d\omega_{i\ j} - \omega_{i\ k} \wedge \omega_{k\ j} = \sum_{k<l}^{2n+1} R_{ijkl}\omega_k \wedge \omega_l$ to compute the curvature components;

$$\begin{aligned}
&R_{2i-1\ 2j-1\ 2j-1\ 2i-1} \\
&= (-d\omega_{2i-1\ 2j-1} + \omega_{2i-1\ s} \wedge \omega_{s\ 2j-1},\ \ \omega_{2i-1} \wedge \omega_{2j-1})_g \\
&= -\frac{f_{i,jj}}{f_i f_j^2} + \frac{f_{i,j} f_{j,j}}{f_i f_j^3} - \frac{f_{j,ii}}{f_j f_i^2} + \frac{f_{j,i} f_{i,i}}{f_j f_i^3} - \sum_{k \neq i,j}^{n} \frac{f_{i,k} f_{j,k}}{f_i f_j f_k^2} - \frac{f_{i,2n+1}}{f_i} \cdot \frac{f_{j,2n+1}}{f_j},
\end{aligned}$$

$$\begin{aligned}
&R_{2i\ 2j\ 2j\ 2i} \\
&= (-d\omega_{2i\ 2j},\ \omega_{2i} \wedge \omega_{2j})_g + (\omega_{2i\ s} \wedge \omega_{s\ 2j},\ \omega_{2i} \wedge \omega_{2j})_g \\
&= \frac{f_{j,i}}{r_i f_i^2 f_j} + \frac{f_{i,j}}{r_j f_j^2 f_i} - \sum_{k=1}^{n} \frac{f_{i,k} f_{j,k}}{f_i f_j f_k^2} - \frac{f_{i,2n+1}}{f_i} \cdot \frac{f_{j,2n+1}}{f_j},
\end{aligned}$$

$$\begin{aligned}
&R_{2i-1\ 2j\ 2j\ 2i-1} \\
&= (-d\omega_{2i-1\ 2j},\ \omega_{2i-1} \wedge \omega_{2j})_g + (\omega_{2i-1\ s} \wedge \omega_{s\ 2j},\ \omega_{2i-1} \wedge \omega_{2j})_g \\
&= \frac{\delta_{ij} f_{i,i}}{r_j f_i^3} + \frac{f_{j,ii}}{f_i^2 f_j} - \frac{f_{j,i} f_{i,i}}{f_i^3 f_j} - \frac{2 f_{j,i}^2}{f_i^2 f_j^2} + \frac{2\delta_{ij} f_{j,i}}{r_j f_i^2 f_j} - \sum_{k \neq i}^{n} \frac{\delta_{jk} f_{i,k}}{r_j f_i f_k^2} + \sum_{k \neq i}^{n} \frac{f_{i,k} f_{j,k}}{f_i f_j f_k^2} \\
&\quad + \frac{f_{i,2n+1}}{f_i} \cdot \frac{f_{j,2n+1}}{f_j},
\end{aligned}$$

$$R_{2n+1\ 2i-1\ 2i-1\ 2n+1} = -(\frac{f_{i,2n+1}}{f_i})_{2n+1} - (\frac{f_{i,2n+1}}{f_i})^2,$$
$$R_{2n+1\ 2i\ 2i\ 2n+1} = (\frac{f_{i,2n+1}}{f_i})_{2n+1} - (\frac{f_{i,2n+1}}{f_i})^2.$$

The scalar curvature is as follows;

$$\begin{aligned}
\frac{s_{\tilde{g}}}{2} &= \sum_{1\le s<t}^{2n+1} R_{stts} \\
&= \sum_{1\le t}^{2n} R_{2n+1\ t\ t\ 2n+1} + \sum_{1\le i<j}^{n} (R_{2i-1\ 2j-1\ 2j-1\ 2i-1} + R_{2i\ 2j\ 2j\ 2i}) \\
&\quad + \sum_{1\le i<j}^{n} (R_{2i-1\ 2j\ 2j\ 2i-1} + R_{2j-1\ 2i\ 2i\ 2j-1}) + \sum_{i=1}^{n} R_{2i-1\ 2i\ 2i\ 2i-1} \\
&= -\sum_{i=1}^{n} (\frac{f_{i,2n+1}}{f_i})^2 + \sum_{i=1}^{n} (\frac{f_{i,ii}}{f_i^3} + 3\frac{f_{i,i}}{r_i f_i^3} - 3\frac{f_{i,i}^2}{f_i^4}) - \sum_{i<j} \frac{f_{i,j}^2 + f_{j,i}^2}{f_i^2 f_j^2} \\
&= -\frac{1}{2}\sum_{i=1}^{n} \{(f_i^{-2})_{ii} + \frac{3}{r_i}(f_i^{-2})_i\} - \sum_{i<j} \frac{f_{i,j}^2 + f_{j,i}^2}{f_i^2 f_j^2} - \sum_{i=1}^{n} (\frac{f_{i,2n+1}}{f_i})^2.
\end{aligned}$$

Set $F_i = f_i^{-2}$, $i = 1, \dots, n$. We shall find the functions F_i so that they satisfy

$$\sum_{i=1}^{n} (F_{i,ii} + \frac{3}{r_i} F_{i,i}) = 0. \tag{2}$$

We consider smooth functions $\beta(z)$ and $\alpha_j^i(r)$, $i = 1, \dots, n-1$, $j = 1, \dots, n$ on $\mathbb{R}$ which satisfy at least

$$\beta(z) = 0 \quad \text{for} \ \ z \le -1, \ \text{or} \ \ z \ge 1, \qquad \text{and} \qquad \beta(z) > 0 \ \ \text{on} \ \ -1 < z < 1,$$

$$\alpha_j^i(r) = 0 \ \ \text{for} \ r \le 0, \ \text{or} \ \ r \ge 1.$$

The functions α_j^i's need to be specified more. Let $k_j^i(r)$ be smooth functions on $\mathbb{R}$ satisfying

$$\begin{cases}
\text{a)} \ \ k_j^i(r) = 0 \quad \text{for} \quad r \le 0, \ r \ge 1, \\
\text{b)} \ \ |(k_j^i)'(r)|_{C_0} \ll |r^3|_{C_0}, \\
\text{c)} \ \ \int_0^1 \frac{k_j^i(r)}{r^3}\, dr = 0, \\
\text{d)} \ \ 0 < \int_0^c \frac{k_j^i(r)}{r^3}\, dr < 1 \quad \text{for any } c \ \ \text{with} \ \ 0 < c < 1.
\end{cases}$$

Set $\alpha_j^i(r) = \frac{1}{r^3}\frac{dk_j^i}{dr}(r)$, which will be smooth on $\mathbb{R}$.

Graphs of typical α_j^i and β are given in Figures 1 and 2 below.

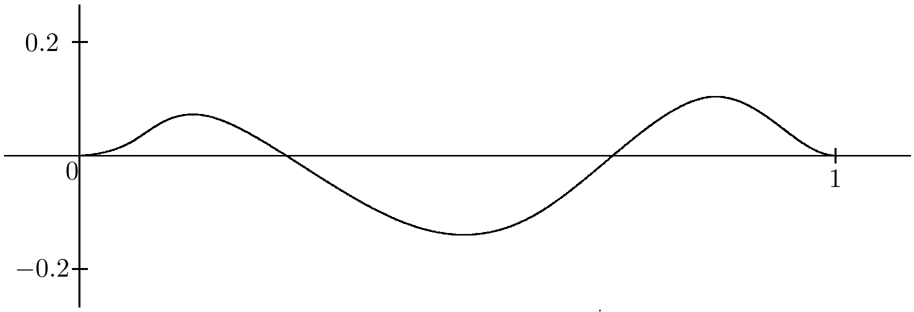

Figure 1. The graph of α^i_j.

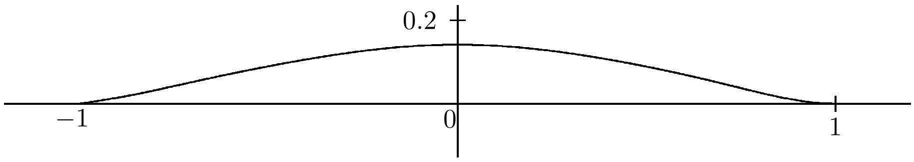

Figure 2. The graph of β.

Define the functions F_i, $i = 1, \dots, n-1$, and F_n by

$$F_i(r_1, \dots, r_n, z) = 1 + \beta(z) \cdot \alpha^i_1(r_1) \cdots \hat{\alpha^i_i(r_i)} \cdots \alpha^i_n(r_n) \int_0^{r_i} (\frac{1}{y^3} \int_0^y x^3 \alpha^i_i(x)\, dx)\, dy,$$

where $\hat{}$ denotes the missing factor in that position,

$$F_n(r_1, \dots, r_n, z) = 1 - \beta(z) \cdot \sum_{i=1}^{n-1} \alpha^i_1(r_1) \cdots \alpha^i_{n-1}(r_{n-1}) \int_0^{r_n} (\frac{1}{y^3} \int_0^y x^3 \alpha^i_n(x)\, dx)\, dy.$$

We consider F_i's and F_n defined on $\mathbb{R}^{2n+1} = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R}$. Then they satisfy the equation (2) and

$$F_i,\ F_n \equiv 1 \quad \text{if} \quad r_k \le 0 \text{ or } r_k \ge 1 \quad \text{for some } k, \quad \text{or } |z| > 1,$$
$$F_i,\ F_n > 0 \quad \text{everywhere.}$$

We set $\mathbf{C} = \{(r_1, \theta_1, \dots, r_n, \theta_n, z) \mid |z| < 1,\ 0 \le r_i < 1,\ 0 \le \theta_i < 2\pi\}$. We now see that $\tilde{g}$ is Euclidean away from $\mathbf{C}$ and that its scalar curvature $s_{\tilde{g}}$ is negative inside $\mathbf{C}$ except the thin subset $\mathfrak{T} := \{(r_1, \theta_1, \dots, r_n, \theta_n, z) \in \mathbf{C} \mid F_{i,j} = 0,\ F_{i,2n+1} = 0,\ 1 \le i \ne j \le n\}$.

Proposition 2.1. *There exist Riemannian metrics on* $\mathbb{R}^{2n+1}$, $n \ge 2$ *such that their scalar curvatures are negative on the pre-compact subset* $\mathbf{C} \backslash \mathfrak{T}$ *and they are Euclidean away from* $\mathbf{C}$.

We need to recall the similar result in even dimensions from Sections 3 and 5 of [7].

Proposition 2.2. *There exist Riemannian metrics on $\mathbb{R}^{2n}$, $n \geq 2$ such that their scalar curvatures are negative on a pre-compact subset* $\mathbf{K}$ *and they are Euclidean away from* $\mathbf{K}$.

3. Decreasing property of the scalar curvature of metrics

We are going to show that there is a C^∞-continuous path $\tilde{g}_t$ among the metrics in the previous section such that its scalar curvature $s(\tilde{g}_t)$ is decreasing in $\mathbf{C}\backslash\mathfrak{T}$ and $\tilde{g}_t$ is Euclidean in the complement of $\mathbf{C}$.

We set

$$F_i^t(r_1,\ldots,r_n,z) = 1 + t\cdot\beta(z)\cdots\alpha_1^i(r_1)\cdots\hat{\alpha_i^i(r_i)}\cdots\alpha_n^i(r_n)\int_0^{r_i}(\frac{1}{y^3}\int_0^y x^3\alpha_i^i(x)\,dx)\,dy,$$

where $\hat{}$ denotes the missing factor in that position,

$$F_n^t(r_1,\ldots,r_n,z) = 1 - t\cdot\beta(z)\cdot\sum_{i=1}^{n-1}\alpha_1^i(r_1)\cdots\alpha_{n-1}^i(r_{n-1})\int_0^{r_n}(\frac{1}{y^3}\int_0^y x^3\alpha_n^i(x)\,dx)\,dy.$$

Still under the relation $F_i^t = (f_i^t)^{-2}$, $i = 1,\ldots,n$, we let

$$\tilde{g}_t = dz^2 + \sum_{i=1}^n (f_i^t)^2 dr_i^2 + \frac{r_i^2}{(f_i^t)^2}d\theta_i^2. \tag{3}$$

The scalar curvature is

$$s_{\tilde{g}_t}(r_1,\ldots,r_n) = -\frac{1}{4}\sum_{i<j}\{(\frac{F_{i,j}^t}{F_i^t})^2F_j^t + (\frac{F_{j,i}^t}{F_j^t})^2F_i^t\} - \frac{1}{4}\sum_{i=1}^n(\frac{F_{i,2n+1}^t}{F_i^t})^2.$$

One can easily check $\frac{d(s(\tilde{g}_t))}{dt}|_{t=0} = 0$ and

$$\begin{aligned}\frac{d^2(s(\tilde{g}_t))}{dt^2}|_{t=0} =& -\frac{1}{4}\sum_{i<j}\{\frac{d^2(F_{i,j}^t)^2}{dt^2}|_{t=0} + \frac{d^2(F_{j,i}^t)^2}{dt^2}|_{t=0}\}\\ &-\frac{1}{4}\sum_{i=1}^n\frac{d^2(F_{i,2n+1}^t)^2}{dt^2}|_{t=0}\\ =& -\frac{1}{2}\sum_{i<j}\{(F_{i,j})^2 + (F_{j,i})^2\} - \frac{1}{2}\sum_{i=1}^n(F_{i,2n+1})^2 \leq 0.\end{aligned}$$

Note that inside $\mathbf{C}$ the set of points with $\frac{d^2}{dt^2}(s(\tilde{g}_t))|_{t=0} = 0$ is identical to the set $\mathfrak{T}$. We see that $s(\tilde{g}_t)$ is strictly decreasing only on $\mathbf{C}\backslash\mathfrak{T}$. In order to have the right decreasing property, we need to diffuse the negativity (of scalar curvature) onto a ball containing $\mathbf{C}\backslash\mathfrak{T}$.

4. Diffusion of negative scalar curvature onto a ball

Our argument in this section is similar to that in [8, Section 4], so we avoid some details. We use the following functions; $F_{t,m}(\rho) \in C^\infty(\mathbb{R}, \mathbb{R}^{\geq 0})$ for $m > 0$, $t \geq 0$ defined by $F_{t,m}(\rho) = m \cdot t^2 \cdot \exp(-\frac{100}{\rho})$ on $\mathbb{R}^{>0}$ and $F_{t,m} = 0$ on $\mathbb{R}^{\leq 0}$. Also choose an $H \in C^\infty(\mathbb{R}, [0,1])$ with $H = 0$ on $\mathbb{R}^{\geq 1}$, $H = 1$ on $\mathbb{R}^{\leq 0}$ and $H^b_\epsilon(\rho) = H(\frac{1}{\epsilon}(\rho - b))$ for $b > 0$, $\epsilon > 0$.

Let $B_r(x)$ be the open ball of radius r with respect to g_0 centered at x. We choose a point p and a number $\epsilon_1 < 0.1$ so that $B_{2\epsilon_1}(p) \subset \mathbf{C}\backslash\mathfrak{T}$. Then $s(\tilde{g}_t) < 0$ on $B_{\epsilon_1}(p)$ when $0 < t < c$ for some number c.

Let $f_{t,m} \in C^\infty(\mathbb{R}^{2n+1}, \mathbb{R}^{\geq 0})$ be $f_{t,m}(q) = F_{t,m}(\rho(q))$, where ρ is the g_0-distance from the above point p to $q \in \mathbb{R}^{2n+1}$ and let $h^b_\epsilon \in C^\infty(\mathbb{R}^{2n+1}, \mathbb{R}^{\geq 0})$ be $h^b_\epsilon(q) = H^b_\epsilon(\rho(q))$. We choose $b = 9$ and $\epsilon = \epsilon_1$. We consider the Riemannian metric $e^{2\phi_t}\tilde{g}_t$, where

$$\phi_t(\rho) = f_{t,m}(9 + \epsilon_1 - \rho) \cdot h^9_{\epsilon_1}(9 + \epsilon_1 - \rho) = mt^2 e^{-\frac{100}{9+\epsilon_1-\rho}} h^9_{\epsilon_1}(9 + \epsilon_1 - \rho).$$

We consider the scalar curvature $s(e^{2\phi_t}\tilde{g}_t)$. We easily get $\frac{ds(e^{2\phi_t}\tilde{g}_t)}{dt}|_{t=0} = 0$. Using the conformal deformation formula $s(e^{2\phi_t}g_t) = e^{-2\phi_t}(s_{g_t} + 4n\Delta_{g_t}\phi_t - 2n(2n-1)|\nabla_{g_t}\phi_t|^2)$, we calculate as in [8, Section 4] to show that $\frac{d^2 s(e^{2\phi_t}\tilde{g}_t)}{dt^2}|_{t=0} < 0$ on $B_{9+\epsilon_1}(p)$ for small $m > 0$. Note that $e^{2\phi_t}\tilde{g}_t = g_0$ on $\mathbb{R}^{2n+1}\backslash B_{9+\epsilon_1}(p)$.

But due to the boundary $\partial B_{9+\epsilon_1}(p)$, we can not yet conclude the existence of a constant ε such that $s(e^{2\phi_t}\tilde{g}_t)$ is strictly decreasing in the ball $B_{9+\epsilon_1}(p)$ for $0 \leq t \leq \varepsilon$.

We continue to follow the argument in [8, Section 4] to show that $\frac{ds(e^{2\phi_t}\tilde{g}_t)}{dt} < 0$ on $B_{9+\epsilon_1}(p)\backslash\overline{B_9(p)}$ when $0 < t \leq t_0$ for some number $t_0 > 0$.

This yields a scalar-curvature melting $g_t = e^{2\phi_t}\tilde{g}_t$ on $B_{9+\epsilon_1}(p)$. By pulling it back by an affine transformation, we can get a scalar-curvature melting on the unit ball.

In even dimensions, we start with the metrics in Proposition 2.2 and proceed similarly as in Section 3 and Section 4. Then we can get a scalar-curvature melting on the unit ball in $\mathbb{R}^{2n}$, $n \geq 2$. This proves Theorem 1.1.

Remark 4.1. The odd dimensional metric in Proposition 2.1 is in fact a contact metric compatible with the standard contact structure on $\mathbb{R}^{2n+1}$. We suspect our melting can be done in the space of contact metrics. It is very interesting to find a scalar curvature melting of a general metric on a ball, not to mention a Ricci-curvature melting.

5. Fubini-Study metric

In this section we demonstrate that the arguments for Euclidean metrics can work similarly for the Fubini-Study metric.

We need to discuss in the context of almost Kähler metrics, which are Riemannian metrics g compatible with a symplectic structure ω, i.e., $\omega(X, Y) =$

$g(X, JY)$ for an almost complex structure J, where X, Y are tangent vectors. Here ω and g determine J. One may refer to [2] for some knowledge of almost Kähler geometry needed in this section. In this geometry, for the canonical hermitian connection ∇ determined by J we have the corresponding *hermitian scalar* curvature s^∇. It proves to be equal to $\frac{1}{2}(s^* + s)$, where s^* is the *star-scalar* curvature. It is known that $s^* - s = \frac{1}{2}|DJ|^2$, where D is the Levi-Civita connection. So $s^\nabla \geq s$, with equality if and only if (ω, g) is Kähler .

In [9, Subsection 4.1], for a toric symplectic manifold (M^{2n}, ω), i.e., a symplectic manifold equipped with an effective Hamiltonian action of an n-dimensional torus T, M. Lejmi considered ω-compatible T-invariant almost Kähler metrics g which have the local expression

$$g = \sum_{i,j=1}^{n} G_{ij}(z) dz_i \otimes dz_j + H_{ij}(z) dt_i \otimes dt_j, \tag{4}$$

where $z_1, \ldots, z_n$ are moment coordinates corresponding to Hamiltonian vector fields generating T action and $H = (H_{ij})$ is a symmetric positive-definite matrix-valued function and $G = (G_{ij})$ is the inverse matrix of H. In z, t coordinates, $\omega = \sum dz_i \wedge dt_i$. Any metric of the form (4) is ω-compatible almost Kähler. He computed that $s^\nabla = \frac{1}{2}(s + s^*) = -\sum_{i,j=1}^{n} H_{ij,ij}$, where $(\cdot)_{,ij} = \frac{\partial^2(\cdot)}{\partial z_j \partial z_i}$.

Example ([1])**.** Consider the complex projective space $\mathbb{CP}_n$ with the Fubini-Study metric g_{FS} in homogeneous coordinates $[z_0, z_1, \ldots, z_n]$. We denote the Kähler form by ω_{FS}. The T^n-action on $\mathbb{CP}_n$ given by $(y_1, \ldots, y_n) \cdot [z_0, z_1, \ldots, z_n] = [z_0, e^{-y_1 i} z_1, \ldots, e^{-y_n i} z_n]$, is Hamiltonian, with moment map $\mu : \mathbb{CP}_n \to \mathbb{R}^n$ given by $\mu([z_0, z_1, \ldots, z_n]) = \frac{1}{\|z\|^2}(\|z_1\|^2, \ldots, \|z_n\|^2)$.

Set $S_t := \{(x_1, \ldots, x_n) \mid \text{each } x_i > 0,\ \sum_{i=1}^{n} x_i < t\} \subset \mathbb{R}^n$. Then the image of μ is the closure of S_1. g_{FS} can be expressed as (4) with some $H^0_{ij}(z)$.

Proposition 5.1. *Given an open set S_c, $0 < c < 1$, there exists a family of T^n-invariant almost-Kähler metrics $(\omega_{FS}, \bar{g}_t)$ on $\mathbb{CP}_n$, $0 \leq t < \epsilon_2$ for some number ϵ_2, such that*

(i) *on $\mathbb{CP}_n - \mu^{-1}(S_c)$; $\bar{g}_t = g_{FS}$ for $0 \leq t < \epsilon_2$,*
(ii) *on $\mathbb{CP}_n$; $\bar{g}_0 = g_{FS}$, $s^{\nabla_{\bar{g}_t}} = s^{\nabla_{\bar{g}_0}}$ and $s(\bar{g}_t) \leq s(\bar{g}_0)$ for $0 \leq t < \epsilon_2$,*
(iii) *$s(\bar{g}_t) < s(\bar{g}_0)$ for $0 < t < \epsilon_2$ on some open subset W of $\mu^{-1}(S_c)$.*

Proof. Set $H^t_{ij}(z) = H^0_{ij}(z) + tU_{ij}(z)$ and we denote the corresponding metric in (4) by $\bar{g}_t$. The condition $s^{\nabla_{\bar{g}_t}} = s^{\nabla_{\bar{g}_0}}$ is equivalent to $\sum_{i,j=1}^{n} \{U_{ij}\}_{,ij} = 0$. For its solution, choose $U = (U_{ij})$ as the diagonal matrix with diagonal entries

$$U_{ii}(z) = \alpha^i_1(z_1) \cdots \alpha^i_i\hat{(}r_i) \cdots \alpha^i_n(z_n) \int_0^{z_i} (\int_0^y \alpha^i_i(x)\, dx)\, dy \text{ for } i = 1, \ldots, n-1,$$

where ˆ denotes the missing factor in that position,

$$U_{nn}(z) = -\sum_{i=1}^{n-1} \alpha_1^i(z_1)\cdots\alpha_{n-1}^i(z_{n-1}) \int_0^{z_n} (\int_0^y \alpha_n^i(x)\,dx)\,dy,$$

where $\alpha_j^i(z_j)$, $i = 1,\ldots,n-1$, $j = 1,\ldots,n$ are smooth functions on $\mathbb{R}$ which satisfy at least $\alpha_j^i(z_j) = 0$ for $z_j \leq 0$, or $z_j \geq \tilde{c}$ for some $\tilde{c} > 0$. This is similar to the solution of the equation (2). Again, one can properly choose $\tilde{c}$ small and α_j^i so that U_{ij} become smooth functions with compact support in $\mu^{-1}(S_c)$ and that $\bar{g}_t$, $t > 0$, is an almost Kähler metric which is non-Kähler, i.e., $\frac{1}{2}|DJ|^2 = s^* - s \neq 0$ somewhere. Indeed, either by direct computation on a component of DJ or by an argument using [5, Section 4], one can find $\{U_{ij}\}$ so that near some chosen point $\bar{g}_t$ is non-Kähler for any small t.

As (ω_{FS}, g_{FS}) is Kähler, $s(g_{FS}) = s^{\nabla^{\bar{g}_0}}$. But then, $s^{\nabla^{\bar{g}_0}} = s^{\nabla^{\bar{g}_t}} \geq s(\bar{g}_t)$ with equality exactly where $(\omega, \bar{g}_t)$ is Kähler. This proves that $s(\bar{g}_t) < s(\bar{g}_0)$ for $0 < t < \epsilon_2$ on an open pre-compact subset W of $\mu^{-1}(S_c)$. □

The metrics $\bar{g}_t$ play the same role as those in Propositions 2.1 or 2.2.

Theorem 5.2. *Suppose we are given a point $p_0 \in \mathbb{CP}_n$ and a number r_0 with $0 < r_0 < \frac{1}{2}$diameter(g_{FS}). Then there exists a C^∞-continuous path of Riemannian metrics g_t on $\mathbb{CP}_n$, which exists for $0 \leq t < \varepsilon$ for some number ε with the following property: $g_0 = g_{FS}$, $s(g_{\tilde{t}}) < s(g_t)$ for $0 \leq t < \tilde{t} < \varepsilon$ in the ball $B_{r_0}^{g_{FS}}(p_0)$ of g_{FS}-radius r_0 centered at p_0 and g_t is isometric to g_{FS} in the complement of the ball.*

Proof. Since $(\mathbb{CP}_n, g_{FS})$ is homogeneous, we may choose the coordinates and hamiltonian T^n action so that $p_0 = \mu^{-1}(0,\ldots,0)$. We choose c so that $\mu^{-1}(S_c) \subset B_{\frac{r_0}{2}}^{g_{FS}}(p_0)$ and get $\bar{g}_t$ in Proposition 5.1. Choose the smallest natural number k such that $\frac{d^k s_{\bar{g}_t}}{dt^k}|_{t=0}$ is not identically zero. This k exists because at each point $s_{\bar{g}_t}$ is a rational function of t. Then $\frac{d^j s_{\bar{g}_t}}{dt^j}|_{t=0} \equiv 0$ for $j = 1,\ldots,k-1$. $\frac{d^k s_{\bar{g}_t}}{dt^k}|_{t=0} \leq 0$ and $\frac{d^k s_{\bar{g}_t}}{dt^k}|_{t=0}(p) < 0$ at some $p \in W$. We now apply the argument of Section 4.

We consider a smooth coordinates system $y := y_1,\ldots,y_{2n}$ on $B_{\frac{3}{2}r_0}^{g_{FS}}(p_0)$, which is a topological ball, such that $y(0) = p$ and $B_{r_0}^{g_{FS}}(p_0)$ becomes a y-coordinates ball of radius, say R. Let g^0 be the Euclidean metric $g^0 = dy_1^2 + \cdots + dy_{2n}^2$ and $\rho = \sqrt{\sum_{i=1}^{2n} y_i^2}$.

From now on, $B_r(\cdot)$ means a ball of g^0-radius r with center at $\cdot$. For some positive number $\epsilon < \frac{R}{10}$, $B_{2\epsilon}(p)$ should satisfy $B_{2\epsilon}(p) \cap \{q \mid \frac{d^k s_{\bar{g}_t}}{dt^k}(q)|_{t=0} = 0\} = \emptyset$. Choosing ϵ further small if necessary, we assume that $B_{R-\epsilon}(p) \supset B_{\frac{r_0}{2}}^{g_{FS}}(p_0)$.

Define $F_{t,m}^d(x) = mt^k e^{-\frac{d}{x}}$. We consider $g_t := e^{2\phi_t}\bar{g}_t$, where $\phi_t(\rho) = F_{t,m}^d(b + \epsilon - \rho) \cdot h_\epsilon^b(b + \epsilon - \rho)$. We set $b = R - \epsilon$. m and d shall be determined below.

The scalar curvature is as follows; $s(g_t) = e^{-2\phi_t}B$, where $B = s_{\bar{g}_t} + a_n\Delta_{\bar{g}_t}\phi_t - b_n|\nabla_{\bar{g}_t}\phi_t|^2$ for some positive numbers a_n, b_n depending on n. Then

$$\frac{ds(g_t)}{dt} = e^{-2\phi_t}(-2\frac{d\phi_t}{dt}B + \frac{ds_{\bar{g}_t}}{dt} + a_n\frac{d\Delta_{\bar{g}_t}\phi_t}{dt} - b_n\frac{d|\nabla_{\bar{g}_t}\phi_t|^2}{dt}). \tag{5}$$

We easily get $\frac{d^j s(g_t)}{dt^j}|_{t=0} = 0$ for $j = 1, \ldots, k-1$ and

$$\frac{d^k s(g_t)}{dt^k}|_{t=0} = -2k!ms_{g_0}e^{-\frac{d}{b+\epsilon-\rho}}h^b_\epsilon(b+\epsilon-\rho) + \frac{d^k s_{\bar{g}_t}}{dt^k}|_{t=0} + a_n\frac{d^k\Delta_{\bar{g}_t}\phi_t}{dt^k}|_{t=0}\ .$$

On $B_{b+\epsilon}(p) - B_\epsilon(p)$, since $h^b_\epsilon(b+\epsilon-\rho) = 1$ we have

$$\begin{aligned}\frac{d^k s(g_t)}{dt^k}|_{t=0} &\leq -2k!ms_{g_0}e^{-\frac{d}{b+\epsilon-\rho}} + a_n\frac{d^k\Delta_{\bar{g}_t}\phi_t}{dt^k}|_{t=0}\\ &= mk!(-2s_{g_0}e^{-\frac{d}{b+\epsilon-\rho}} + a_n\Delta_{g_0}e^{-\frac{d}{b+\epsilon-\rho}})\\ &\leq mk!(-2s_{g_0}G - \alpha_1 G^{''} - \alpha_2 G^{'}) < 0,\ \text{when } d \text{ is large,}\end{aligned}$$

where $G(\rho) = e^{-\frac{d}{b+\epsilon-\rho}}$ and α_1, α_2 are some positive numbers and we used Lemmas 5.3 and 5.4 below. On $B_\epsilon(p)$, $\frac{d^k s_{\bar{g}_t}}{dt^k}|_{t=0} < -c_1 < 0$ for some number $c_1 > 0$, so choose $m > 0$ small so that $-2k!ms_{g_0}e^{-\frac{d}{b+\epsilon-\rho}}h^b_\epsilon(b+\epsilon-\rho) + \frac{d^k s_{\bar{g}_t}}{dt^k}|_{t=0} + a_n\frac{d^k\Delta_{\bar{g}_t}\phi_t}{dt^k}|_{t=0} < 0$.

In sum, we have $\frac{d^j s(g_t)}{dt^j}|_{t=0} = 0$ for $j = 1, \ldots, k-1$ and $\frac{d^k s(g_t)}{dt^k}|_{t=0} < 0$ on $B_{b+\epsilon}(p)$ and $g_t = g_0$ on $M - B_{b+\epsilon}(p)$. On $\overline{B_b(p)}$, there exists $\epsilon_3 > 0$ such that $s(g_t)$ is strictly decreasing for $0 \leq t \leq \epsilon_3$.

On $B_{b+\epsilon}(p) - \overline{B_b(p)}$, $\bar{g}_t = g_0$. From (5), Lemmas 5.3, 5.4 and 5.5, for large d,

$$\begin{aligned}&e^{2\phi_t}\frac{ds(g_t)}{dt}\\ &= -2\frac{d\phi_t}{dt}(s_{g_0} + a_n\Delta_{g_0}\phi_t - b_n|\nabla_{g_0}\phi_t|^2) + a_n\frac{d\Delta_{g_0}\phi_t}{dt} - b_n\frac{d|\nabla_{g_0}\phi_t|^2}{dt}\\ &= -2kmt^{k-1}e^{-\frac{d}{b+\epsilon-\rho}}(s_{g_0} + a_n mt^k\Delta_{g_0}e^{-\frac{d}{b+\epsilon-\rho}} - b_n m^2 t^{2k}|\nabla_{g_0}e^{-\frac{d}{b+\epsilon-\rho}}|^2)\\ &\quad + ka_n mt^{k-1}\Delta_{g_0}e^{-\frac{d}{b+\epsilon-\rho}} - 2kb_n m^2 t^{2k-1}|\nabla_{g_0}e^{-\frac{d}{b+\epsilon-\rho}}|^2\\ &\leq kmt^{k-1}\{-2e^{-\frac{d}{b+\epsilon-\rho}}(s_{g_0} + a_n mt^k\Delta_{g_0}e^{-\frac{d}{b+\epsilon-\rho}}) + a_n\Delta_{g_0}e^{-\frac{d}{b+\epsilon-\rho}}\}\\ &\leq kmt^{k-1}(-2s_{g_0}e^{-\frac{d}{b+\epsilon-\rho}} + \frac{a_n}{2}\Delta_{g_0}e^{-\frac{d}{b+\epsilon-\rho}})\\ &\leq kmt^{k-1}(-2s_{g_0}G - \tilde{\alpha_1}G^{''} - \tilde{\alpha_2}G^{'}) < 0 \text{ for numbers } \tilde{\alpha_1}, \tilde{\alpha_2} > 0,\end{aligned}$$

while $0 < t < \epsilon_4$ for some ϵ_4. This implies that $s(g_t)$ is strictly decreasing for $0 \leq t < \epsilon_4$ on $B_{b+\epsilon}(p) - \overline{B_b(p)}$. So, $s(g_t)$ is strictly decreasing for $0 \leq t < \varepsilon = \min\{\epsilon_3, \epsilon_4\}$ on $B_{b+\epsilon}(p)$. This proves Theorem 5.2. □

For the function $F(t) = e^{-\frac{d}{t}}$ on $\mathbb{R}^{>0}$, one can modify easily Lemma 1.2 in [10] as follows; for $m_0, m_1 \in \mathbb{R}$ and $m_2, b \in \mathbb{R}^{>0}$ there exist numbers $d_0(b) > 0$ and $d_1(m_0, m_1, m_2, b) > 0$ such that $F^{(j)} := \frac{d^j F}{dt^j} > 0$ on $(0, b)$ for $j = 0, 1, 2, 3$ if $d \geq d_0(b)$ and $m_2 F^{''} + m_1 F^{'} + m_0 F > 0$ on $(0, b)$ if $d \geq d_1(m_0, m_1, m_2, b)$. Since $G^{(j)}(\rho) = (-1)^j F^{(j)}(b + \epsilon - \rho)$, we get:

Lemma 5.3. *For $m_0, m_1 \in \mathbb{R}$ and $m_2, b \in \mathbb{R}^{>0}$, there exists $d_2(m_0, m_1, m_2, b) > 0$ such that $m_2 G^{''} + m_1 G^{'} + m_0 G > 0$ on $(\epsilon, b + \epsilon)$ if $d \geq d_2(m_0, m_1, m_2, b)$. And $(-1)^j G^{(j)} > 0$ on $(\epsilon, b + \epsilon)$ for $j = 0, 1, 2, 3$ if $d \geq d_0(b)$.*

Next, we modify Corollary 2.3 in [10] as follows. Assume that g on a domain $D \subset \mathbb{R}^{n+1}$ fulfill the following two conditions for some $k > 1$: (i) $g_{\text{Eucl}}(\nu, \nu) \leq k^2 \cdot g(\nu, \nu)$. (ii) The C^3-norm $\|g\|_{C^3_{g_{Eucl}}}(D) \leq k$. Let $H \in C^\infty(\mathbb{R}, \mathbb{R})$ be a function with $H^{'} \leq 0$, $H^{''} \geq 0$. Then there are constants $a_1, a_2 > 0$ depending only on n and k such that $(a_1 H^{''} + a_2 H^{'}) \circ \pi \leq -\Delta_g(H \circ \pi)$ on D, where $\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the projection. This can be easily verified, following the argument in pp. 660–661 in [10].

We can choose a coordinates system $(u_1, \ldots, u_{2n})$ with $u_1 = \rho$ on a proper subdomain $\tilde{D}$ of $B_{b+\epsilon}(p) - B_\epsilon(p)$ so that (i) and (ii) holds with $g_{Eucl} := d\rho^2 + du_2^2 + \cdots + du_{2n}^2$. Applying the above paragraph to $g_0|_{\tilde{D}}$ and G, we get:

Lemma 5.4. *If $d \geq d_0(b)$, there are constants $a_1, a_2 > 0$ such that $\Delta_{g_0} G(\rho) \leq -a_1 G^{''} - a_2 G^{'}$ on $B_{b+\epsilon}(p) - B_\epsilon(p)$.*

Putting Lemmas 5.3 and 5.4 together;

Lemma 5.5. *$\Delta_{g_0} G(\rho) < 0$ on $B_{b+\epsilon}(p) - B_\epsilon(p)$ if d is large.*

Remark 5.6. For the Fubini-Study metric, the kernel of L_g^* on $\mathbb{CP}_n$ is trivial. But we do not know if the kernel of L_g^* is trivial when restricted to a ball. In any case, our construction gives a large amount of deformation, compared to the small deformation of Corvino's, as the latter is based on Implicit Function Theorem.

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