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SURFACES OF REVOLUTION SATISFYING $\Delta^{II}\mathbf{G} = f(\mathbf{G} + C)$

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Abstract. In this paper, we study surfaces of revolution without parabolic points in 3-Euclidean space $\mathbb{R}^3$, satisfying the condition $\Delta^{II}\mathbf{G} = f(\mathbf{G}+C)$, where Δ^{II} is the Laplace operator with respect to the second fundamental form, f is a smooth function on the surface and C is a constant vector. Our main results state that surfaces of revolution without parabolic points in $\mathbb{R}^3$ which satisfy the condition $\Delta^{II}\mathbf{G} = f\mathbf{G}$, coincide with surfaces of revolution with non-zero constant Gaussian curvature.

1. Introduction

During the late 1970's, B.-Y. Chen introduced the notion of Euclidean immersions of finite type. Basically, these are submanifolds whose immersion $r : M \to \mathbb{R}^m$ into $\mathbb{R}^m$ is constructed by making use of finite number of $\mathbb{R}^m$-valued eigenfunctions of their Laplacian [3, 4]. Many works were done to characterize the classification of submanifolds in terms of finite type and many interesting results have been obtained (see [5] for a report on this subject).

Later, B.-Y. Chen and P. Piccinni [7] extented this notion to differential maps defined on the submanifold M, in particular to Gauss map $\mathbf{G}$ on submanifolds in Euclidean space.

Then, in [1, 2, 11] submanifolds with 1-type Gauss map are studied. On the other hand, one can notice that the Laplacian of the Gauss map of a helicoid, a catenoid and a right cone in $\mathbb{R}^3$ take the form

$$\Delta\mathbf{G} = f(\mathbf{G} + C), \tag{1.1}$$

where Δ is the Laplace operator with respect to the first fundamental form, f is a smooth function on the surface and C is a constant vector. This led several geometers to classify all submanifolds in a Euclidean m-space $\mathbb{R}^m$ (or in the Lorentz-Minkowski space $\mathbb{R}^m_1$) satisfying the condition (1.1).

A submanifold M of a Euclidean space $\mathbb{R}^m$ is said to have *pointwise 1-type Gauss map* if its Gauss map $\mathbf{G}$ satisfies the condition (1.1) for some non-zero smooth function f on M and a constant vector C. A submanifold with

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pointwise 1-type Gauss map is said to be *of the first kind* if C is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be *of the second kind* [6, 8, 12, 13].

In [8], M. Choi and Y. H. Kim obtained a characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map in $\mathbb{R}^3$.

Moreover, in the paper [14], A. Niang studied rotation surfaces in $\mathbb{R}^3$ under the condition

$$\Delta\mathbf{G} = f\mathbf{G},$$

and obtained a characterization theorem for rotation surfaces of constant mean curvature. In [15] he investigated the Lorentz version of the rotation surfaces in $\mathbb{R}^3_1$ under the same condition. Also, B-Y. Chen, M. Choi and Y. H. Kim [6] proved that surfaces of revolution in $\mathbb{R}^3$ with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature. Furthermore, they characterized the rational surfaces of revolution with pointwise 1-type Gauss map in $\mathbb{R}^3$.

Later, M. Choi, D.-S. Kim and Y. H. Kim [9] studied the helicoidal surfaces with pointwise 1-type or harmonic Gauss map in $\mathbb{R}^3$. They gave again a characterization of rational helicoidal surfaces with pointwise 1-type or harmonic Gauss map.

If a surface in $\mathbb{R}^3$ has no parabolic points, then the second fundamental form can be regarded as a new (pseudo) Riemannian metric.

In this paper, we consider surfaces of revolution without parabolic points in 3-Euclidean space $\mathbb{R}^3$, satisfying the condition

$$\Delta^{II}\mathbf{G} = f(\mathbf{G} + C), \tag{1.2}$$

where Δ^{II} is the Laplace operator with respect to the second fundamental form, f is a smooth function on the surface and C is a constant vector. Our main results state that surfaces of revolution without parabolic points in $\mathbb{R}^3$ which satisfy the condition

$$\Delta^{II}\mathbf{G} = f\mathbf{G}, \tag{1.3}$$

for non-zero function f, coincide with surfaces of revolution with non-zero constant Gaussian curvature. Also, we affirm that there is no surface of revolution in $\mathbb{R}^3$ which satisfies $\Delta^{II}\mathbf{G} = 0$.

Throughout this paper, we assume that all surfaces are connected and all objects are smooth, unless otherwise specified.

2. Preliminaries

Let M^2 be a 2-dimensional surface,without parabolic points, of the Euclidean 3-space $\mathbb{R}^3$. The map $\mathbf{G} : M^2 \to \mathbf{S}^2(1) \subset \mathbb{R}^3$ which sends each point of M^2 to the unit normal vector to M^2 at the point is called the *Gauss map* of the surface M^2, where $\mathbf{S}^2(1)$ is the unit sphere of $\mathbb{R}^3$.

On the other hand, we denote by $E, F, G; L, M, N$ the coefficients of the first and second fundamental form, respectively, of this surface.

If $\phi : M^2 \to \mathbb{R}, (u,v) \mapsto \phi(u,v)$ is a smooth function and Δ^{II} is the Laplace operator with respect to the second fundamental form of M^2, then from [16] we have

$$\Delta^{II}\phi = -\frac{1}{\sqrt{|LN - M^2|}}\left[\left(\frac{N\phi_u - M\phi_v}{\sqrt{|LN - M^2|}}\right)_u - \left(\frac{M\phi_u - L\phi_v}{\sqrt{|LN - M^2|}}\right)_v\right], \tag{2.1}$$

where $LN - M^2 \neq 0$ since the surface has no parabolic points.

The mean curvature H and the Gaussian curvature K_G can be computed by the well-known classical formulas

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)}, \quad K_G = \frac{LN - M^2}{EG - F^2}. \tag{2.2}$$

Now, we give a definition of a surface of revolution in $\mathbb{R}^3$.

For an open interval J, let $\gamma : J \to \Pi$ be a curve in a plane Π in $\mathbb{R}^3$ and let l be a straight line in Π which does not intersect the curve γ. A surface of revolution M^2 in $\mathbb{R}^3$ is defined to be a non-degenerate surface revolving a *profile curve* γ around the *axis* l. In other words, a surface M^2 of revolution with axis l in $\mathbb{R}^3$ is invariant under the action of the group of motions in $\mathbb{R}^3$ which fixes each point of the line l.

We may suppose that the axis is the z-axis and without loss of generality, we may suppose that Π is the xz-plane. Then the profile curve γ is given by

$$\gamma(u) = (\varphi(u), 0, \psi(u)),$$

where φ is a positive function and ψ is a function on J.

So, M^2 can be parameterized by

$$r(u,v) = (\varphi(u)\cos v, \varphi(u)\sin v, \psi(u)); \;\; u \in J, v \in \mathbb{R}. \tag{2.3}$$

For the sake of simplicity, we suppose that the curve γ is parameterized by the arc-length, so

$$\varphi'^2(u) + \psi'^2(u) = 1, \quad \forall u \in J. \tag{2.4}$$

Then we have the natural frame $\{r_u, r_v\}$ given by

$$r_u = (\varphi'(u)\cos v, \varphi'(u)\sin v, \psi'(u)),$$

$$r_v = (-\varphi(u)\sin v, \varphi(u)\cos v, 0).$$

Accordingly we see

$$E = 1, F = 0, G = \varphi^2.$$

The unit normal vector to M^2 is defined by

$$\mathbf{G} = \frac{r_u \times r_v}{\varphi},$$

so we get

$$\mathbf{G} = (-\psi'(u)\cos v, -\psi'(u)\sin v, \varphi'(u)). \tag{2.5}$$

Then $\mathbf{G}$ is the unit normal vector to M^2 and hence it can be regarded as a Gauss map of M^2 into the 2-dimensional unit sphere $\mathbf{S}^2(1)$. Moreover, we get

$$L = \varphi'\psi'' - \psi'\varphi'',\ M = 0,\ N = \varphi\psi',$$
$$2H = \varphi'\psi'' - \psi'\varphi'' + \frac{\psi'}{\varphi},\ K_G = \frac{-\varphi''}{\varphi}.$$

3. Main theorems

First of all, we need the following lemma.

Lemma 1. *Let M^2 be a surface of revolution in $\mathbb{R}^3$ whose Gauss map satisfies $\Delta^{II}\mathbf{G} = f(\mathbf{G} + C)$ for some non-zero function f. Then, f depends only on u and the vector C is parallel to the axis of the surface of revolution.*

Proof. By using (2.5), (2.1) and the second fundamental form given above, one shows that

$$\Delta^{II}G = \begin{pmatrix} \left(-\varphi'' + \frac{1}{2\varphi} + \frac{\psi'^2}{2\varphi} - \frac{\varphi'\varphi'''}{2\varphi''}\right)\cos v \\ \left(-\varphi'' + \frac{1}{2\varphi} + \frac{\psi'^2}{2\varphi} - \frac{\varphi'\varphi'''}{2\varphi''}\right)\sin v \\ -\psi'' - \frac{\varphi'\psi'}{2\varphi} - \frac{\psi'\varphi'''}{2\varphi''} \end{pmatrix}. \tag{3.1}$$

Suppose that M^2 satisfies (1.2) for some non-zero function f and some vector C. Then, from (1.2), (2.5) and (3.1) we deduce that the first two components of C must be zero and

$$\begin{aligned} -\varphi'' + \tfrac{1}{2\varphi} + \tfrac{\psi'^2}{2\varphi} - \tfrac{\varphi'\varphi'''}{2\varphi''} &= -f(u,v)\,\psi'(u), \\ -\psi'' - \tfrac{\varphi'\psi'}{2\varphi} - \tfrac{\psi'\varphi'''}{2\varphi''} &= f(u,v)\,(\varphi'(u) + c), \end{aligned} \tag{3.2}$$

where $C = (0, 0, c)$. Since we have (2.4), the function f depends only on u. $\square$

Henceforth, we assume that the vector C in (1.2) is the zero vector. Thus, we classify the surfaces of revolution M^2 in $\mathbb{R}^3$ satisfying the condition (1.3).

First of all, we consider a catenoid in $\mathbb{R}^3$. It is a surface parameterized by

$$r(u,v) = \left(\sqrt{1+u^2}\cos v, \sqrt{1+u^2}\sin v, \sinh^{-1} u\right).$$

Its Gauss map $\mathbf{G}$ is given by

$$\mathbf{G} = \frac{1}{\sqrt{1+u^2}}\left(-\cos v, -\sin v, u\right).$$

Then, one can get the Laplacian $\Delta^{II}\mathbf{G}$

$$\Delta^{II}\mathbf{G} = \frac{2u}{(1+u^2)^{\frac{3}{2}}}\left(u\cos v, u\sin v, 1\right).$$

Now, we prove the following.

Proposition 2. *If a surface of revolution M^2 in $\mathbb{R}^3$ satisfies*

$$\Delta^{II}\mathbf{G} = f\mathbf{G},$$

then $2H = -f(u)$ where H is the mean curvature of M^2.

Proof. Suppose that the immersed surface of revolution M^2 is given by (2.3). Since the relation (2.4) holds, there exists a smooth function $t = t(u)$ such that

$$\varphi'(u) = \cos t(u),\ \psi'(u) = \sin t(u),\ \forall u \in J.$$

Therefore,

$$LN - M^2 = \varphi(u)t'(u)\sin t(u),$$

$$2H = t'(u) + \frac{\sin t(u)}{\varphi(u)}, \quad K_G = \frac{t'(u)\sin t(u)}{\varphi(u)}, \tag{3.3}$$

$$\Delta^{II}\mathbf{G} = \begin{pmatrix} \left(t'\sin t + \frac{1}{2\varphi} + \frac{\sin^2 t}{2\varphi} - \frac{t''\cos t}{2t'} - \frac{t'\cos^2 t}{2\sin t}\right)\cos v \\ \left(t'\sin t + \frac{1}{2\varphi} + \frac{\sin^2 t}{2\varphi} - \frac{t''\cos t}{2t'} - \frac{t'\cos^2 t}{2\sin t}\right)\sin v \\ -\frac{3}{2}t'\cos t - \frac{\cos t\sin t}{2\varphi} - \frac{t''\sin t}{2t'} \end{pmatrix} \tag{3.4}$$

and since the surface has no parabolic points we must have $\varphi(u)t'(u)\sin t(u) \neq 0$. We may assume that M^2 satisfies (1.3), then

$$\langle \Delta^{II}\mathbf{G}, \mathbf{G}\rangle = f, \tag{3.5}$$

where $\langle\ ,\ \rangle$ is the natural inner product on $\mathbb{R}^3$. On the other hand, a straightforward computation, with help of $(3,3)$ yields

$$\langle \Delta^{II}\mathbf{G}, \mathbf{G}\rangle = -\left(t' + \frac{\sin t}{\varphi}\right) = -2H, \tag{3.6}$$

that is $-2H = f(u)$. □

Case 1: We suppose that f is a constant function.

Firstly, we can state the following corollary.

Corollary 3. *There do not exist surfaces of revolution in $\mathbb{R}^3$ which satisfy the condition $\Delta^{II}\mathbf{G} = 0$.*

Proof. The condition $\Delta^{II}\mathbf{G} = 0$ implies

$$t' + \frac{\sin t}{\varphi} = 0,$$

that is the mean curvature H vanishes identically. Therefore, the surface M^2 is minimal, that is M^2 is a catenoid. But, as was noted above, a catenoid does not satisfy $\Delta^{II}\mathbf{G} = 0$. In fact, the vectors $\Delta^{II}\mathbf{G}$ and $\mathbf{G}$ are orthogonal for a catenoid. □

Secondly, we may assume that the surface of revolution in $\mathbb{R}^3$ satisfies the condition $\Delta^{II}\mathbf{G} = f\mathbf{G}$, where f is a non-zero constant function ($f(u) = \lambda \neq 0$, $\forall u \in J$).

Y. H. Kim, C. W. Lee and D. W. Yoon [10] classified all surfaces of revolution without parabolic points in $\mathbb{R}^3$ satisfying the condition

$$\Delta^{II}\mathbf{G} = A\mathbf{G}, \quad A \in \mathrm{Mat}\,(3, \mathbb{R})$$

In particular, they stated that "*the only surface of revolution in a Euclidean 3-space whose Gauss map satisfies* $\Delta^{II}\mathbf{G} = \lambda\mathbf{G}, \lambda \in \mathbb{R}\setminus\{0\}$, *is locally the sphere* $S^2\left(C, \frac{2}{|\lambda|}\right)$ *where* $C = (0, 0, c)$". Let us note that the sphere has non-zero constant Gaussian curvature.

Case 2: We suppose that f is a non-constant function. If the function f defined by (1.3) is non-constant, we obtain the following theorem.

Theorem 4. *A surface of revolution without parabolic points in a Euclidean 3-space has non-zero constant Gaussian curvature if and only if it satisfies the condition*

$$\Delta^{II}\mathbf{G} = f\mathbf{G}$$

for some non-zero smooth function f on the surface.

Proof. Suppose that the surface of revolution M^2 has no parabolic points and satisfies the condition (1.3). By using the relations (3.4), (3.5) and (3.6) we obtain the system

$$\begin{cases} -\frac{\cos^2 t}{\varphi} + \frac{t'' \cos t}{t'} + \frac{t' \cos^2 t}{\sin t} = 0 \\ t' \cos t - \frac{\cos t \sin t}{\varphi} + \frac{t'' \sin t}{t'} = 0 \end{cases} \tag{3.7}$$

which can be reduced to the unique equation

$$t' \cos t - \frac{\cos t \sin t}{\varphi} + \frac{t'' \sin t}{t'} = 0.$$

Since M^2 has no parabolic points, the last equation is equivalent to

$$\left(\frac{t' \sin t}{\varphi}\right)' = 0$$

which implies that $\frac{t' \sin t}{\varphi}$ is a constant. It follows with the help of (3.3) that the Gaussian curvature K_G is constant ($K_G \neq 0$ because $LN - M^2 \neq 0$).

Conversely, assume the Gaussian curvature K_G be non-zero constant, $K_G' = 0$. By using (3.3), this means that

$$\frac{t'^2 \cos t}{\varphi} + \frac{t'' \sin t}{\varphi} - \frac{t' \cos t \sin t}{\varphi^2} = 0.$$

If we multiply this last equation by $\frac{\varphi \cos t}{t' \sin t}$ and by $\frac{\varphi}{t'}$, respectively, we get the two equations in (3.7). We conclude by noting that (3.7) is equivalent to the fact that M^2 satisfies the condition (1.3). This proves the theorem. □

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