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# Cost analysis on renewable warranty policies subject to imperfect strategies using inter-failure intervals

**Minjae Park***
*College of Business Administration, Hongik University, Seoul, 121-791 Korea*

**Abstract.** In this paper, cost analysis is conducted using inter-failure interval under renewable warranty subject to imperfect repair for multi-component system. One way to model the imperfect repair is to use the quasi-renewal process (Wang and Pham 1996). Two alternative quasi-renewal processes were suggested by Park and Pham (2010) using quasi-renewal process; first is an altered quasi-renewal process with random variable parameter and second is a mixed quasi-renewal process considering replacement service and repair service, simultaneously. In this study, we use the altered and mixed quasi-renewal processes and develop the warranty cost model to obtain the expected value of warranty cost and to help company make important decisions regarding the warranty policy. Numerical examples are used to demonstrate the applicability of the methodology derived in the paper.

**Key Words*:* ***i**mperfect repair, multi-component system, quasi-renewal processes, replacement service, warranty cost*

## NOTATIONS

| | |
|---|---|
| pmf, pdf, cdf | Probability mass function, probability density function, cumulative distribution function, respectively |
| QRP | Quasi-Renewal Processes |
| CV | Coefficient of Variation |
| $w$ | Length of a warranty period |
| $EXP(\lambda)$ | Exponential distribution with parameter $\lambda$ |
| $h$ | Upper limit on the number of repair under warranty |
| $\alpha, \beta$ | Parameters for QRP and altered QRP, respectively |
| $q$ | Number of components in series system |
| $N_a, N_b$ | Number of repair and replacement within a warranty period, respectively |
| $N_s$ | Number of system failures, i.e. $N_s = N_a + N_b$ |

* Corresponding Author.
E-mail address: *mjpark@hongik.ac.kr*

| | |
|---|---|
| $X_i$ | Renewal inter-failure time of repairs between the ($i$-1)th and ($i$)th events of the process |
| $Y_i$ | Renewal inter-failure time of replacements between the ($i$-1)th and ($i$)th events of the process |
| $C$ | Warranty cost for a failed system within a warranty period *w* |
| $c_a, c_b$ | Warranty cost for repairs and replacements, respectively, within a warranty period *w* |
| $N_a(t), N_b(t), N_s(t)$ | Number of repairs, number of replacements and number of system failures, respectively, at *r.v.* time *T* |
| $f_s(\cdot), F_s(\cdot), R_s(\cdot)$ | pdf, cdf and reliability function of system failure times, respectively, within a warranty period *w* |
| $f_{ij}(\cdot), F_{ij}(\cdot), R_{ij}(\cdot)$ | pdf, cdf and reliability function of component *j*'s failure times after *i*-1 repair/replacement, respectively, within a warranty period *w* |
| $f_{is}(\cdot), F_{is}(\cdot), R_{is}(\cdot)$ | pdf, cdf and reliability function of system failure times after *i*-1 repair/replacement, respectively, within a warranty period *w* |
| $f_{ijk}(\cdot), F_{ijk}(\cdot), R_{ijk}(\cdot)$ | pdf, cdf and reliability function of $k^{th}$ components in cluster *j* and after *i*-1 repair/replacement, respectively, within a warranty period *w* |

## 1. INTRODUCTION

Many multinational companies offer global competitive warranty policy to capture the interest of consumers in a competitive market. Warranty policy guarantees consumer a replacement or a repair service for a product in case of failure while it is functioning. The manufacturer can use warranty policy as a marketing tool to increase sales, brand recognition, customer loyalty and etc. Also, there is a significant cost related to the policy, therefore analyzing the cost structure for the warranty policy is important. Developing a warranty policy that could maximize the benefit of the customer with the minimal cost borne by the manufacturer would be ideal therefore the study of cost analysis becomes more interesting. The range of warranty cost analysis needs to consider not only the details/characteristic and replacement/repair cost but also the distribution of the number of product failures.

Researchers have studied different warranty models. For general descriptions about the variations of warranty policies and mathematical models, Blischke and Murthy (1996) are referred. Wang and Pham (1996; 2001) introduce the quasi-renewal processes and successfully apply it in the software engineering modeling. Recently, Bai and Pham (2005) develop the truncated and censored quasi-renewal processes using the quasi-renewal processes. They suggest *repair-limit-risk-free* warranty policies where there are more than a certain number of system failures within a warranty period, and thereafter the failed product would be replaced instead of being repaired. They consider that only repair service could occur in the upper limit of the number of repair under warranty, whenever there were failed products. Two alternate quasi-renewal processes are suggested using the quasi-renewal process (Park and Pham 2008; 2010). The first is called a mixed quasi-renewal processes. By an assumption of imperfect repair, both repairs and replacements are considered simultaneously to obtain the mixed quasi-renewal processes. Within a warranty period, a mal-functioning product would be replaced immediately when there is a severe and premature failure although the numbers of failures are less than the threshold level. $\{N(t),\ t>0\}$ is said to be the altered QRP associated with the distribution $F$ and the random variable parameter $\beta_n$, $\beta_n > 0$, a constant, if $X_n = \beta_n \cdot X$ , $n$=1,2,… where $X \sim F,\ \beta_i,\ i=1,2,\cdots$ are not necessarily equal and $\{N(t),\ t>0\}$ is a counting process. Define $f_i$ and $F_i$ as the pdf and cdf of $X_i$. And for altered QRP, cdf and pdf are given by (Park and Pham 2008)

$$F_i(x) = F(\beta_i x),\ \ f_i(x) = \beta_i f(\beta_i x) \tag{1}$$

Additionally, the component *j*'s cdfs and pdfs of inter-failure interval *i* are given by (Park and Pham 2008)

$$F_{ij}(x) = F(\beta_{ij} x),\ \ f_{ij}(x) = \beta_{ij} f(\beta_{ij} x) \tag{2}$$

In this paper, the altered QRP is used to obtain cdf and pdf of the number of system failures. The quasi-renewal processes model uses a parameter $\alpha$, which causes inter-failure intervals to display a fixed pattern. Therefore, altered quasi-renewal process is used to mitigate such patterns. Bai and Pham (2005) show that the relationships between the number of repairs and the number of replacements. Let *h* be an upper limit of the number of repairs under warranty. $N_a$ and $N_b$ are correlated and $\text{cov}(N_a, N_b) \neq 0$. They can be summarized as follows (Park and Pham 2008; 2010);

$$N_a(w) < h \rightarrow N_b(w) = 0,\ \ N_b(w) > 0 \rightarrow N_a(w) = h \tag{3}$$

The relations (3) was altered as (4) for the mixed QRP (Park and Pham 2008; 2010).

$$N_a(w) < h \rightarrow N_b(w) \geq 0,\ \ N_b(w) > 0 \rightarrow N_a(w) \leq h \tag{4}$$

In this model, despite there being less than *h* failures, replacement can occur using mixed QRP. In times when repair services occur while replacement services do not, the indicator function of repair is 1, otherwise, 0. Similar to this, when replacement services occur but repair services do not, the indicator function of replacement is 1, otherwise, 0. Also, let $f_{N_a}$ and $f_{N_b}$ be pdfs of repairs and replacements respectively. Then,

$$f_{N_{ab}} = f_{N_a} \cdot I_{repair} + f_{N_b} \cdot I_{replacement}$$

$$f_{N_{ab}} = \begin{cases} f_{N_a} & \textit{when a repair service happens} \\ f_{N_b} & \textit{when a replacement service happens} \end{cases} \tag{5}$$

Using these two concepts, the cost analyses are conducted for multi-components systems such as series and series-parallel in this study. For the warranty cost analysis, it is necessary to get the distribution function of failures. To do so, we assume that the warranty policy is renewable, in other words, if the product is replaced, then the warranty period is renewable. It is also assumed that repair and replacement do not occur simultaneously. Let $N_s(t)$ be the number of system failures at *r.v.* time $T$ within a warranty period *w*. We assume that the warranty policy is renewable after each repair/replacement. Using this concept, the distribution of the number of system failures within warranty period can be derived. Under the perfect repair assumption, we obtain the system's pdf as follows (Park and Pham 2008; 2010);

$$f_s(w) = P\left[N_s(t) = n_s\right] = \left[F_s(w)\right]^{n_s} R_s(w), \quad for\ n_s = 1, 2, 3, \cdots \tag{6}$$

Under imperfect repair, every $F_{is}(w)$ is different, so pdf of the system is given by (Park and Pham 2008; 2010)

$$f_s(w) = \left[\prod_{i=1}^{n_s} F_{is}(w)\right] R_{n_s+1}(w) \tag{7}$$

The outline of the paper is as follows. Section 2 presents the distribution of the number of systems failures and warranty cost analysis for multi-component system such as series and series-parallel. Numerical examples are given in Section 3 to illustrative two cases with different distributions are shown. Concluding remarks are presented in Section 4.

## 2. WARRANTY COST MODEL FOR MULTI-COMPONENT SYSTEMS

In this section, warranty cost analyses are conducted by computing the expected warranty cost as well as the variance of the warranty for multi-component systems such as series and series-parallel. We extend Park and Pham's (2010) model.

### 2.1. Cost Analysis For Series System

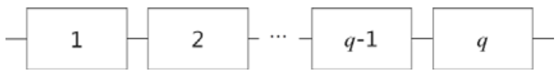


Fig. 1. Series system with *q* components

We use the inter-failure intervals to analyze warranty cost. We assume that the system has *q* components and it is series system like Fig. 1. Then, the reliability function of $j^{th}$ component and $i^{th}$ inter-failure interval using the altered QRP is as follows;

$$R_{ij}(x) = 1 - \int \beta_{ij} f(\beta_{ij} x) dx \tag{8}$$

The reliability function of $i^{th}$ inter-failure interval for the series system is given by

$$R_{is}(x) = \prod_{j=1}^{q} \left(1 - \int \beta_{ij} f(\beta_{ij} x) dx\right) \tag{9}$$

The reliability function of the series system is given by

$$R_s(x) = \prod_{i=1}^{n_s}\prod_{j=1}^{q} R_{ij}(x) = \prod_{i=1}^{n_s}\prod_{j=1}^{q}\left(1-\int \beta_{ij} f(\beta_{ij}x)dx\right) \tag{10}$$

For the cost analysis, the expectation of the cost and the variance of the cost is derived. To obtain the expectation, pdf of the system failure times under the warranty, $f_s(n)$, is used. For series system, pdf is given by

$$\begin{aligned} f(n) &= P[N_s = n] \\ &= P[N_s \geq n] - P[N_s \geq n+1] \\ &= \left(\prod_{i=1}^{n_s} F_{is}\right)\left(R_{n_s+1,s}\right) \\ &= \left(\prod_{i=1}^{n_s}\left(1-\prod_{j=1}^{q}\left(1-\int \beta_{ij} f(\beta_{ij}x)dx\right)\right)\right)\left(\prod_{j=1}^{q}\left(1-\int \beta_{n_s+1,j} f(\beta_{n_s+1,j}x)dx\right)\right) \end{aligned} \tag{11}$$

The expected number of repair warranty service is given by

$$\begin{aligned} E(N_a) &= \int n_a \cdot f(n_a)\cdot dn_a \\ &= \int n_a \cdot \left(\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{q}\left(1-\int \beta_{ij} f(\beta_{ij}x)dx\right)\right)\right)\left(\prod_{j=1}^{q}\left(1-\int \beta_{n_s+1,j} f(\beta_{n_s+1,j}x)dx\right)\right)\cdot dn_a \end{aligned} \tag{12}$$

Next we will derive the variance of the number of repair service. First we calculate the second moment.

$$\begin{aligned} E(N_a^2) &= \int n_a^2 \cdot f(n_a)\cdot dn_a \\ &= \int n_a^2 \cdot \left(\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{q}\left(1-\int \beta_{ij} f(\beta_{ij}x)dx\right)\right)\right)\left(\prod_{j=1}^{q}\left(1-\int \beta_{n_s+1,j} f(\beta_{n_s+1,j}x)dx\right)\right)\cdot dn_a \end{aligned} \tag{13}$$

Therefore, the variance of the number of repair service is given by

$$\begin{aligned} Var(N_a) &= E(N_a{}^2) - [E(N_a)]^2 \\ &= \int n_a{}^2 \cdot \left(\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{q}\left(1-\int \beta_{ij} f(\beta_{ij}x)dx\right)\right)\right)\left(\prod_{j=1}^{q}\left(1-\int \beta_{n_s+1,j} f(\beta_{n_s+1,j}x)dx\right)\right)\cdot dn_a \\ &\quad -\left(\int n_a \cdot \left(\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{q}\left(1-\int \beta_{ij} f(\beta_{ij}x)dx\right)\right)\right)\left(\prod_{j=1}^{q}\left(1-\int \beta_{n_s+1,j} f(\beta_{n_s+1,j}x)dx\right)\right)\cdot dn_a\right)^2 \end{aligned} \tag{14}$$

The expected warranty cost for series system is given by

$$
\begin{aligned}
E(C) &= c_a E\left(N_a(t)\right) + c_b E\left(N_b(t)\right) \\
&= c_a \int n_a \cdot \left( \prod_{i=1}^{n_a} \left( 1 - \prod_{j=1}^{q} \left( 1 - \int \beta_{ij} f\left(\beta_{ij} x\right) dx \right) \right) \right) \left( \prod_{j=1}^{q} \left( 1 - \int \beta_{n_s+1,j} f\left(\beta_{n_s+1,j} x\right) dx \right) \right) \cdot dn_a \\
&\quad + c_b \int n_b \cdot \left( \prod_{i=1}^{n_b} \left( 1 - \prod_{j=1}^{q} \left( 1 - \int f(y) dy \right) \right) \right) \left( \prod_{j=1}^{q} \left( 1 - \int f(y) dy \right) \right) \cdot dn_b
\end{aligned}
\tag{15}
$$

Next, we will derive the covariance of the repair services and the replacement services. The covariance of those are given by

$$
Cov(N_a, N_b) = E[N_a N_b] - E[N_a]E[N_b] \tag{16}
$$

If we obtain the expected value of the first term, then,

$$
E[N_a N_b] = \sum_{n_b=0}^{\infty} \sum_{n_a=0}^{h} n_a n_b \prod_{i=1}^{n_a+n_b} \left( 1 - \prod_{j=1}^{q} \left( 1 - \int \beta_{ij} f\left(\beta_{ij} x\right) dx \right) \right) \prod_{j=1}^{q} \left( 1 - \int \beta_{n_a+n_b+1,j} f\left(\beta_{n_a+n_b+1,j} x\right) dx \right) \tag{17}
$$

Therefore, the covariance of the repair service and the replacement service is given by

$$
\begin{aligned}
Cov(N_a, N_b) &= E[N_a N_b] - E[N_a]E[N_b] \\
&= \sum_{n_b=0}^{\infty} \sum_{n_a=0}^{h} n_a n_b \prod_{i=1}^{n_a+n_b} \left( 1 - \prod_{j=1}^{q} \left( 1 - \int \beta_{ij} f\left(\beta_{ij} x\right) dx \right) \right) \prod_{j=1}^{q} \left( 1 - \int \beta_{n_a+n_b+1,j} f\left(\beta_{n_a+n_b+1,j} x\right) dx \right) \\
&\quad - \left( \int n_a \cdot \left( \prod_{i=1}^{n_a} \left( 1 - \prod_{j=1}^{q} \left( 1 - \int \beta_{ij} f\left(\beta_{ij} x\right) dx \right) \right) \right) \left( \prod_{j=1}^{q} \left( 1 - \int \beta_{n_s+1,j} f\left(\beta_{n_s+1,j} x\right) dx \right) \right) \cdot dn_a \right) \cdot \\
&\quad \left( \int n_b \cdot \left( \prod_{i=1}^{n_b} \left( 1 - \prod_{j=1}^{q} \left( 1 - \int f(y) dy \right) \right) \right) \left( \prod_{j=1}^{q} \left( 1 - \int f(y) dy \right) \right) \cdot dn_b \right)
\end{aligned}
\tag{18}
$$

Using Eqs. (14) and (18), the variance of warranty cost is given by

$$
\begin{aligned}
Var(C) &= c_a^2 \cdot Var\left(N_a(t)\right) + c_b^2 \cdot Var\left(N_b(t)\right) + 2c_a c_b \cdot Cov\left(N_a(t), N_b(t)\right) \\
&= c_a^2 \cdot \left( \begin{aligned} &\int n_a^2 \cdot \left( \prod_{i=1}^{n_s} \left( 1 - \prod_{j=1}^{q} \left( 1 - \int \beta_{ij} f\left(\beta_{ij} x\right) dx \right) \right) \right) \left( \prod_{j=1}^{q} \left( 1 - \int \beta_{n_s+1,j} f\left(\beta_{n_s+1,j} x\right) dx \right) \right) \cdot dn_a \\ &- \left( \int n_a \cdot \left( \prod_{i=1}^{n_s} \left( 1 - \prod_{j=1}^{q} \left( 1 - \int \beta_{ij} f\left(\beta_{ij} x\right) dx \right) \right) \right) \left( \prod_{j=1}^{q} \left( 1 - \int \beta_{n_s+1,j} f\left(\beta_{n_s+1,j} x\right) dx \right) \right) \cdot dn_a \right)^2 \end{aligned} \right) \\
&\quad + c_b^2 \cdot \left( \begin{aligned} &\int n_b^2 \cdot \left( \prod_{i=1}^{n_s} \left( 1 - \prod_{j=1}^{q} \left( 1 - \int f(y) dy \right) \right) \right) \left( \prod_{j=1}^{q} \left( 1 - \int f(y) dy \right) \right) \cdot dn_b \\ &- \left( \int n_b \cdot \left( \prod_{i=1}^{n_s} \left( 1 - \prod_{j=1}^{q} \left( 1 - \int f(y) dy \right) \right) \right) \left( \prod_{j=1}^{q} \left( 1 - \int f(y) dy \right) \right) \cdot dn_b \right)^2 \end{aligned} \right)
\end{aligned}
$$

$$+2c_ac_b\cdot\left(\begin{array}{l}\sum_{n_b=0}^{\infty}\sum_{n_a=0}^{h}n_an_b\prod_{i=1}^{n_a+n_b}\left(1-\prod_{j=1}^{q}\left(1-\int\beta_{ij}f\left(\beta_{ij}x\right)dx\right)\right)\prod_{j=1}^{q}\left(1-\int\beta_{n_a+n_b+1,j}f\left(\beta_{n_a+n_b+1,j}x\right)dx\right)\\-\left(\int n_a\cdot\left(\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{q}\left(1-\int\beta_{ij}f\left(\beta_{ij}x\right)dx\right)\right)\right)\left(\prod_{j=1}^{q}\left(1-\int\beta_{n_s+1,j}f\left(\beta_{n_s+1,j}x\right)dx\right)\right)\cdot dn_a\right)\cdot\\\left(\int n_b\cdot\left(\prod_{i=1}^{n_b}\left(1-\prod_{j=1}^{q}\left(1-\int f\left(y\right)dy\right)\right)\right)\left(\prod_{j=1}^{q}\left(1-\int f\left(y\right)dy\right)\right)\cdot dn_b\right)\end{array}\right) \tag{19}$$

Similar to the series system, we can do the warranty cost analyses for parallel system. In the next section, we consider series-parallel system.

## 2.2. Cost Analysis For Series-parallel System

Series-parallel system consists of *n* subsystems in series with *m* units in parallel in each subsystem as shown in Fig. 2. The system has *m***n* components.

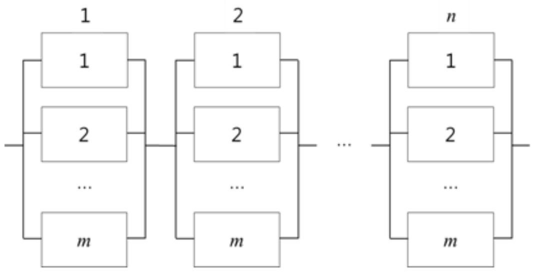


Fig. 2. Series-parallel system with m*n components

The reliability function of $k^{th}$ component in cluster *j* and $i^{th}$ inter-failure interval is given by

$$\begin{aligned}R_{i..}\left(x\right)&=\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(1-R_{ijk}\left(x\right)\right)\right]\\&=\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\int\beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right]\end{aligned} \tag{20}$$

Then the system reliability function is given by

$$R_s\left(x\right)=\prod_{i=1}^{n_s}\left(\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\int\beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right]\right) \tag{21}$$

We obtain expected warranty cost and variance of the warranty cost. We use the pdf of the system failure times under the warranty for expected warranty cost.

$$\begin{aligned} f(n_s) &= P[N_s = n_s] \\ &= P[N_s \ge n_s] - P[N_s \ge n_s + 1] \\ &= \left(\prod_{i=1}^{n_s}\left(F_{i..}(w)\right)\right)\left(R_{n_s+1..}(w)\right) \\ &= \left[\prod_{i=1}^{n_s}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\int \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right]\right)\right]\cdot\left[\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int \beta_{n_s+1,jk} f_{n_s+1,jk}\left(\beta_{n_s+1,jk}x\right)dx\right)\right]\right] \end{aligned} \tag{22}$$

The expected number of repair warranty service is given by

$$\begin{aligned} E(N_a) &= \int n_a \cdot f(n_a)\cdot dn_a \\ &= \int n_a \cdot \left[\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right)\right]\right)\right]\cdot\left[\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\int \beta_{n_s+1,jk} f_{n_s+1,jk}\left(\beta_{n_s+1,jk}x\right)dx\right]\right]dn_a \end{aligned} \tag{23}$$

Next, we derive the variance of the number of repair service. We calculate the second moment.

$$\begin{aligned} E\left(N_a^2\right) &= \int n_a^2 \cdot f(n_a)\cdot dn_a \\ &= \int n_a^2 \left[\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right)\right]\right)\right]\left[\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\int \beta_{n_s+1,jk} f_{n_s+1,jk}\left(\beta_{n_s+1,jk}x\right)dx\right]\right]dn_a \end{aligned} \tag{24}$$

Therefore, the variance of the number of repair service is given by

$$\begin{aligned} Var(N_a) &= E\left(N_a{}^2\right) - \left[E(N_a)\right]^2 \\ &= \int n_a^2 \left[\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right)\right]\right)\right]\left[\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\int \beta_{n_s+1,jk} f_{n_s+1,jk}\left(\beta_{n_s+1,jk}x\right)dx\right]\right]dn_a \\ &\quad -\left(\int n_a \left[\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right)\right]\right)\right]\left[\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\int \beta_{n_s+1,jk} f_{n_s+1,jk}\left(\beta_{n_s+1,jk}x\right)dx\right]\right]dn_a\right)^2 \end{aligned} \tag{25}$$

The expected warranty cost for series-parallel system is given by

$$\begin{aligned} E(C) &= c_a E\left(N_a(t)\right) + c_b E\left(N_b(t)\right) \\ &= c_a\left(\int n_a \prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right)\right]\right)\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int \beta_{n_s+1,jk} f_{n_s+1,jk}\left(\beta_{n_s+1,jk}x\right)dx\right)\right]dn_a\right) \\ &\quad + c_b\left(\int n_b\left[\prod_{i=1}^{n_b}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int f_{ijk}(y)dy\right)\right]\right)\right]\left[\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int f_{n_s+1,jk}(y)dy\right)\right]\right]dn_b\right) \end{aligned} \tag{26}$$

If we obtain $E[N_a N_b]$, then it is good enough to know covariance of the repair services and the replacement services.

$$E\left[N_aN_b\right]=\sum_{n_b=0}^{\infty}\sum_{n_a=0}^{h}\begin{pmatrix} n_an_b\prod_{i=1}^{n_a+n_b}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\int\ \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right]\right) \\ \cdot\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ \beta_{n_a+n_b+1,jk}f_{n_a+n_b+1,jk}\left(\beta_{n_a+n_b+1,jk}x\right)dx\right)\right]\end{pmatrix} \tag{27}$$

Using Eqs. (25) and (27), the variance of warranty system cost is given by

$$Var\left(C\right)=c_a^{\ 2}\cdot Var\left(N_a\left(t\right)\right)+c_b^{\ 2}\cdot Var\left(N_b\left(t\right)\right)+2c_ac_b\cdot Cov\left(N_a\left(t\right),N_b\left(t\right)\right)$$

$$=c_a^{\ 2}\cdot\begin{pmatrix}\int\ n_a^{\ 2}\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right)\right]\right)\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ \beta_{n_s+1,jk}f_{n_s+1,jk}\left(\beta_{n_s+1,jk}x\right)dx\right)\right]dn_a \\ -\left(\int\ n_a\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right)\right]\right)\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ \beta_{n_s+1,jk}f_{n_s+1,jk}\left(\beta_{n_s+1,jk}x\right)dx\right)\right]dn_a\right)^2\end{pmatrix}$$

$$+c_b^{\ 2}\cdot\begin{pmatrix}\int\ n_b^{\ 2}\prod_{i=1}^{n_b}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ f_{ijk}\left(y\right)dy\right)\right]\right)\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ f_{n_s+1,jk}\left(y\right)dy\right)\right]dn_b \\ -\left(\int\ n_b\prod_{i=1}^{n_b}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ f_{ijk}\left(y\right)dy\right)\right]\right)\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ f_{n_s+1,jk}\left(y\right)dy\right)\right]dn_b\right)^2\end{pmatrix}$$

$$+2c_ac_b\cdot\begin{pmatrix}\sum_{n_b=0}^{\infty}\sum_{n_a=0}^{h}\begin{pmatrix} n_an_b\prod_{i=1}^{n_a+n_b}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\int\ \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right]\right) \\ \prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ \beta_{n_a+n_b+1,jk}f_{n_a+n_b+1,jk}\left(\beta_{n_a+n_b+1,jk}x\right)dx\right)\right]\end{pmatrix} \\ -\left(\int\ n_a\prod_{i=1}^{n_a}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ \beta_{ijk}\cdot f_{ijk}\left(\beta_{ijk}x\right)dx\right)\right]\right)\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\int\ \beta_{n_s+1,jk}f_{n_s+1,jk}\left(\beta_{n_s+1,jk}x\right)dx\right]dn_a\right)\cdot \\ \left(\int\ n_b\prod_{i=1}^{n_b}\left(1-\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ f_{ijk}\left(y\right)dy\right)\right]\right)\prod_{j=1}^{n}\left[1-\prod_{k=1}^{m}\left(\int\ f_{n_s+1,jk}\left(y\right)dy\right)\right]dn_b\right)\end{pmatrix} \tag{28}$$

Similar to the series-parallel system, we can do the warranty cost analyses for parallel-series system.

## 3. NUMERICAL EXAMPLES AND SENSITIVITY ANALYSIS

In this section, two cases are presented for sensitivity analysis. For the first case, we show 2*2 series-parallel system using exponential distribution for each component. For the second case, same system is investigated using Weibull and exponential distributions.

*Case I.*

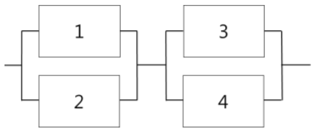


Fig. 3. Series-parallel system with 4 components

Consider the series-parallel system such as Fig. 3. We assume that the failure times of every component follow the exponential distribution. Other assumptions summarized up in Table 1. We consider expected warranty cost, standard deviation of warranty cost and coefficient of variation (CV) during 20 warranty period units which start at 0.1 and finish at 2.0. Time *T* would be assumed as total warranty period *w*.

Table 1. pdf of components' failure time

| Component | Distribution | f(x) |
|---|---|---|
| 1 | Exponential | $\exp(-x)$ |
| 2 | Exponential | $\exp(-2x)$ |
| 3 | Exponential | $\exp(-3x)$ |
| 4 | Exponential | $\exp(-3x)$ |

Table 2. E(C), SD(C) and CV for Case I

| $w$ | $E(C)$ | $SD(C)*$ | $CV*$ | $SD(C)**$ | $CV**$ |
|---|---|---|---|---|---|
| 0.1 | 18 | 53 | 2.94 | 874 | 48.15 |
| 0.2 | 62 | 55 | 0.90 | 1628 | 26.46 |
| 0.3 | 122 | 56 | 0.45 | 2341 | 19.12 |
| 0.4 | 197 | 56 | 0.28 | 2839 | 14.43 |
| 0.5 | 282 | 55 | 0.20 | 3322 | 11.80 |
| 0.6 | 375 | 55 | 0.15 | 3738 | 9.97 |
| 0.7 | 475 | 54 | 0.11 | 4096 | 8.63 |
| 0.8 | 580 | 53 | 0.09 | 4404 | 7.59 |
| 0.9 | 690 | 53 | 0.08 | 4670 | 6.77 |
| 1 | 803 | 52 | 0.06 | 4898 | 6.10 |
| 1.1 | 919 | 51 | 0.06 | 5095 | 5.55 |
| 1.2 | 1036 | 51 | 0.05 | 5264 | 5.08 |
| 1.3 | 1156 | 50 | 0.04 | 5409 | 4.68 |
| 1.4 | 1276 | 50 | 0.04 | 5534 | 4.34 |
| 1.5 | 1398 | 49 | 0.04 | 5641 | 4.04 |
| 1.6 | 1519 | 49 | 0.03 | 5732 | 3.77 |
| 1.7 | 1641 | 48 | 0.03 | 5810 | 3.54 |
| 1.8 | 1762 | 48 | 0.03 | 5876 | 3.34 |
| 1.9 | 1882 | 48 | 0.03 | 5932 | 3.15 |
| 2 | 2001 | 47 | 0.02 | 5980 | 2.99 |

$*$ : Repair and replacement are independent
$**$ : Repair and replacement are dependent

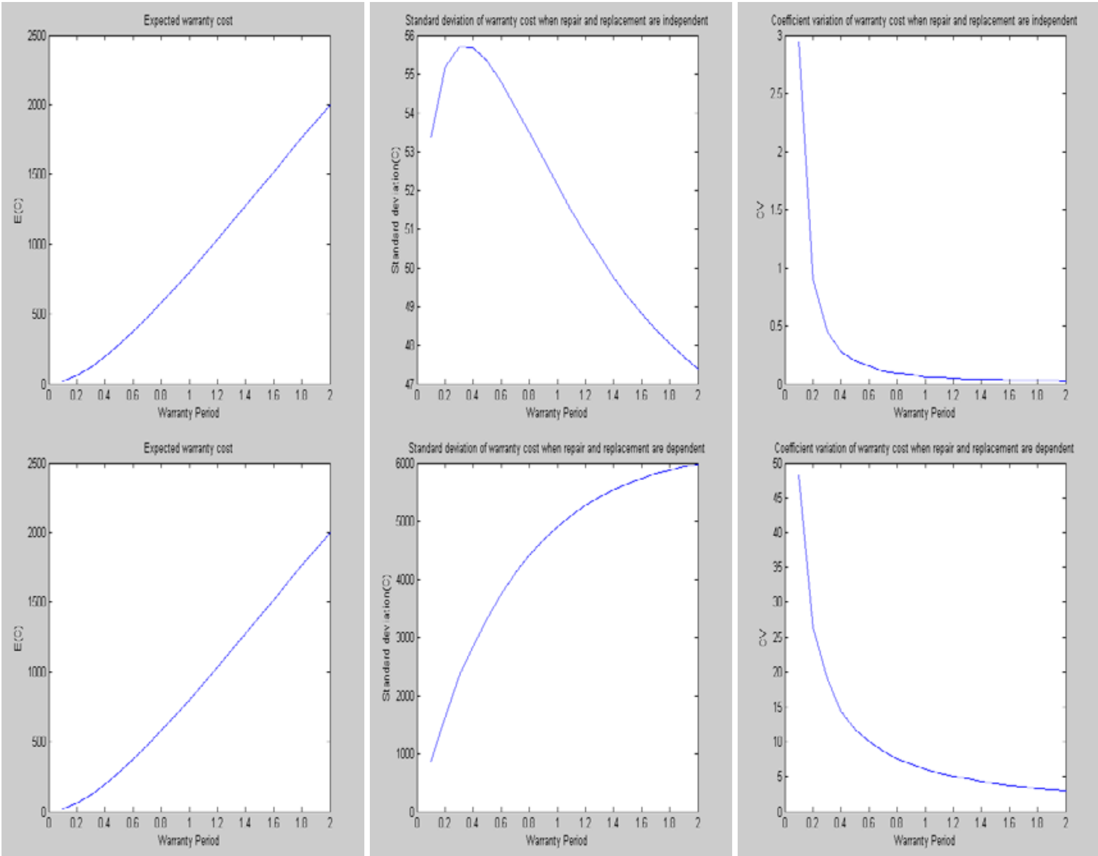


Fig. 4. E(C), SD(C) and CV for Case I

And *n*=2, *m*=2 and *h*=2. We assumed that the warranty cost as follows;

$$c_a = \$2{,}000 \;\; and \;\; c_b = \$3{,}000$$

Also, we assume $\beta$ values as follows;

$$\beta_{i11} = 0.99, \beta_{i12} = 0.98, \beta_{i21} = 0.97, \beta_{i22} = 0.98$$

Expected warranty cost, standard deviation of warranty cost and coefficient of variation (CV) are obtained in Table 2.

In Fig. 4, we separate into cases when repair and replacement are independent and dependent. When repair and replacement are independent, the expected warranty cost increases as warranty time passes and standard deviation of warranty cost increases early in the warranty period and thereafter decreases. In case when repair and replacement are dependent, it can be seen that both the expectation and the standard deviation of warranty cost increase over *w* monotonically. This can be verified since the distribution of *n* follows a geometric distribution whose standard deviation is always higher than expected warranty cost. Coefficient of variation (CV) is the ratio of the standard deviation to the mean and describes the magnitude sample values and the variation within them. When warranty time passes, CV decreases. This indicates smaller variations.

*Case II.*

We consider the 2*2 series-parallel system with different distributions. It is assumed that the failure times of every component do not follow exponential distribution. 2nd and 4th components' failure times follow Weibull distribution and 1st and 3rd components' failure times follow exponential distribution. Other assumptions summarize up in Table 3.

Table 3. pdf of components' failure time.

| Component | Distribution | f(x) |
|---|---|---|
| 1 | Exponential | $e^{-x}$ |
| 2 | Weibull | $e^{-x^{0.1}}$ |
| 3 | Exponential | $e^{-3x}$ |
| 4 | Weibull | $e^{-0.2x^{0.1}}$ |

In this system, $E(C)$, standard deviation and coefficient of variation are obtained in Table 4.

Table 4. E(C), SD(C) and CV for Case II

| $w$ | $E(C)$ | $SD(C)*$ | $CV*$ | $SD(C)**$ | $CV**$ |
|---|---|---|---|---|---|
| 0.1 | 20 | 56 | 2.73 | 627 | 30.64 |
| 0.2 | 58 | 130 | 2.24 | 1201 | 20.70 |
| 0.3 | 114 | 217 | 1.91 | 1740 | 15.25 |
| 0.4 | 183 | 311 | 1.70 | 2248 | 12.31 |
| 0.5 | 262 | 408 | 1.56 | 2728 | 10.41 |
| 0.6 | 351 | 508 | 1.45 | 3183 | 9.07 |
| 0.7 | 449 | 612 | 1.36 | 3614 | 8.05 |
| 0.8 | 555 | 718 | 1.29 | 4025 | 7.25 |
| 0.9 | 669 | 753 | 1.12 | 4403 | 6.58 |
| 1 | 790 | 936 | 1.19 | 4789 | 6.07 |
| 1.1 | 916 | 1048 | 1.14 | 5146 | 5.62 |
| 1.2 | 1047 | 1161 | 1.11 | 5488 | 5.24 |
| 1.3 | 1181 | 1276 | 1.08 | 5815 | 4.93 |
| 1.4 | 1316 | 1392 | 1.06 | 6130 | 4.66 |
| 1.5 | 1451 | 1509 | 1.04 | 6434 | 4.43 |
| 1.6 | 1584 | 1626 | 1.03 | 6727 | 4.25 |
| 1.7 | 1710 | 1743 | 1.02 | 7011 | 4.10 |
| 1.8 | 1827 | 1859 | 1.02 | 7286 | 3.99 |
| 1.9 | 1930 | 1973 | 1.02 | 7554 | 3.91 |
| 2 | 2013 | 2084 | 1.04 | 7814 | 3.88 |

* : Repair and replacement are independent

** : Repair and replacement are dependent

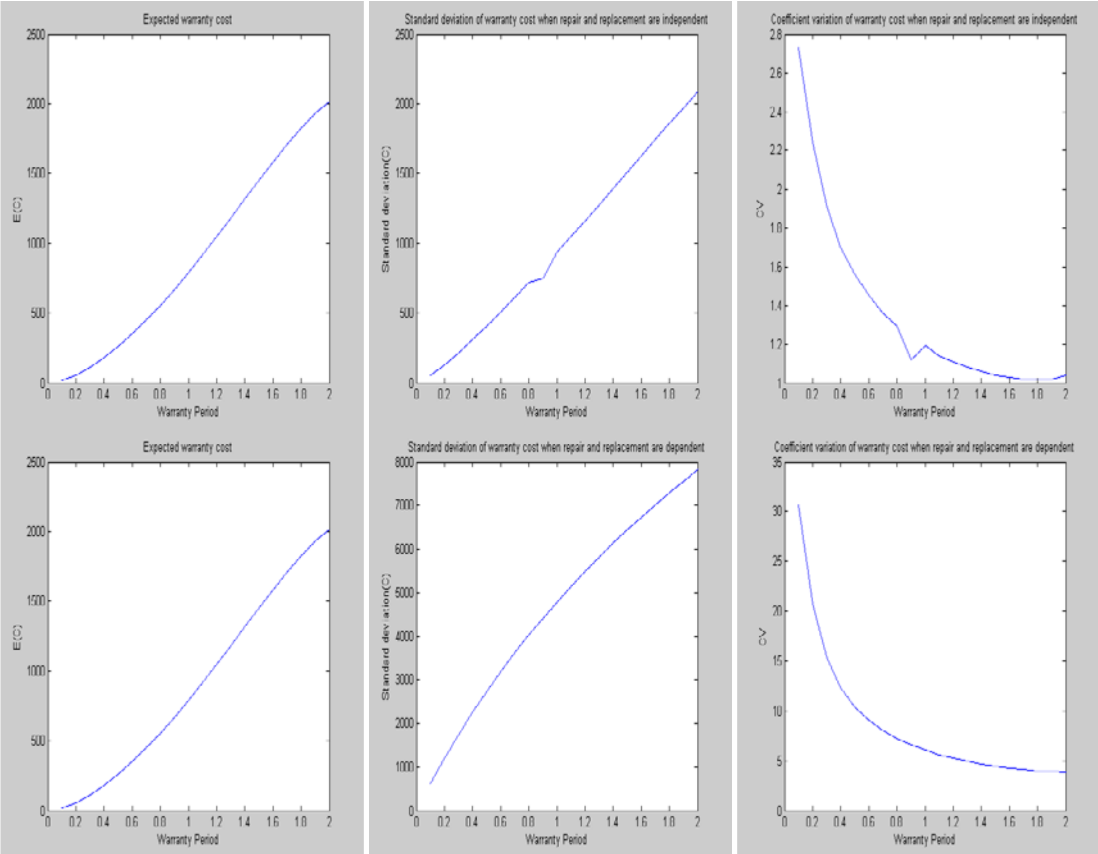


Fig. 5. E(C), SD(C) and CV for Case II

As shown in Fig. 5, both the warranty expected cost and standard deviation of the warranty cost increases, not impacted by the dependency of repair and replacement policy.

## 4. CONCLUDING REMARKS

In this paper, cost analyses for systems are conducted using the altered quasi-renewal processes and the mixed quasi-renewal processes. We obtain the expected value and variance of warranty cost. Also, we consider multi-component system such as series and series-parallel systems and show sensitivity analysis and numerical examples. Based on this study, we believe that these mixed and altered quasi-renewal processes are useful for warranty cost analysis of multi-component system.

There are several potential extensions to the warranty study using alternative quasi-renewal processes on complex systems if we relax several assumptions. For example, in this study we assume that the replacement time and repair time are negligible but we may factor in the time increase in our future study.

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