Korean J. Math. **21** (2013), No. 2, pp. 197–201
http://dx.doi.org/10.11568/kjm.2013.21.2.197

A SIMPLE PROOF OF HILBERT BASIS THEOREM FOR $*_w$-NOETHERIAN DOMAINS

Jung Wook Lim and Dong Yeol Oh*

Abstract. Let D be an integral domain with quotient field K, $*$ a star-operation on D, $\mathrm{GV}^*(D)$ the set of nonzero finitely generated ideals J of D such that $J_* = D$, and $*_w$ a star-operation on D defined by $I_{*_w} = \{x \in K \mid Jx \subseteq I$ for some $J \in \mathrm{GV}^*(D)\}$ for all nonzero fractional ideals I of D. In this article, we give a simple proof of Hilbert basis theorem for $*_w$-Noetherian domains.

1. Introduction

One of the most beautiful results in commutative algebra is Hilbert basis theorem, which states that if D is a Noetherian domain, then so is the polynomial ring $D[X]$. In [4, Theorem 3.2], the authors generalized and proved the $*_w$-Noetherian domain version of Hilbert basis theorem, which mentions that if $*$ is a star-operation on $D[X]$, then D being a $\overline{*}_w$-Noetherian domain implies $D[X]$ being a $*_w$-Noetherian domain.

Received April 7, 2013. Revised May 27, 2013. Accepted June 1, 2013.
2010 Mathematics Subject Classification: 13E99, 13A15, 13B25, 13G05.
Key words and phrases: $*_w$-Noetherian domain, Hilbert basis theorem.
Words and phrases : $*_w$-Noetherian domain, Hilbert basis theorem.
*Corresponding author.

(Relevant definitions and notation are reviewed in the sequel.) The standard proofs are very complicated and so it is not easy to understand them. The purpose of this article is to give a simple proof of Hilbert basis theorem for $*_w$-Noetherian domains.

For the reader's better understanding, we now review some definitions and notation. Let D be an integral domain with quotient field K and $\mathbf{F}(D)$ the set of nonzero fractional ideals of D. A *star-operation* on D is a mapping $I \mapsto I_*$ from $\mathbf{F}(D)$ into itself which satisfies the following three conditions for all $0 \neq a \in K$ and all $I, J \in \mathbf{F}(D)$:

(1) $(aD)_* = aD$ and $(aI)_* = aI_*$;
(2) $I \subseteq I_*$, and if $I \subseteq J$, then $I_* \subseteq J_*$; and
(3) $(I_*)_* = I_*$.

An $I \in \mathbf{F}(D)$ is called a $*$-*ideal* if $I = I_*$. A $*$-ideal I is said to be *of finite type* if $I = J_*$ for some nonzero finitely generated subideal J of I. Given any star-operation $*$ on D, we can construct a new star-operation $*_w$ induced by $*$. For all $I \in \mathbf{F}(D)$, the $*_w$-*operation* is defined by $I_{*_w} = \{x \in K \mid Jx \subseteq I \text{ for some } J \in \mathrm{GV}^*(D)\}$, where $\mathrm{GV}^*(D)$ is the set of nonzero finitely generated ideals J of D with $J_* = D$.

Recall that D is a $*$-*Noetherian domain* if D satisfies the ascending chain condition on integral $*$-ideals of D. It is well known that D is a $*$-Noetherian domain if and only if every integral $*$-ideal of D is of finite type [7, Theorem 1.1].

The readers can refer to [2] for star-operations on integral domains and to [1] for $*_w$-operations on integral domains.

2. Main result

Let $D[X]$ denote the polynomial ring over D and $*$ a star-operation on $D[X]$. Then $*$ induces a new star-operation $\overline{*}$ on D defined by $I \mapsto$

$(ID[X])_* \cap K$ for each $I \in \mathbf{F}(D)$ [5, Proposition 2.1]. From now on, we refer to this induced star-operation as $\overline{*}$.

LEMMA 2.1. *Let $*$ be a star-operation on $D[X]$. If $J \in \mathrm{GV}^{\overline{*}}(D)$, then $JD[X] \in \mathrm{GV}^{*}(D[X])$.*

Proof. This was shown in [4, Lemma 3.1]. □

THEOREM 2.2. *Let $*$ be a star-operation on $D[X]$. If D is a $\overline{*}_w$-Noetherian domain, then $D[X]$ is a $*_w$-Noetherian domain.*

Proof. Suppose to the contrary that $D[X]$ is not a $*_w$-Noetherian domain, and let I be a $*_w$-ideal of $D[X]$ which is not of finite type. Let n_1 be the least degree of polynomials in I and take any $f_1 \in I$ with $\deg(f_1) = n_1$. If n_k and f_k have already been chosen for an integer $k \geq 1$, let n_{k+1} be the least degree of polynomials in $I \backslash (f_1, \ldots, f_k)_{*_w}$ and take an $f_{k+1} \in I \backslash (f_1, \ldots, f_k)_{*_w}$ such that $\deg(f_{k+1}) = n_{k+1}$. Note that $n_k \leq n_{k+1}$ for all integers $k \geq 1$. For each $k \geq 1$, let a_k be the leading coefficient of f_k. We claim that the chain $(a_1)_{\overline{*}_w} \subseteq (a_1, a_2)_{\overline{*}_w} \subseteq \cdots$ is not stationary. If the chain stops after finitely many steps, then we can find a positive integer m such that $(a_1, \ldots, a_m)_{\overline{*}_w} = (a_1, \ldots, a_{m+1})_{\overline{*}_w}$. Now, $a_{m+1} \in (a_1, \ldots, a_m)_{\overline{*}_w}$; so $Ja_{m+1} \subseteq (a_1, \ldots, a_m)$ for some $J \in \mathrm{GV}^{\overline{*}}(D)$, which indicates that for any $0 \neq j \in J$, we have $ja_{m+1} = d_{j1}a_1 + \cdots + d_{jm}a_m$ for some $d_{j1}, \ldots, d_{jm} \in D$. Let $g_j = jf_{m+1} - \sum_{i=1}^{m} d_{ji} f_i X^{n_{m+1}-n_i}$. Then $g_j \in I$ and $\deg(g_j) < n_{m+1}$. Note that $g_j \in (f_1, \ldots, f_m)_{*_w}$ if and only if $jf_{m+1} \in (f_1, \ldots, f_m)_{*_w}$. If $g_j \in (f_1, \ldots, f_m)_{*_w}$ for all $j \in J$, then $Jf_{m+1} \subseteq (f_1, \ldots, f_m)_{*_w}$; so $JD[X]f_{m+1} \subseteq (f_1, \ldots, f_m)_{*_w}$. By Lemma 2.1, $JD[X] \in \mathrm{GV}^{*}(D[X])$; so we have

$$\begin{aligned} f_{m+1} &\in ((f_1, \ldots, f_m)_{*_w})_{*_w} \\ &= (f_1, \ldots, f_m)_{*_w}, \end{aligned}$$

which contradicts the choice of f_{m+1}. Therefore there exists a $j \in J$ such that $g_j \in I \setminus (f_1, \ldots, f_m)_{*_w}$. However, this is also impossible because of the minimality of n_{m+1}. Hence the claim is proved. But, this is absurd since D is a $\overline{*}_w$-Noetherian domain. Thus we conclude that $D[X]$ is a $*_w$-Noetherian domain. $\square$

For an $I \in \mathbf{F}(D)$, set $I^{-1} = \{a \in K \mid aI \subseteq D\}$ and $I_v = (I^{-1})^{-1}$. Then v is a star-operation on D [2, Theorem 34.1(2)]. When $* = v$, $*_w$ is the so-called w-operation and a $*_w$-Noetherian domain is the so-called a strong Mori domain (SM-domain). Also, it was shown in [5, Remark 2.2] (cf. [3, Proposition 4.3]) that if $*_w$ is the w-operation on $D[X]$, then $\overline{*}_w$ is the w-operation on D. Hence by Theorem 2.2, we recover

Corollary 2.3. ([6, Theorem 1.13]) *If D is an SM-domain, then $D[X]$ is also an SM-domain.*

References

[1] D.D. Anderson and S.J. Cook, *Two star-operations and their induced lattices*, Comm. Algebra **28** (2000), 2461–2475.

[2] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Appl. Math., vol. **90**, Queen's University, Kingston, Ontario, Canada, 1992.

[3] J. Hedstrom and E. Houston, *Some remarks on star-operations*, J. Pure Appl. Algebra **18** (1980), 37–44.

[4] C.J. Hwang and J.W. Lim, *A note on $*_w$-Noetherian domains*, Proc. Amer. Math. Soc. **141** (2013), 1199–1209.

[5] A. Mimouni, *Note on star operations over polynomial rings*, Comm. Algebra **36** (2008), 4249–4256.

[6] F. Wang and R.L. McCasland, *On strong Mori domains*, J. Pure Appl. Algebra **135** (1999), 155–165.

[7] M. Zafrullah, *Ascending chain conditions and star operations*, Comm. Algebra **17** (1989), 1523–1533.

Department of Mathematics
Kyungpook National University
Daegu 702-701, Republic of Korea
E-mail: jwlim@knu.ac.kr

Division of Liberal Arts
Hanbat National University
Daejeon 305-719, Republic of Korea
E-mail: dongyeol70@gmail.com