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A STUDY ON THE CARTESIAN CLOSED CATEGORY POSM

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Abstract. PosM is a category whose objects are ample spaces and morphisms are possibility mappings. We study some properties of the Category PosM . So we show that Category PosM is a cartesian closed category, and it forms a topos with some condition.

1. Introduction

Yuan [4] showed that PosM, whose objects are ample spaces and morphisms are possibility mappings, is a category.

In this paper, we study some properties of the Category PosM. In particular, terminal object, equalizer, finite product, pull-back and exponentials exist in the Category PosM. So Category PosM is a cartesian closed category. Also with some condition, it forms a topos.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments to prove our results.

Definition 2.1. Let X be a set and $\mathcal{A}$ be a subset of power set $P(X)$ of X.

If

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(1) $X \in \mathcal{A}$
(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
(3) For any index set I, $A_i \in \mathcal{A} \Rightarrow \cup A_i \in \mathcal{A}$.

Then A is called an ample field over X and $(X, \mathcal{A})$ is called an ample space.

Definition 2.2. Let $(X, \mathcal{A})$ be an ample space, then
$[x] = \bigcap\{A|x \in A \in \mathcal{A}\}$
is called an atom of $\mathcal{A}$ containing the element $x \in X$.

Proposition 2.3. *Let $(X, \mathcal{A})$ be an ample space, then*

(1) $[x] \subseteq A$ *or* $[x] \cap A = \varnothing$ *for any* $A \in \mathcal{A}$
(2) $A \in \mathcal{A} \Leftrightarrow A = \cup[x]$

Proof. See [4], [5]. □

Definition 2.4. Let $(X, \mathcal{A})$ be an ample space. If a mapping $\Pi : \mathcal{A} \to [0, 1]$ satisfies

(1) $\Pi(\emptyset) = 0$; $\Pi(X) = 1$;
(2) $\Pi(\cup A_i)$ =sup $\Pi(A_i)$.

Then Π is called a possibility measure over $\mathcal{A}$, and $M(x) = \Pi([x])$ is called a possibility distribution of Π.

Definition 2.5. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two ample spaces. If a mapping $f : X \to Y$ satisfies $B \in \mathcal{B} \Rightarrow f^{-1}(B) \in \mathcal{A}$
then, f is called a fuzzy variable from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$.

Proposition 2.6. *Let $(X, \mathcal{A})$ be an ample space and Π is a possibility measure over $\mathcal{A}$. If f is a fuzzy variable from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$, then $\Pi(f^{-1}(B)), \forall B \in \mathcal{B}$ is a possibility measure over $\mathcal{B}$.*

Proof. See [2], [4]. □

Definition 2.7. A cartesian closed category is a category $\mathcal{E}$ that satisfies the following;

(1) $\mathcal{E}$ is finitely complete,
(2) $\mathcal{E}$ has exponentiation. A topos is a cartesian closed category $\mathcal{E}$ that satisfies the following;
(3) $\mathcal{E}$ has a subobject classifier.

Example 2.8. ([1], [3]) Category *Fuz* of fuzzy sets is a cartesian closed category whose object is (A, α_A) where A is an object and $\alpha_A : A \to I$ is a morphism with $I = (0, 1]$ in *Set* and morphism from (A, α_A) to (B, α_B) is a function $f : A \to B$ in *Set* such that $\alpha_A(a) \leqq \alpha_B \circ f(a)$.

Example 2.9. ([1], [3]) If M_2 is a monoid with two elements, then the category $M_2 - \mathcal{S}et$ is a topos.

Consider $(M_2, \circ, e)$ where $M_2 = \{e, a\}$ and $\circ$ is defined by $e \circ e = e$, $e \circ a = a \circ e = a \circ a = a$. Then M_2 is a monoid with identity e, in which a has no inverse. The set L_2 of left ideals of M_2 has three elements, that is, M_2, $\emptyset$, and $\{a\}$. Thus in $M_2 - \mathcal{S}et$, $\Omega = (L_2, \omega)$, where the action $\omega : M_2 \times L_2 \to L_2$ is defined by $\omega(m, B) = \{n | n \circ m \in B\}$.

Definition 2.10. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two ample spaces. Let the mapping $f : X \times \mathcal{B} \to [0, 1]$ satisfy

(1) $\forall x \in X, f(x, -) : \mathcal{B} \to [0, 1]$ is a possibility measure.
(2) $\forall B \in \mathcal{B}, f(-, B) : X \to [0, 1]$ is a fuzzy variable, where ample field on [0,1] is $P([0, 1])$. Then f is called a possibility mapping from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$.

3. Some properties of the category PosM

Theorem 3.1. *Terminal object exists in the Category PosM.*

Proof. Let $1 = (\{*\}, \mathcal{A})$ where $\mathcal{A} = \{\phi, \{*\}\}$. Then, for any $(Y, \mathcal{B})$, there exists a mapping $f : Y \times \mathcal{A} \to [0, 1]$ defined by $f(y, \{*\}) = 1$ and $f(y, \phi) = 0$ for all $y \in Y$. So, we get $f(y, \bigcup A_i) = \sup f(y, A_i)$. Thus $f(y, -)$ is a possibility measure. Also $f(-, A) : Y \to [0, 1]$ is a fuzzy variable. Since

$f(-, \phi)^{-1}(B) = Y$ if $0 \in B$
$f(-, \phi)^{-1}(B) = \phi$ otherwise. And
$f(-, \{*\})^{-1}(B) = Y$ if $1 \in B$
$f(-, \{*\})^{-1}(B) = \phi$ otherwise.
Therefore $f : Y \times \mathcal{A} \to [0, 1]$ is a possibility mapping. □

Theorem 3.2. *Equalizer exists in the Category PosM.*

Proof. $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are two ample spaces and $f, g : (X, \mathcal{A}) \to (Y, \mathcal{B})$ are two possibility mappings. Let $E = \{x \in X | f(-, B)(x) =$

$g(-,B)(x)\forall B \in \mathcal{B}\}$ and $\mathcal{E} = \mathcal{A}$. And we construct a mapping $e : E \times \mathcal{A} \to [0,1]$ defined by

$e(a,A) = 1$ if $a \in A$

$e(a,A) = 0$ otherwise.

Then $e : (E,\mathcal{E}) \to (X,\mathcal{A})$ is a possibility mapping. Since $e(a,-) : \mathcal{A} \to [0,1]$ is a possibility measure,

$e(a,-)(\phi) = 0$,

$e(a,-)(X) = 1$ and

$e(a,-)(\bigcup E_i) = \sup e(a,-)(E_i)$,

$e(-,A) : E \to [0,1]$ is a fuzzy variable,

$e(-,A)^{-1}(E_i) = \{a \in E | e(a,A) = 1\}$

$= \{a \in E | a \in A\}$ if $1 \in E_i$ and

$e(-,A)^{-1}(E_j) = \{a \in E | e(a,A) = 0\}$

$= \{a \in E | a \in A^c\}$ if $0 \in E_j$.

Since

$f \circ e(x,B) = \vee(e(x,[a]) \wedge f(a,B)) = f(a,B)$

by $(e(x,[a]) = 1$ or $(e(x,[a]) = 0$ and

$g \circ e(x,B) = \vee(e(x,[a]) \wedge g(a,B)) = g(a,B)$

by $(e(x,[a]) = 1$ or $(e(x,[a]) = 0$,

we have $f \circ e = g \circ e$. □

THEOREM 3.3. *Finite products exist in the Category PosM.*

Proof. For any two ample spaces $(X,\mathcal{A})$ and $(Y,\mathcal{B})$, we construct an ample space $(Z,\mathcal{Z})$ where $Z = X \times Y$ and $\mathcal{E} = P(X) \times P(Y)$ with $p_X : (Z,\mathcal{Z}) \to (X,\mathcal{A})$ and $p_Y : (Z,\mathcal{Z}) \to (Y,\mathcal{B})$. Then $((Z,\mathcal{Z}), p_X, p_Y)$ is a product object of $(X,\mathcal{A})$ and $(Y,\mathcal{B})$.

We construct $p_X : (Z,\mathcal{Z}) \to (X,\mathcal{A})$ defined by

$p_X((x,y),A) = 1$ if $x \in A$

$p_X((x,y),A) = 0$ otherwise.

Then we have that $p_X : (Z,\mathcal{Z}) \to (X,\mathcal{A})$ is a possibility mapping. Since $p_X((a,b),-) : \mathcal{A} \to [0,1]$ is a possibility measure,

$p_X((a,b),-)(\phi) = 0$,

$p_X((a,b),-)(X) = 1$ and

$p_X((a,b),-)(\bigcup A_i) = \sup p_X((a,b),-)(A_i)$,

also $p_X(-,A) : Z \to [0,1]$ is a fuzzy variable,

$p_X(-,A)^{-1}(E_i) = \{(x,y) | x \in A\}$ if $1 \in E_i$ and

$p_X(-,A)^{-1}(E_j) = \{(x,y) | x \notin A\}$ if $0 \in E_j$.

For any possibility mappings $f : (K,\mathcal{K}) \to (X,\mathcal{A})$ and $g : (K,\mathcal{K}) \to (Y,\mathcal{B})$, there exists a mapping $< f, g >: (K,\mathcal{K}) \to (Z,\mathcal{E})$ defined by

$< f.g > (k, (A, B)) = f(k, A)$ if B is fixed
$< f.g > (k, (A, B)) = g(k, B)$ if A is fixed.
Then $< f, g >$ is a possibility mapping. Since $x \notin A$ implies $p_X((x, y), A) = 0$, $x \in A$ implies $p_X((x, y), A) = 1$ and y is fixed we get
$\bigvee[< f, g > (k, [(x, y)]) \bigwedge p_X((x, y), A)] = f(f, A)$,
we have $p_X \circ < f, g >= f$.

□

Theorem 3.4. *Exponentiation exists in the Category PosM.*

Proof. For any two ample spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, we define $Y^X = \{f|f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a possibility mapping $\}$ and $\mathcal{D} = P(Y^X)$. Then $(Y^X, \mathcal{D})$ is an object in the Category PosM. For any $Y^X \times X = \{(f(-, Y), x)|f(-, Y)$ is a fuzzy variable $\}$ and $P(Y^X) \times P(X) = \mathcal{E}$, there exists a mapplng $ev : Y^X \times X \to Y$ defined by
$ev((f(-, Y_i), x), Y_j) = 0$ if $Y_i \cap Y_j = \phi$
$ev((f(-, Y_i), x), Y_j) = 1$ if $Y_i \cap Y_j \neq \phi$.
Then $ev : (Y^X \times X) \times \mathcal{B} \to [0, 1]$ is a possibility measure, since
$ev((f(-, Y_i), x), \phi) = 0$
$ev((f(-, Y_i), x), Y) = 0$
$ev((f(-, Y_i), x), \bigcup Y_i) =$sup $ev((f(-, Y_i), x), Y_i)$.
also $ev : (Y^X \times X) \times \mathcal{B} \to [0, 1]$ is a fuzzy variable, since
$ev(-, Y_j)^-1(0) = (f(-, Y_i), a) \in \mathcal{E}$ with $Y_i \cap Y_j = \phi$
$ev(-, Y_j)^-1(1) = (f(-, Y_i), a) \in \mathcal{E}$ with $Y_i \cap Y_j \neq \phi$.
So $ev : (Y^X \times X) \times \mathcal{B} \to [0, 1]$ is a possibility mapping.

For any possibility mapping $g : Z \times X \to Y$ where $((Z, \mathcal{Z})$ is an ample space, there exists a mapping $\bar{g} : (Z, \mathcal{Z}) \to (Y^X, P(Y^X))$ defined by $\bar{g}(z, (f(-, Y_i), x) = g((z, x), Y_i)$. Then $\bar{g} : (Z, \mathcal{Z}) \to (Y^X, P(Y^X))$ is a possibility mapping. So we have
$\bar{g} \times id \circ ev((z, x), Y_i)$
$= \vee(\bar{g} \times id((z, x), [(f(-, Y_i), x)]) \wedge ev(((f(-, Y_i), x), Y_j))$
$=\bar{g} \times id(z, [(f(-, Y_i), x)])$
$=\bar{g}(z, [(f(-, Y_i), x)])$
$=g((z, x), Y_i)$ □

Corollary 3.5. Category PosM is Cartesian closed.

Theorem 3.6. *For each monic $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ where $f[X] \in \mathcal{D}$, a subobject classifier exists in the Category PosM.*

Proof. Let $2 = \{0,1\}$ and $\mathcal{O} = \{\phi, \{0\}, \{1\}, 2\}$. Then $\Omega = (2, \mathcal{O})$ is an ample space. We construct $\top : (\{*\}, \mathcal{A}) \to (2, \mathcal{O})$ defined by $\top(*, \phi) = \top(*, \{0\}) = 0, \top(*, \{1\}) = \top(*, 2) = 1$. So, we get that $\top(*, \bigcup O_i) = \sup \top(*, O_i)$. Thus $\top(*, -)$ is a possibility measure. Also $\top(-, O) : \{*\} \to [0,1]$ is a fuzzy variable since

$\top^{-1}(-, 2)(0) = \phi \in \mathcal{A}$
$\top^{-1}(-, \{1\})(0) = \phi \in \mathcal{A}$
$\top^{-1}(-, \{0\})(0) = * \in \mathcal{A}$
$\top^{-1}(-, \phi)(0) = * \in \mathcal{A}$ and
$\top^{-1}(-, 2)(1) = * \in \mathcal{A}$
$\top^{-1}(-, \{1\})(1) = * \in \mathcal{A}$
$\top^{-1}(-, \{0\})(1) = \phi \in \mathcal{A}$
$\top^{-1}(-, \phi)(1) = \phi \in \mathcal{A}$.

Hence $\top$ is a possibility mapping. For any possibility mapping $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ where $f[X] \in \mathcal{D}$, we construct a mapping $\chi_f : (Y, \mathcal{D}) \to (2, \mathcal{O})$ defined by $\chi_f(y, \phi) = \chi_f(f(x), \{0\}) = \chi_f(Y - f(x), \{1\}) = 0$ and $\chi_f(y, 2) = \chi_f(f(x), \{1\}) = \chi_f(Y - f(x), \{0\}) = 1$. So we get

$\chi_f(Y, \bigcup A_i) = \sup \chi_f(Y, A_i)$.

Thus $\chi_f(Y, -)$ is a possibility measure. Also $\chi_f(-, O)$ is a fuzzy variable since

$\chi_f^{-1}(-, 2)(1) = Y \in \mathcal{D}$
$\chi_f^{-1}(-, \{1\})(1) = f[X] \in \mathcal{D}$
$\chi_f^{-1}(-, \{0\})(1) = Y - f[X] \in \mathcal{D}$
$\chi_f^{-1}(-, \phi)(1) = \phi \in \mathcal{D}$ and
$\chi_f^{-1}(-, 2)(0) = \phi \in \mathcal{D}$
$\chi_f^{-1}(-, \{1\})(0) = Y - f[X] \in \mathcal{D}$
$\chi_f^{-1}(-, \{0\})(0) = f[X] \in \mathcal{D}$

Hence χ_f is a possibility mapping. Also we get $\top \circ ! = \chi_f \circ f$ since

$\top \circ !(x, \phi) = \chi_f \circ f(x, \phi) = 0$
$\top \circ !(x, \{0\}) = \chi_f \circ f(x, \{0\}) = 0$
$\top \circ !(x, \{1\}) = \chi_f \circ f(x, \{1\}) = 1$
$\top \circ !(x, 2) = \chi_f \circ f(x, 2) = 1$.

□

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