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APPROXIMATE BI-HOMOMORPHISMS AND BI-DERIVATIONS IN C^*-TERNARY ALGEBRAS: REVISITED

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Abstract. Bae and W. Park [3] proved the Hyers-Ulam stability of bi-homomorphisms and bi-derivations in C^*-ternary algebras.

It is easy to show that the definitions of bi-homomorphisms and bi-derivations, given in [3], are meaningless. So we correct the definitions of bi-homomorphisms and bi-derivations. Under the conditions in the main theorems, we can show that the related mappings must be zero. In this paper, we correct the statements and the proofs of the results, and prove the corrected theorems.

1. Introduction and preliminaries

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the

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outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [1, 16]).

Definition 1.1. ([3]) Let A and B are C^*-ternary algebras. A $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is called a *C^*-ternary bi-homomorphism* if it satisfies

$$\begin{aligned} H([x, y, z], w) &= [H(x, w), H(y, w), H(z, w)], \\ H(x, [y, z, w]) &= [H(x, y), H(x, z), H(x, w)] \end{aligned}$$

for all $x, y, z, w \in A$.

A $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is called a *C^*-ternary bi-derivation* if it satisfies

$$\begin{aligned} \delta([x, y, z], w) &= [\delta(x, w), y, z] + [x, \delta(y, w), z] + [x, y, \delta(z, w)], \\ \delta(x, [y, z, w]) &= [\delta(x, y), z, w] + [y, \delta(x, z), w] + [y, z, \delta(x, w)] \end{aligned}$$

for all $x, y, z, w \in A$.

Note that if we replace w by $2w$ in the first equality of the definition of bi-homomorphism then $2H([x, y, z], w) = 8[H(x, w), H(y, w), H(z, w)]$ and so $H([x, y, z], w) = 0$ for all $x, y, z, w \in A$. Similarly, one can show that $H(x, [y, z, w]) = 0$ for all $x, y, z, w \in A$.

The w-variable of the left side in the first equality is $\mathbb{C}$-linear and the x-variable of the left side in the second equality is $\mathbb{C}$-linear. But the w-variable of the right side in the first equality is not $\mathbb{C}$-linear and the x-variable of the right side in the second equality is not $\mathbb{C}$-linear. Thus we correct the definitions of bi-homomorphism and bi-derivation as follows.

Definition 1.2. Let A and B are C^*-ternary algebras. A $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is called a *C^*-ternary bi-homomorphism* if it satisfies

$$\begin{aligned} H([x, y, z], w^3) &= [H(x, w), H(y, w^*), H(z, w)], \\ H(x^3, [y, z, w]) &= [H(x, y), H(x^*, z), H(x, w)] \end{aligned}$$

for all $x, y, z, w \in A$.

A $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is called a *C^*-ternary bi-derivation* if it satisfies

$$\begin{aligned} \delta([x, y, z], w) &= [\delta(x, w), y, z] + [x, \delta(y, w^*), z] + [x, y, \delta(z, w)], \\ \delta(x, [y, z, w]) &= [\delta(x, y), z, w] + [y, \delta(x^*, z), w] + [y, z, \delta(x, w)] \end{aligned}$$

for all $x, y, z, w \in A$.

The stability problem of functional equations originated from a question of Ulam [15] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [13] for linear mappings by considering an unbounded Cauchy difference. J.M. Rassias [12] followed the innovative approach of the Th.M. Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of various functional equations have been extensively investigated by a number of authors (see [4, 5, 6, 7, 8, 9, 10, 14]).

Under the conditions in the main results given in [3], we can show that the related mappings must be zero. In this paper, we correct the statements and the proofs of the results, and prove the corrected theorems.

2. Hyers-Ulam stability of C^*-ternary bi-homomorphisms in C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$ and that B is a C^*-ternary algebra with norm $\|\cdot\|_B$.

For a given mapping $f : A \times A \to B$, we define

$$\begin{aligned} D_{\lambda,\mu} f(x, y, z, w) :=& f(\lambda x + \lambda y, \mu z - \mu w) \\ &+ f(\lambda x - \lambda y, \mu z + \mu w) - 2\lambda\mu f(x, z) + 2\lambda\mu f(y, w) \end{aligned}$$

for all $\lambda, \mu \in \mathbb{T}^1 := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and all $x, y, z, w \in A$.

We need the following lemma to obtain the main results.

Lemma 2.1. ([3]) *Let $f : A \times A \to B$ be a mapping satisfying $D_{\lambda,\mu} f(x, y, z, w) = 0$ for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. Then the mapping $f : A \times A \to B$ is $\mathbb{C}$-bilinear.*

We prove the Hyers-Ulam stability of bi-homomorphisms in C^*-ternary algebras for the functional equation $D_{\lambda,\mu} f(x, y, z, w) = 0$.

THEOREM 2.2. *Let p and θ be positive real numbers with $p < 2$, and let $f : A \times A \to B$ be a mapping such that*

$$\|D_{\lambda,\mu} f(x, y, z, w)\|_B \leq \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p), \tag{1}$$

$$\begin{aligned} &\|f([x, y, z], w^3) - [f(x, w), f(y, w^*), f(z, w)]\|_B \\ &\quad +\|f(x^3, [y, z, w]) - [f(x, y), f(x^*, z), f(x, w)]\|_B \\ &\quad \leq \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p) \end{aligned} \tag{2}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique C^*-ternary bi-homomorphism $H : A \times A \to B$ such that

$$\|f(x, y) - H(x, y)\|_B \leq \frac{6\theta}{4 - 2^p}(\|x\|_A^p + \|y\|_A^p) + \frac{4}{3}\|f(0, 0)\|_B \tag{3}$$

for all $x, y \in A$.

Proof. By the same reasoning as in the proof of [3, Theorem 2.3], there exists a unique $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ satisfying (3). The $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is given by

$$H(x, y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

for all $x, y \in A$.

It follows from (2) that

$$\begin{aligned} &\|H([x, y, z], w^3) - [H(x, w), H(y, w^*), H(z, w)]\|_B \\ &\quad + \|H(x^3, [y, z, w]) - [H(x, y), H(x^*, z), H(x, w)]\|_B \\ &= \lim_{n\to\infty} \frac{1}{64^n}\big(\|f([2^n x, 2^n y, 2^n z], 8^n w^3) \\ &\quad - [f(2^n x, 2^n w), f(2^n y, 2^n w^*), f(2^n z, 2^n w)]\|_B \\ &\quad + \|f(8^n x^3, [2^n y, 2^n z, 2^n w]) \\ &\quad - [f(2^n x, 2^n y), f(2^n x^*, 2^n z), f(2^n x, 2^n w)]\|_B\big) \\ &\leq \lim_{n\to\infty} \frac{2^{np}}{64^n}\theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p) = 0 \end{aligned}$$

for all $x, y, z, w \in A$. So

$$H([x, y, z], w^3) = [H(x, w), H(y, w^*), H(z, w)]$$

and

$$H(x^3, [y, z, w]) = [H(x, y), H(x^*, z), H(x, w)]$$

for all $x, y, z, w \in A$.

Therefore, the mapping H is a unique C^*-ternary bi-homomorphism satisfying (3). $\square$

Theorem 2.3. *Let p and θ be positive real numbers with $p > 6$, and let $f : A\times A \to B$ be a mapping satisfying (1), (2) and $f(0,0) = 0$. Then there exists a unique C^*-ternary bi-homomorphism $H : A\times A \to B$ such that*

$$\|f(x,y) - H(x,y)\|_B \leq \frac{6\theta}{2^p - 4}(\|x\|_A^p + \|y\|_A^p)$$

for all $x, y \in A$.

Proof. The proof is similar to the proof of Theorem 2.2. $\square$

Theorem 2.4. *Let p and θ be positive real numbers with $p < \frac{1}{2}$, and let $f : A \times A \to B$ be a mapping such that*

$$\big\|D_{\lambda,\mu} f(x,y,z,w)\big\|_B \leq \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \cdot \|z\|_A^p \cdot \|w\|_A^p, \tag{4}$$

$$\begin{aligned} &\big\|f([x,y,z],w^3) - [f(x,w), f(y,w^*), f(z,w)]\big\|_B \\ &\quad + \big\|f(x^3,[y,z,w]) - [f(x,y), f(x^*,z), f(x,w)]\big\|_B \\ &\quad\quad \leq \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \cdot \|z\|_A^p \cdot \|w\|_A^p \end{aligned} \tag{5}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique C^*-ternary bi-homomorphism $H : A \times A \to B$ such that

$$\|f(x,y) - H(x,y)\|_B \leq \frac{2\theta}{4 - 2^{4p}}\|x\|_A^{2p}\|y\|_A^{2p} + \frac{4}{3}\|f(0,0)\|_B \tag{6}$$

for all $x, y \in A$.

Proof. By the same reasoning as in the proof of [3, Theorem 2.6], there exists a unique $\mathbb{C}$-bilinear mapping $H : A \times A \to A$ satisfying (6). The $\mathbb{C}$-bilinear mapping $H : A \times A \to A$ is given by

$$H(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

for all $x, y \in A$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Theorem 2.5. *Let p and θ be positive real numbers with $p > \frac{3}{2}$, and let $f : A\times A \to B$ be a mapping satisfying (4), (5) and $f(0,0) = 0$. Then there exists a unique C^*-ternary bi-homomorphism $H : A\times A \to B$ such that*

$$\|f(x,y) - H(x,y)\|_B \leq \frac{2\theta}{2^{4p} - 4}\|x\|_A^{2p}\|y\|_A^{2p}$$

for all $x, y \in A$.

Proof. The proof is similar to the proofs of Theorems 2.2 and 2.4. $\square$

3. Hyers-Ulam stability of C^*-ternary bi-derivations on C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$.

We prove the Hyers-Ulam stability of C^*-ternary bi-derivations on C^*-ternary algebras for the functional equation $D_{\lambda,\mu}f(x,y,z,w) = 0$.

THEOREM 3.1. *Let p and θ be positive real numbers with $p < 2$, and let $f : A \times A \to A$ be a uniformly continuous mapping such that*

$$\|D_{\lambda,\mu}f(x,y,z,w)\|_A \le \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p), \tag{7}$$

$$\begin{aligned}&\|f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w^*),z] - [x,y,f(z,w)]\|_A \\ &\quad +\|f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x^*,z),w] - [y,z,f(x,w)]\|_A \\ &\quad \le \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p)\end{aligned} \tag{8}$$

for all $x,y,z,w \in A$. Then there is a unique C^*-ternary bi-derivation $\delta : A \times A \to A$ such that

$$\|f(x,y) - \delta(x,y)\|_A \le \frac{6\theta}{4-2^p}(\|x\|_A^p + \|y\|_A^p) + \frac{4}{3}\|f(0,0)\|_A \tag{9}$$

for all $x, y \in A$.

Proof. By the same reasoning as in the proof of [3, Theorems 2.3 and 3.1], there exists a unique $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ satisfying (9). The $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is given by

$$\delta(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

for all $x, y \in A$.

It is easy to show that

$$\delta(x,y) = \lim_{n\to\infty} \frac{1}{16^n} f(8^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{16^n} f(2^n x, 8^n y)$$

for all $x, y \in A$, since δ is bi-additive and f is uniformly continuous.

It follows from (8) that

$$\begin{aligned}
&\|\delta([x,y,z],w)-[\delta(x,w),y,z]-[x,\delta(y,w^*),z]-[x,y,\delta(z,w)]\|_A\\
&\quad+\|\delta(x,[y,z,w])-[\delta(x^*,y),z,w]-[y,\delta(x,z),w]-[y,z,\delta(x,w)]\|_A\\
&=\lim_{n\to\infty}\Bigg(\bigg\|\frac{1}{16^n}f\big(2^{3n}[x,y,z],2^nw\big)-\left[\frac{1}{4^n}f(2^nx,2^nw),y,z\right]\\
&\qquad-\left[x,\frac{1}{4^n}f(2^ny,2^nw^*),z\right]-\left[x,y,\frac{1}{4^n}f(2^nz,2^nw)\right]\bigg\|_A\\
&\qquad+\bigg\|\frac{1}{16^n}f\big(2^nx,2^{3n}[y,z,w]\big)-\left[\frac{1}{4^n}f(2^nx,2^ny),z,w\right]\\
&\qquad-\left[y,\frac{1}{4^n}f(2^nx^*,2^nz),w\right]-\left[y,z,\frac{1}{4^n}f(2^nx,2^nw)\right]\bigg\|_A\Bigg)\\
&\le\lim_{n\to\infty}\frac{2^{np}}{16^n}\theta(\|x\|_A^p+\|y\|_A^p+\|z\|_A^p+\|w\|_A^p)=0
\end{aligned}$$

for all $x,y,z,w\in A$. So

$$\delta([x,y,z],w)=[\delta(x,w),y,z]+[x,\delta(y,w^*),z]+[x,y,\delta(z,w)],$$

$$\delta(x,[y,z,w])=[\delta(x,y),z,w]+[y,\delta(x^*,z),w]+[y,z,\delta(x,w)]$$

for all $x,y,z,w\in A$.

Therefore, the mapping $\delta : A\times A\to A$ is a unique C^*-ternary bi-derivation satisfying (9), as desired. □

For the case $p>4$, one can obtain a similar result.

THEOREM 3.2. *Let p and θ be positive real numbers with $p>4$, and let $f : A\times A\to A$ be a uniformly continuous mapping satisfying (7), (8) and $f(0,0)=0$. Then there is a unique C^*-ternary bi-derivation $\delta : A\times A\to A$ such that*

$$\|f(x,y)-\delta(x,y)\|_A\le\frac{6\theta}{2^p-4}\left(\|x\|_A^p+\|y\|_A^p\right)$$

for all $x,y\in A$.

THEOREM 3.3. *Let p and θ be positive real numbers with $p<\frac{1}{2}$, and let $f : A\times A\to A$ be a uniformly continuous mapping such that*

$$\big\|D_{\lambda,\mu}f(x,y,z,w)\big\|_A\le\theta\cdot\|x\|_A^p\cdot\|y\|_A^p\cdot\|z\|_A^p\cdot\|w\|_A^p,\tag{10}$$

$$
\begin{aligned}
&\|f([x,y,z],w)-[f(x,w),y,z]-[x,f(y,w^*),z]-[x,y,f(z,w)]\|_A\\
&\quad+\|f(x,[y,z,w])-[f(x,y),z,w]-[y,f(x^*,z),w]-[y,z,f(x,w)]\|_A\\
&\quad\leq\theta\cdot\|x\|_A^p\cdot\|y\|_A^p\cdot\|z\|_A^p\cdot\|w\|_A^p \qquad (11)
\end{aligned}
$$

for all $x, y, z, w \in A$. Then there exists a unique C^*-ternary bi-derivation $\delta : A \times A \to A$ such that

$$\|f(x,y)-\delta(x,y)\|_A \leq \frac{2\theta}{4-2^{4p}}\|x\|_A^{2p}\|y\|_A^{2p}+\frac{4}{3}\|f(0,0)\|_A$$

for all $x, y \in A$.

Proof. The proof is similar to the proof of Theorem 3.1. □

For the case $p > 1$, one can obtain a similar result.

THEOREM 3.4. *Let p and θ be positive real numbers with $p > 1$, and let $f : A \times A \to A$ be a uniformly continuous mapping satisfying (10), (**??**) and $f(0, 0) = 0$. Then there exists a unique C^*-ternary bi-derivation $\delta : A \times A \to A$ such that*

$$\|f(x,y)-\delta(x,y)\|_A \leq \frac{2\theta}{2^{4p}-4}\|x\|_A^{2p}\|y\|_A^{2p}$$

for all $x, y \in A$.

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