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NAVIER-STOKES EQUATIONS IN BESOV SPACE $B^{-s}_{\infty,\infty}(\mathbb{R}^n_+)$

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Abstract. In this paper we consider the Navier-Stokes equations in the half space. Our aim is to construct a mild solution for initial data in $B^{-\alpha}_{\infty,\infty}(\mathbb{R}^n_+)$, $0 < \alpha < 1$. To do this, we derive the estimate of the Stokes flow with singular initial data in $B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)$, $0 < \alpha < 1$, $1 < q \leq \infty$.

1. Introduction

Let us consider the Navier-Stokes equations in the n-dimensional half space $\mathbb{R}^n_+$, $n \geq 2$:

$$\text{(1)} \qquad \begin{cases} \operatorname{div}\mathbf{u} = 0 & \text{for } \mathbf{x} \in \mathbb{R}^n_+, t > 0, \\ \frac{\partial}{\partial t}\mathbf{u} - \Delta\mathbf{u} + \nabla p = -\operatorname{div}(\mathbf{u} \otimes \mathbf{u}) & \text{for } \mathbf{x} \in \mathbb{R}^n_+, t > 0, \\ \mathbf{u}|_{t=0} = \mathbf{a} & \text{for } \mathbf{x} \in \mathbb{R}^n_+, \\ \mathbf{u}|_{x_n=0} = 0 & \text{for } t > 0. \end{cases}$$

Here, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), \ldots, u_n(\mathbf{x}, t))$ and $p = p(\mathbf{x}, t)$ stand for the unknown velocity vector field and the pressure of the fluid, respectively, and $\mathbf{a}$ is the given initial velocity vector field.

We transform (1) into the abstract ordinary differential equation

$$\text{(2)} \qquad \frac{\partial}{\partial t}\mathbf{u} + A\mathbf{u} = -\mathbb{P}\operatorname{div}(\mathbf{u} \otimes \mathbf{u}),$$

where $A = -\mathbb{P}\Delta$ is the Stokes operator and $\mathbb{P}$ is the Helmholtz-Weyl projection operator.

Using the solution operator of the Stokes equations in the half space, the solution of (2) is formally expressed in the integral form

$$\text{(3)} \qquad \mathbf{u}(t) = e^{-At}\mathbf{a} - \int_0^t e^{-A(t-s)}\mathbb{P}\operatorname{div}(\mathbf{u} \otimes \mathbf{u})(s)ds.$$

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A mild solution of the Navier-Stokes equations in the half space is defined as follows.

Definition 1.1. Let $\mathbf{a} \in S'(\mathbb{R}^n_+)$ with div$\mathbf{a} = 0$. A measurable function $\mathbf{u} \in L^\infty_{loc}((0,T) \times \mathbb{R}^n_+)$ is called a mild solution of the Navier-Stokes equations (1) on $(0,T)$ if $\mathbf{u}$ satisfies (3) in the sense of distribution. Here, $S'(\mathbb{R}^n_+)$ is the set of distributions supported in $\mathbb{R}^n_+$.

Borchers and Miyakawa [4] showed the local in time existence of a strong solution for $\mathbf{a} \in L^p(\mathbb{R}^n_+)$, $n < p < \infty$. Solonnikov [22] showed the local in time existence of a mild solution of the Navier-Stokes equations for $\mathbf{a} \in C(\mathbb{R}^n_+)$ and P. Maremonti [17] constructed a strong solution of the Navier-Stokes equations for $\mathbf{a} \in C(\mathbb{R}^n_+)$ (see also [18]).

Several mathematicians have considered Besov spaces of nonpositive differential order as the space of initial data. H. Amman [3] constructed a strong solution for initial data in $B^{-1+\frac{n}{p}}_{p,q}(\Omega)$, $n < p < \infty, 1 \le q \le \infty$ for any standard domain such as the whole space, the half space, exterior domains, etc.

O. Sawada [19] considered the critical case when $p = \infty$ for the Cauchy problem. He constructed a strong solution for initial data in $B^{-\epsilon}_{p,q}(\mathbb{R}^n)$, $0 \le \epsilon < 1 - \frac{n}{p}$ and $1 \le q \le \infty$. H. Kozono, T. Ogawa and Y. Taniuchi [15] also showed the local in time existence of a strong solution for initial data in $B^{0;\infty}_{\infty}(\mathbb{R}^n)$ by a different argument from [19] (We also refer to the papers of Giga [9, 10, 11], which consider initial data in $L^\infty(\mathbb{R}^n)$, and the paper of G. Koch and D. Tataru [14], which considers initial data in $BMO^{-1}(\mathbb{R}^n)$).

In this paper we consider the critical case when $p = \infty$ for the half space problem. The aim of this paper is to construct a mild solution for initial data in $B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)$, $0 < \epsilon < 1$. To do this, it is necessary to derive the estimate of the Stokes flow with singular initial data in $B^{-s}_{\infty,q}(\Omega)$, $0 < s < 1$, $1 < q \le \infty$.

When the domain is the whole space $\mathbb{R}^n$, the solution operator is a convolution type of the Heat kernel $\Gamma(\mathbf{x}, t)$, n-dimensional Riesz operators R_i, and their compositions. This makes it easy to apply well-known theories on singular integral operators for the estimate of solution operators. The functions in the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ can be defined via Littlewood-Payley decomposition of unity, and all the Besov norm estimates of the solution operator can be done using Littlewood-Payley decomposition of unity even for $s \le 0$.

When the domain involves a nonempty boundary, it is no longer possible to apply such technique, even for the half space problem. Luckily, the kernels of the (linear and bilinear) operators for the half space problem are well known. However, the solution operators are no longer convolution typed nor Littlewood-Payley decomposition of unity can be applied.

In this paper, our idea is based on the fact that $B^{-s}_{\infty,q}(\mathbb{R}^n_+)$ is the dual space of $B^s_{1,q'}(\mathbb{R}^n_+)$ for $0 < s < 1$ and $1 < q \le \infty$. Instead of the estimate of $e^{-tA} : B^{-s}_{\infty,q}(\mathbb{R}^n_+) \to B^{-s}_{\infty,q}(\mathbb{R}^n_+)$, we derive the estimate of $e^{-tA^*} : B^s_{1,q'}(\mathbb{R}^n_+) \to B^s_{1,q'}(\mathbb{R}^n_+)$, where e^{-tA^*} is the dual operator of e^{-tA}.

Cannone, Planchon and Schonbek [6] derived the Stokes estimate for the data in $B^{-s}_{p,\infty}(\mathbb{R}^n_+)$, $0 < s < 1 - \frac{1}{p}$, $1 < p < \infty$. Their main idea relies on the interpolation theory and the fact that Riesz transform is L^p-bounded for $1 < p < \infty$. Lemarié-Rieusset and A. Zhioua [16] considered the critical case when $p = \infty$. They gave an alternative description of Ukai's formula that does not rely on the use of Riesz transforms and introduced tangential Besov spaces.

Our main theorem is stated as below.

Theorem 1.2 (Main Theorem)**.** *Assume $n \geq 2$ and $0 < \epsilon < 1$. Assume that $a \in B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)$ satisfies div$\mathbf{a} = 0$. Then there exist a positive time T_* and a unique mild solution $\mathbf{u}$ of* (1) *satisfying*

$$\begin{aligned} &\mathbf{u} \in C_w([0, T_*); B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)),\\ &t^{\frac{\epsilon}{2}}\mathbf{u} \in L^\infty(0, T_*; L^\infty(\mathbb{R}^n_+)),\\ &t^{\frac{\epsilon+1}{2}}\mathbf{u} \in L^\infty(0, T_*; B^1_{\infty,\infty}(\mathbb{R}^n_+)). \end{aligned}$$

Moreover,

$$\|\mathbf{u} - e^{-tA}\mathbf{a}\|_{B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)} = O(t^{\frac{1-\epsilon}{2}}) \text{ as } t \to 0+ .$$

Here, C_w denotes the space of all weakly continuous functions.

2. Notations, Besov spaces and Stokes formula

Throughout this paper, we use the following notations for vectors, tensors, and Lebesgue measures: $\bar{\mathbf{f}} = (f_1, \ldots, f_{n-1})$, $\mathbf{f} = (f_1, \ldots, f_n) = (\bar{\mathbf{f}}, f_n)$, $\mathcal{F} = (F_{jk})^n_{j,k=1}$, $d\bar{x} = dx_1 \cdots dx_{n-1}$, $dx = dx_1 \cdots dx_n = d\bar{x}dx_n$.

We denote by C the various constants which depend only on n, the dimension of the domain.

2.1. Besov spaces defined in the half space

Let ϕ_j be the Littlewood-Paley dyadic decomposition of unity. Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$. A Besov space $B^s_{p,q}(\mathbb{R}^n)$ is a set of functions f satisfying

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \|\psi * f\|_{L^p} + [\sum_{j=1}^\infty 2^{jsq}\|\phi_j * f\|^q_{L^p}]^{1/q} < \infty \text{ if } q < \infty \\ \|\psi * f\|_{L^p} + \sup_{j\geq 1} 2^{js}\|\phi_j * f\|_{L^p} < \infty \text{ if } q = \infty. \end{cases}$$

Now consider a half space $\mathbb{R}^n_+$. Let $s > 0$. A Besov space $B^s_{p,q}(\mathbb{R}^n_+)$ is defined by the set of functions $\{g|_{\mathbb{R}^n_+} : g \in B^s_{p,q}(\mathbb{R}^n)\}$, which is a Banach space with norm $\|f\|_{B^s_{p,q}(\mathbb{R}^n_+)} = \inf_{g|\Omega=f} \|g\|_{B^s_{p,q}(\mathbb{R}^n)}$.

Denote by $\overset{\circ}{B}{}^s_{p,q}(\mathbb{R}^n_+)$ the complete closure of $C^\infty_0(\mathbb{R}^n_+)$ with the norm of $B^s_{p,q}(\mathbb{R}^n_+)$. It is well-known that $\overset{\circ}{B}{}^s_{p,q}(\mathbb{R}^n_+) = B^s_{p,q}(\mathbb{R}^n_+)$ if $0 \leq sp < 1$, $1 \leq p, q < \infty$; $\overset{\circ}{B}{}^s_{p,q}(\mathbb{R}^n_+) = \{f \in B^s_{p,q}(\mathbb{R}^n_+) : f|_{\partial\Omega} = 0\}$ if $sp > 1$, $1 \leq p, q < \infty$ (see [13, 5] for reference).

Let $B^{-s}_{p,q}(\mathbb{R}^n_+) = (\overset{\circ}{B}{}^s_{p',q'}(\mathbb{R}^n_+))'$ for $1 < p, q \le \infty$, where X' denotes the dual space of X and $p' = \frac{p}{p-1}, q' = \frac{q}{q-1}$. Note that $B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+) = (B^{\alpha}_{1,q'}(\mathbb{R}^n_+))'$ for $0 < \alpha < 1$ and $1 < q \le \infty$ since $\overset{\circ}{B}{}^{\alpha}_{1,q'}(\mathbb{R}^n_+) = B^{\alpha}_{1,q'}(\mathbb{R}^n_+)$ for $0 < \alpha < 1$ and $1 \le q' < \infty$.

It is well-known that $(L^p(\mathbb{R}^n_+), W^{1,p}(\mathbb{R}^n_+))_{s,q} \equiv B^s_{p,q}(\mathbb{R}^n_+)$ for $0 < s < 1$ and $1 \le p, q \le \infty$, where $(X, Y)_{s,q}$ denotes the real interpolation between the Banach spaces X and Y (see [2] and the references therein).

It is well-known that for $1 \le p, q < \infty$,

$$\|f\|_{B^{\alpha}_{p,q}(\mathbb{R}^n_+)} \equiv \|f\|_{L^p(\mathbb{R}^n_+)} + \left(\int_{\mathbb{R}^n_+}\left(\int_{\mathbb{R}^n_+}\frac{|f(\mathbf{x}) - f(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{n\frac{p}{q}+\alpha p}}dx\right)^{\frac{q}{p}}dy\right)^{\frac{1}{q}}$$

(see [5, 2] and the references therein). Here, $A \equiv B$ means the norm equivalence, that is, $\frac{1}{C}A \le B \le CA$ for some positive constant $C > 0$.

It is also well-known that for $1 \le p < \infty$,

$$\begin{aligned}\|f\|_{B^{\alpha,p}_p(\mathbb{R}^n_+)} &\equiv \|f\|_{L^p(\mathbb{R}^n_+)} \\ &+ \left(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\int_0^\infty \frac{|f(\bar{\mathbf{x}}, x_n) - f(\bar{\mathbf{y}}, x_n)|^p}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^{n-1+\alpha p}}dx_n d\bar{x} d\bar{y}\right)^{1/p} \\ &+ \left(\int_0^\infty\int_0^\infty\int_{\mathbb{R}^{n-1}} \frac{|f(\bar{\mathbf{x}}, x_n) - f(\bar{\mathbf{x}}, y_n)|^p}{|x_n - y_n|^{1+\alpha p}}d\bar{x} dx_n dy_n\right)^{1/p}\end{aligned}$$

(see [5, 1]).

2.2. Solution formula of Stokes equations

Let us consider the Stokes problem in the n-dimensional half space $\mathbb{R}^n_+$, $n \ge 2$:

$$\begin{cases} \mathrm{div}\mathbf{u} = 0 & \text{for } \mathbf{x} \in \mathbb{R}^n_+, t > 0, \\ \mathbf{u}_t - \Delta\mathbf{u} + \nabla p = \mathrm{div}\mathcal{F} & \text{for } \mathbf{x} \in \mathbb{R}^n_+, t > 0, \\ \mathbf{u}|_{t=0} = \mathbf{a} & \text{for } \mathbf{x} \in \mathbb{R}^n_+, \\ \mathbf{u}|_{x_n=0} = 0 & \text{for } t > 0. \end{cases} \tag{4}$$

Here, $\mathbf{a}$ is a given initial velocity and $\mathrm{div}\mathcal{F}$ is a given external force.

In [22], the solution $\mathbf{u}$ of the Stokes problem (4) is formally given by

$$\mathbf{u}(\mathbf{x}, t) = \int_{\mathbb{R}^n_+}\mathfrak{G}(\mathbf{x}, \mathbf{y}, t)\mathbf{a}(\mathbf{y})d\mathbf{y} + \int_0^t d\tau\int_{\mathbb{R}^n_+}\mathfrak{G}(\mathbf{x}, \mathbf{y}, t-\tau)\mathbb{P}\mathrm{div}\mathcal{F}(\mathbf{y}, \tau)d\mathbf{y}, \tag{5}$$

where $\mathfrak{G} = (G_{ij})_{i,j=1,\dots,n}$ are the Green tensors of the Stokes equation defined by

$$\begin{aligned}G_{ij}(\mathbf{x}, \mathbf{y}, t) &= \delta_{ij}(\Gamma(\mathbf{x}-\mathbf{y}, t) - \Gamma(\mathbf{x}-\mathbf{y}^*, t)) \\ &+ 4(1-\delta_{jn})\frac{\partial}{\partial y_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial}{\partial z_i}N(\mathbf{x}-\mathbf{z})\Gamma(\mathbf{z}-\mathbf{y}^*, t)dz\end{aligned}$$

$$= \delta_{ij}\Gamma(\mathbf{x}-\mathbf{y},t) + G^*_{ij}(\mathbf{x},\mathbf{y},t). \tag{6}$$

Here, $\Gamma(\mathbf{x},t) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}$ for $t > 0$, $N(\mathbf{x}) = \begin{cases} \frac{1}{n(2-n)\omega_n}|\mathbf{x}|^{2-n}, & \text{if } n \geq 3, \\ \frac{1}{2\pi}\ln|\mathbf{x}|, & \text{if } n = 2. \end{cases}$

Let $G(\mathbf{x},\mathbf{y})$ be the Green function of the Laplace equation defined by

$$G(\mathbf{x},\mathbf{y}) = N(\mathbf{x}-\mathbf{y}) + N(\mathbf{x}-\mathbf{y}^*). \tag{7}$$

The Helmholz-Weyl projection operator is given by

$$\mathbb{P}\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \nabla_x \int_{\mathbb{R}^n_+} \nabla_y G(\mathbf{x},\mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \tag{8}$$

when $f_n|_{x_n=0} = 0$.

Set $e^{-At}\mathbf{f}(\mathbf{x}) := \int_{\mathbb{R}^n_+} \mathfrak{G}(\mathbf{x},\mathbf{y},t)\mathbf{f}(\mathbf{y})d\mathbf{y} = (\int_{\mathbb{R}^n_+} G_{ij}(\mathbf{x},\mathbf{y},t) f_j(\mathbf{y}) d\mathbf{y})^n_{i=1}$. Then $\mathbf{u}$ can be rewritten as

$$\mathbf{u}(\mathbf{x},t) = e^{-tA}\mathbf{a}(\mathbf{x}) + \int_0^t e^{-(t-\tau)A}\mathbb{P}\text{div}\mathcal{F}(\mathbf{x},\tau)d\tau.$$

3. Estimate of the Stokes operator

It is a well-known fact that

$$\|e^{-tA}\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)} \leq C\|\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)}, \tag{9}$$

$$\|\nabla e^{-tA}\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-1/2}\|\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)} \tag{10}$$

(see [20] and [22] for the estimate

$$\|\nabla^k e^{-tA}\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-k/2}\|\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)}, \ k = 0,1,2,\ldots). \tag{11}$$

In this section, we will show that the following estimates for the Stokes operator hold.

Theorem 3.1. *Let* $0 < \alpha < 1$ *and* $1 < q \leq \infty$. *Then for* $0 < t \leq 1$, *we have*

$$\|e^{-tA}\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha}{2}}\|\mathbf{a}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)}, \tag{12}$$

$$\|e^{-tA}\mathbf{a}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)} \leq C\|\mathbf{a}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)}. \tag{13}$$

Interpolating (9) and (11), we obtain the estimates

$$\|e^{-tA}a\|_{B^\beta_{\infty,r}(\mathbb{R}^n_+)} \leq Ct^{-\beta/2}\|a\|_{L^\infty(\mathbb{R}^n_+)}, \ 0 < t \leq 1 \tag{14}$$

for $0 < \beta$ and $1 \leq r \leq \infty$. Combining (14) and (12), we have the following corollary.

Corollary 3.2. *Let* $0 < \alpha < 1$ *and* $1 < q \leq \infty$. *Then for* $0 < t \leq 1$, *we have*

$$\|e^{-tA}\mathbf{a}\|_{B^\beta_{\infty,r}(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha+\beta}{2}}\|\mathbf{a}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)}, \ 0 < \beta, \ 1 \leq r \leq \infty. \tag{15}$$

Since $\mathbf{a}$ in the above theorem is a distribution in a half space, it is unlikely that $e^{-tA}\mathbf{a}$ is easy to estimate directly. Since $L^\infty(\mathbb{R}^n_+) = (L^1(\mathbb{R}^n_+))'$ and $B^{-s}_{p,q}(\mathbb{R}^n_+) = (\overset{\circ}{B}{}^s_{p',q'}(\mathbb{R}^n_+))'$ for $0 < s$ and $1 < p, q \leq \infty$, the proof of Theorem 3.1 will follow from the estimate of $e^{-tA^*}\mathbf{b}$ for $\mathbf{b} \in L^1(\mathbb{R}^n_+))$ and $\mathbf{b} \in \overset{\circ}{B}{}^\alpha_{1,q'}(\mathbb{R}^n_+)$, respectively, where e^{-tA^*} denotes the dual operator of e^{-tA}. More precisely, since

$$\|e^{-tA}\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)} = \sup_{\mathbf{b}\in C_0^\infty(\mathbb{R}^n_+)} \frac{\langle \mathbf{a}, e^{-tA^*}\mathbf{b}\rangle}{\|\mathbf{b}\|_{L^1(\mathbb{R}^n_+)}} \tag{16}$$

and

$$\|e^{-tA}\mathbf{a}\|_{B^{-s}_{\infty,q}(\mathbb{R}^n_+)} = \sup_{\mathbf{b}\in C_0^\infty(\mathbb{R}^n_+)} \frac{\langle \mathbf{a}, e^{-tA^*}\mathbf{b}\rangle}{\|\mathbf{b}\|_{B^s_{1,q'}(\mathbb{R}^n_+)}}, \tag{17}$$

the following estimate is enough to show (12) and (13):

$$\|e^{-tA^*}\mathbf{b}\|_{B^\alpha_{1,q}(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha}{2}}\|\mathbf{b}\|_{L^1(\mathbb{R}^n_+)} \tag{18}$$

and

$$\|e^{-tA^*}\mathbf{b}\|_{B^\alpha_{1,q}(\mathbb{R}^n_+)} \leq C\|\mathbf{b}\|_{B^\alpha_{1,q}(\mathbb{R}^n_+)}. \tag{19}$$

First, we show the following two inequalities.

Lemma 3.3. *For $t > 0$, we have*

$$\|e^{-tA^*}\mathbf{b}\|_{L^1(\mathbb{R}^n_+)} \leq C\|\mathbf{b}\|_{L^1(\mathbb{R}^n_+)}, \tag{20}$$

$$\|\nabla e^{-tA^*}\mathbf{b}\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-1/2}\|\mathbf{b}\|_{L^1(\mathbb{R}^n_+)}. \tag{21}$$

By the real interpolation, $(L^1(\mathbb{R}^n_+), W^{1,1}(\mathbb{R}^n_+))_{\alpha,q} = B^\alpha_{1,q}(\mathbb{R}^n_+)$ for $0 < \alpha < 1$ and $1 \leq q \leq \infty$. Interpolating (20) and (21), we have

$$\|e^{-tA^*}\mathbf{b}\|_{B^\alpha_{1,q}(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha}{2}}\|\mathbf{b}\|_{L^1(\mathbb{R}^n_+)},\ 0 < t \leq 1$$

for $0 < \alpha < 1$ and $1 \leq q \leq \infty$. Therefore, we obtain the inequality (18).

Second, we show the following inequality.

Lemma 3.4. *Let $0 < \alpha < 1$. Then for $t > 0$, we have*

$$\|e^{-tA^*}\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)} \leq C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}. \tag{22}$$

(It would be simpler if we could estimate $\|e^{-tA^*}\mathbf{b}\|_{W^{1,1}(\mathbb{R}^n_+)}$ for $\mathbf{b} \in W^{1,1}(\mathbb{R}^n_+)$. However, for the Stokes kernel G_{ij}, it is not likely to obtain the estimate $\|e^{-tA^*}\mathbf{b}\|_{W^{1,1}(\mathbb{R}^n_+)}$ for $\mathbf{b} \in W^{1,1}(\mathbb{R}^n_+)$, but the estimate of $\|e^{-tA^*}\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}$ for $\mathbf{b} \in B^\alpha_{1,1}(\mathbb{R}^n_+)$ seems to be optimal.)

By the real interpolation, $(L^1(\mathbb{R}^n_+), B^\alpha_{1,1}(\mathbb{R}^n_+))_{\theta,q} = B^{\alpha\theta}_{1,q}(\mathbb{R}^n_+)$. Interpolating (20) and (22), we have

$$\|e^{-tA^*}\mathbf{b}\|_{B^{\alpha\theta}_{1,q}(\mathbb{R}^n_+)} \leq C\|\mathbf{b}\|_{B^{\alpha\theta}_{1,q}(\mathbb{R}^n_+)},\ 0 < t \leq 1. \tag{23}$$

Setting $\alpha\theta = \beta$, (23) can be rewritten as

$$\|e^{-tA^*}\mathbf{b}\|_{B^{\beta}_{1,q}(\mathbb{R}^n_+)} \le C\|\mathbf{b}\|_{B^{\beta}_{1,q}(\mathbb{R}^n_+)}$$

for $0 < \beta < 1$. Therefore, we obtain the inequality (19).

Now we will give the proofs of Lemma 3.3 and Lemma 3.4. But before, note that $e^{-tA^*}\mathbf{b}$ can be written as

$$(e^{-tA^*}\mathbf{b})_j(\mathbf{y}) = \int_{\mathbb{R}^n_+} G_{ij}(\mathbf{x},\mathbf{y},t)b_i(\mathbf{x})dx$$

since

$$\int_{\mathbb{R}^n_+} b_i(\mathbf{x})[\int_{\mathbb{R}^n_+} G_{ij}(\mathbf{x},\mathbf{y},t)a_j(\mathbf{y})dy]dx = \int_{\mathbb{R}^n_+} a_j(\mathbf{y})[\int_{\mathbb{R}^n_+} G_{ij}(\mathbf{x},\mathbf{y},t)b_i(\mathbf{x})dx]dy$$

for any $\mathbf{a},\mathbf{b} \in C^\infty_0(\mathbb{R}^n_+)$.

Proof of Lemma 3.3. By the definition (6) of G_{ij}, we note that

$$(D^k_y e^{-tA^*}\mathbf{b})_j(\mathbf{y}) = \int_{\mathbb{R}^n_+} D^k_y G_{ij}(\mathbf{x},\mathbf{y},t)b_i(\mathbf{x})dx.$$

Here, D^k_y denotes $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$ for any multi-index α with $|\alpha| = k$.

In [22], it is known that

$$|D^s_t D^k_x D^m_y G^*_{ij}(\mathbf{x},\mathbf{y},t)| \le ct^{-s-m_n/2}(t+x_n^2)^{-k_n/2}(|\mathbf{x}-\mathbf{y}^*|^2+t)^{-\frac{n+|k'|+|m'|}{2}}e^{-\frac{cy_n^2}{t}}.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^n_+} |D^k_y G^*_{ij}(\mathbf{x},\mathbf{y},t)|dy &\le c\int_{\mathbb{R}^n_+} t^{-k_n/2}(|\mathbf{x}-\mathbf{y}^*|^2+t)^{-\frac{n+|k'|}{2}}e^{-\frac{cy_n^2}{t}}dy\\ &\le ct^{-k_n/2}\int_0^\infty[\int_{\mathbb{R}^{n-1}}(|\mathbf{x}-\mathbf{y}^*|^2+t)^{-\frac{n+|k'|}{2}}d\bar{y}]e^{-\frac{cy_n^2}{t}}dy_n\\ &\le ct^{-k_n/2+\frac{1}{2}}(|x_n|^2+t)^{-\frac{1+|k'|}{2}} \le ct^{-\frac{|k|}{2}}.\end{aligned}$$

On the other hand, it is easy to see that

$$\int_{\mathbb{R}^n_+} |D^k_y\Gamma(\mathbf{x}-\mathbf{y},t)|dy \le ct^{-\frac{|k|}{2}}.$$

Since $G_{ij}(\mathbf{x},\mathbf{y},t) = \delta_{ij}\Gamma(\mathbf{x}-\mathbf{y},t) + G^*_{ij}(\mathbf{x},\mathbf{y},t)$, we conclude that

$$\int_{\mathbb{R}^n_+} |D^k_y G_{ij}(\mathbf{x},\mathbf{y},t)|dy \le ct^{-\frac{|k|}{2}}. \tag{24}$$

Then (20) and (21) follows directly from (24). □

Proof of Lemma 3.4. For the proof of the theorem, it is enough to show

$$\int_0^\infty\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{|e^{-tA^*}\mathbf{b}(\bar{\mathbf{y}},y_n) - e^{-tA^*}\mathbf{b}(\bar{\mathbf{z}},y_n)|}{|\bar{\mathbf{y}}-\bar{\mathbf{z}}|^{n-1+\alpha}}d\bar{y}d\bar{z}dy_n$$

$$+\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_0^\infty \frac{|e^{-tA^*}\mathbf{b}(\bar{\mathbf{y}},y_n)-e^{-tA^*}\mathbf{b}(\bar{\mathbf{y}},z_n)|}{|y_n-z_n|^{1+\alpha}}dz_ndy_nd\bar{y}$$
$$\le C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}.$$

• First, we will show that

(25)
$$\int_0^\infty\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|e^{-tA^*}\mathbf{b}(\bar{\mathbf{y}},y_n)-e^{-tA^*}\mathbf{b}(\bar{\mathbf{z}},y_n)|}{|\bar{\mathbf{y}}-\bar{\mathbf{z}}|^{n-1+\alpha}}d\bar{y}d\bar{z}dy_n \le C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}.$$

Since $\int_{\mathbb{R}^n_+} G_{ij}(\mathbf{x},\mathbf{y},t)b_i(\mathbf{x})dx = \int_{\mathbb{R}^n_+} G_{ij}(\bar{\mathbf{x}},x_n,\bar{\mathbf{0}},y_n,t)b_i(\bar{\mathbf{y}}+\bar{\mathbf{x}},x_n)dx$,

$$(e^{-tA^*}\mathbf{b})_j(\bar{\mathbf{y}},y_n)-(e^{-tA^*}\mathbf{b})_j(\bar{\mathbf{z}},y_n)$$
$$= -\int_{\mathbb{R}^n_+} G_{ij}(\bar{\mathbf{x}},x_n,\bar{\mathbf{0}},y_n,t)[b_i(\bar{\mathbf{y}}+\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{z}}+\bar{\mathbf{x}},x_n)]dx.$$

Hence

$$\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|(e^{-tA^*}\mathbf{b})_j(\bar{\mathbf{y}},y_n)-(e^{-tA^*}\mathbf{b})_j(\bar{\mathbf{z}},y_n)|}{|\bar{\mathbf{y}}-\bar{\mathbf{z}}|^{n-1+\alpha}}d\bar{y}d\bar{z}$$
$$\le \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n_+} |G_{ij}(\bar{\mathbf{x}},x_n,\bar{\mathbf{0}},y_n,t)|\frac{|b_i(\bar{\mathbf{y}}+\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{z}}+\bar{\mathbf{x}},x_n)|}{|\bar{\mathbf{y}}-\bar{\mathbf{z}}|^{n-1+\alpha}}dxd\bar{y}d\bar{z}$$
$$= \int_{\mathbb{R}^n_+} |G_{ij}(\bar{\mathbf{x}},x_n,\bar{\mathbf{0}},y_n,t)|\Big(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|b_i(\bar{\mathbf{y}}+\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{z}}+\bar{\mathbf{x}},x_n)|}{|\bar{\mathbf{y}}-\bar{\mathbf{z}}|^{n-1+\alpha}}d\bar{y}d\bar{z}\Big)dx$$
$$= \int_{\mathbb{R}^n_+} |G_{ij}(\bar{\mathbf{x}},x_n,\bar{\mathbf{0}},y_n,t)|\Big(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|b_i(\bar{\mathbf{y}},x_n)-b_i(\bar{\mathbf{z}},x_n)|}{|\bar{\mathbf{y}}-\bar{\mathbf{z}}|^{n-1+\alpha}}d\bar{y}d\bar{z}\Big)dx.$$

Since

$$\int_{\mathbb{R}^{n-1}} |G_{ij}(\bar{\mathbf{x}},x_n,\bar{\mathbf{0}},y_n,t)|d\bar{x} \le \int_{\mathbb{R}^{n-1}} \frac{e^{-c\frac{y_n^2}{t}}}{(|\bar{\mathbf{x}}|^2+(x_n+y_n)^2+t)^{\frac{n}{2}}}d\bar{x}$$
$$\le Ct^{-1/2}e^{-c\frac{y_n^2}{t}},$$

we have the estimate

$$\int_0^\infty\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|(e^{-tA^*}\mathbf{b})_j(\bar{\mathbf{y}},y_n)-(e^{-tA^*}\mathbf{b})_j(\bar{\mathbf{z}},y_n)|}{|\bar{\mathbf{y}}-\bar{\mathbf{z}}|^{n-1+\alpha}}d\bar{y}d\bar{z}dy_n$$
$$\le \Big(\int_0^\infty Ct^{-1/2}e^{-c\frac{y_n^2}{t}}dy_n\Big)\Big(\int_0^\infty\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|b_i(\bar{\mathbf{y}},x_n)-b_i(\bar{\mathbf{z}},x_n)|}{|\bar{\mathbf{y}}-\bar{\mathbf{z}}|^{n-1+\alpha}}d\bar{y}d\bar{z}dx_n\Big)$$
$$\le C\Big(\int_0^\infty\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|b_i(\bar{\mathbf{y}},x_n)-b_i(\bar{\mathbf{z}},x_n)|}{|\bar{\mathbf{y}}-\bar{\mathbf{z}}|^{n-1+\alpha}}d\bar{y}d\bar{z}dx_n\Big) \le C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}.$$

• Second, we will show that

(26)
$$\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_0^\infty \frac{|e^{-tA^*}\mathbf{b}(\bar{\mathbf{y}},y_n)-e^{-tA^*}\mathbf{b}(\bar{\mathbf{y}},z_n)|}{|y_n-z_n|^{1+\alpha}}dz_ndy_nd\bar{y} \le C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}.$$

Set $T_1(t)\mathbf{b} = \int_{\mathbb{R}^n_+} G^1_{ij}(\mathbf{x},\mathbf{y},t)b_i(\mathbf{y})dy$ and $T_2(t)\mathbf{b} = \int_{\mathbb{R}^n_+} G^2_{ij}(\mathbf{x},\mathbf{y},t)b_i(\mathbf{y})dy$, where

$$G^1_{ij}(\mathbf{x},\mathbf{y},t) = -\delta_{ij}[\Gamma_t(\mathbf{x}-\mathbf{y}) - \Gamma_t(\mathbf{x}-\mathbf{y}^*)],$$
$$G^2_{ij}(\mathbf{x},\mathbf{y},t) = 4(1-\delta_{jn})\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial}{\partial x_i}N(\mathbf{x}-\mathbf{z})\Gamma(\mathbf{z}-\mathbf{y}^*,t)dz.$$

Then $e^{-tA^*}\mathbf{b} = T_1(t)\mathbf{b} + T_2(t)\mathbf{b}$.

Instead of $e^{-tA^*}\mathbf{b}(\bar{\mathbf{y}},y_n) - e^{-tA^*}\mathbf{b}(\bar{\mathbf{y}},z_n)$, we consider $T_1(t)\mathbf{b}(\bar{\mathbf{y}},y_n) - T_1(t)\mathbf{b}(\bar{\mathbf{y}},z_n)$ and $T_2(t)\mathbf{b}(\bar{\mathbf{y}},y_n) - T_2(t)\mathbf{b}(\bar{\mathbf{y}},z_n)$ separately.

Step 1. At this step, we derive the estimate

$$\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_0^\infty \frac{|T_1(t)\mathbf{b}(\bar{\mathbf{y}},y_n) - T_1(t)\mathbf{b}(\bar{\mathbf{y}},z_n)|}{|y_n-z_n|^{1+\alpha}}dy_ndz_nd\bar{y} \le C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}. \tag{27}$$

Let $\widetilde{\mathbf{b}}$ be the zero extension of $\mathbf{b}$. By the well-known theory, if $\mathbf{b}\in B^\alpha_{1,1}(\mathbb{R}^n_+)$, $0<\alpha<1$, then

$$\widetilde{\mathbf{b}} \in B^\alpha_{1,1}(\mathbb{R}^n) \text{ with } \|\widetilde{\mathbf{b}}\|_{B^\alpha_{1,1}(\mathbb{R}^n)} \le \|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}$$

(see [13]).

Since

$$\begin{aligned}
T_1(t)\mathbf{b}(\mathbf{y}) &= \int_{\mathbb{R}^n_+}[\Gamma_t(\mathbf{x}-\mathbf{y},t) - \Gamma_t(\mathbf{x}-\mathbf{y}^*)]b_j(\mathbf{x})dx\\
&= \int_{\mathbb{R}^n}[\Gamma_t(\mathbf{x}-\mathbf{y},t) - \Gamma_t(\mathbf{x}-\mathbf{y}^*)]\tilde{b}_j(\mathbf{x})dx\\
&= \int_{\mathbb{R}^n}\Gamma_t(\bar{\mathbf{x}}-\bar{\mathbf{y}},x_n,t)[\tilde{b}_j(\bar{\mathbf{x}},x_n+y_n) - \tilde{b}_j(\bar{\mathbf{x}},x_n-y_n)]dx,
\end{aligned}$$

we have

$$\begin{aligned}
&T_1(t)\mathbf{b}(\bar{\mathbf{y}},y_n) - T_1(t)\mathbf{b}(\bar{\mathbf{y}},z_n)\\
&= \int_{\mathbb{R}^n}\Gamma_t(\bar{\mathbf{x}}-\bar{\mathbf{y}},x_n,t)[\tilde{b}_j(\bar{\mathbf{x}},x_n+y_n) - \tilde{b}_j(\bar{\mathbf{x}},x_n+z_n)]dx\\
&\quad - \int_{\mathbb{R}^n}\Gamma_t(\bar{\mathbf{x}}-\bar{\mathbf{y}},x_n,t)[\tilde{b}_j(\bar{\mathbf{x}},x_n-y_n) - \tilde{b}_j(\bar{\mathbf{x}},x_n-z_n)]dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{|T_1(t)\mathbf{b}(\bar{\mathbf{y}},y_n) - T_1(t)\mathbf{b}(\bar{\mathbf{y}},z_n)|}{|y_n-z_n|^{1+\alpha}}dy_ndz_n\\
&\le \int_0^\infty\int_0^\infty\int_{\mathbb{R}^n}\frac{\Gamma_t(\bar{\mathbf{x}}-\bar{\mathbf{y}},x_n,t)|\tilde{b}_j(\bar{\mathbf{x}},x_n+y_n) - \tilde{b}_j(\bar{\mathbf{x}},x_n+z_n)|}{|y_n-z_n|^{1+\alpha}}dxdy_ndz_n\\
&\quad + \int_0^\infty\int_0^\infty\int_{\mathbb{R}^n}\frac{\Gamma_t(\bar{\mathbf{x}}-\bar{\mathbf{y}},x_n,t)|\tilde{b}_j(\bar{\mathbf{x}},x_n-y_n) - \tilde{b}_j(\bar{\mathbf{x}},x_n-z_n)|}{|y_n-z_n|^{1+\alpha}}dxdy_ndz_n\\
&\le \int_{\mathbb{R}^n}|\Gamma_t(\bar{\mathbf{x}}-\bar{\mathbf{y}},x_n,t)|\Big(\int_0^\infty\int_0^\infty \frac{|\tilde{b}_j(\bar{\mathbf{x}},x_n+y_n) - \tilde{b}_j(\bar{\mathbf{x}},x_n+z_n)|}{|y_n-z_n|^{1+\alpha}}dy_ndz_n\Big)dx
\end{aligned}$$

$$+\int_{\mathbb{R}^n}|\Gamma_t(\bar{\mathbf{x}}-\bar{\mathbf{y}},x_n,t)|\Big(\int_0^\infty\!\!\int_0^\infty\frac{|\tilde{b}_j(\bar{\mathbf{x}},x_n-y_n)-\tilde{b}_j(\bar{\mathbf{x}},x_n-z_n)|}{|y_n-z_n|^{1+\alpha}}dy_ndz_n\Big)dx$$

$$=2\int_{\mathbb{R}^n}|\Gamma_t(\bar{\mathbf{x}}-\bar{\mathbf{y}},x_n,t)|\Big(\int_{-\infty}^\infty\!\!\int_{-\infty}^\infty\frac{|\tilde{b}_j(\bar{\mathbf{x}},y_n)-\tilde{b}_j(\bar{\mathbf{x}},z_n)|}{|y_n-z_n|^{1+\alpha}}dy_ndz_n\Big)dx.$$

Since $\int_{\mathbb{R}^{n-1}}|\Gamma_t(\bar{\mathbf{x}}-\bar{\mathbf{y}},x_n,t)|d\bar{y}=Ct^{-1/2}e^{-\frac{x_n^2}{4t}}$, we have

$$\int_{\mathbb{R}^{n-1}}\!\int_0^\infty\!\!\int_0^\infty\frac{|T_1(t)\mathbf{b}(\bar{\mathbf{y}},y_n)-T_1(t)\mathbf{b}(\bar{\mathbf{y}},z_n)|}{|y_n-z_n|^{1+\alpha}}dy_ndz_nd\bar{y}$$

$$\le 2\Big(\int_0^\infty t^{-1/2}e^{-\frac{x_n^2}{4t}}dx_n\Big)\Big(\int_{\mathbb{R}^{n-1}}\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{|\tilde{b}_j(\bar{\mathbf{x}},y_n)-\tilde{b}_j(\bar{\mathbf{x}},z_n)|}{|y_n-z_n|^{1+\alpha}}dy_ndz_nd\bar{y}\Big)$$

$$\le\Big(\int_{\mathbb{R}^{n-1}}\!\int_{-\infty}^\infty\!\!\int_{-\infty}^\infty\frac{|\tilde{b}_j(\bar{\mathbf{x}},y_n)-\tilde{b}_j(\bar{\mathbf{x}},z_n)|}{|y_n-z_n|^{1+\alpha}}dy_ndz_nd\bar{y}\Big)$$

$$\le C\|\widetilde{\mathbf{b}}\|_{B^\alpha_{1,1}(\mathbb{R}^n)}\le C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}.$$

Step 2. At this step, we derive the estimate

$$\int_{\mathbb{R}^{n-1}}\!\int_0^\infty\!\!\int_0^\infty\frac{|T_2(t)\mathbf{b}(\bar{\mathbf{y}},y_n)-T_2(t)\mathbf{b}(\bar{\mathbf{y}},z_n)|}{|y_n-z_n|^{1+\alpha}}dz_ndy_nd\bar{y}\le C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}. \tag{28}$$

Note that $\int_{\mathbb{R}^{n-1}}G^2_{ij}(\bar{\mathbf{x}},x_n,\mathbf{y},t)d\bar{x}=0$ for each fixed $x_n,\mathbf{y},t$. Hence

$$T_2(t)\mathbf{b}(\mathbf{y})=\int_{\mathbb{R}^n_+}G^2_{ij}(\mathbf{x},\mathbf{y},t)[b_i(\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{y}},x_n)]dx,$$

$$\begin{aligned}&T_2(t)\mathbf{b}(\bar{\mathbf{y}},y_n)-T_2(t)\mathbf{b}(\bar{\mathbf{y}},z_n)\\
=&\int_{\mathbb{R}^n_+}[G^2_{ij}(\mathbf{x},\bar{\mathbf{y}},y_n,t)-G^2_{ij}(\mathbf{x},\bar{\mathbf{y}},z_n,t)][b_i(\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{y}},x_n)]dx\\
=&\int_{|z_n-y_n|\le\frac12|x_n-z_n|}[G^2_{ij}(\mathbf{x},\bar{\mathbf{y}},y_n,t)-G^2_{ij}(\mathbf{x},\bar{\mathbf{y}},z_n,t)][b_i(\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{y}},x_n)]dx\\
&+\int_{|z_n-y_n|\ge\frac12|x_n-z_n|}[G^2_{ij}(\mathbf{x},\bar{\mathbf{y}},y_n,t)-G^2_{ij}(\mathbf{x},\bar{\mathbf{y}},z_n,t)][b_i(\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{y}},x_n)]dx\\
=&\,I+II.\end{aligned}$$

To obtain (28), we show that

$$\int_{\mathbb{R}^{n-1}}\!\int_0^\infty\!\!\int_0^\infty\frac{I}{|y_n-z_n|^{1+\alpha}}dy_ndz_nd\bar{y}\le C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}$$

and

$$\int_{\mathbb{R}^{n-1}}\!\int_0^\infty\!\!\int_0^\infty\frac{II}{|y_n-z_n|^{1+\alpha}}dy_ndz_nd\bar{y}\le C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}.$$

1) First, we estimate $\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_0^\infty\frac{I}{|y_n-z_n|^{1+\alpha}}dy_ndz_nd\bar{y}$.

By the mean value property,

$$G^2_{ij}(\mathbf{x},\bar{\mathbf{y}},y_n,t)-G^2_{ij}(\mathbf{x},\bar{\mathbf{y}},z_n,t)$$

$$= (y_n - z_n)\int_0^1 (D_{y_n}G^2_{ij})(\mathbf{x}, \bar{\mathbf{y}}, z_n + \theta(y_n - z_n), t)d\theta.$$

Since $x_n + z_n + \theta(y_n - z_n) \geq x_n + \min\{y_n, z_n\}$, we have

$$\begin{aligned}
&|(D_{y_n}G^2_{ij})(\mathbf{x}, \bar{\mathbf{y}}, z_n + \theta(y_n - z_n), t)| \\
&\leq c\frac{t^{-1/2}e^{-\frac{c(z_n+\theta(y_n-z_n))^2}{t}}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + z_n + \theta(y_n - z_n))^2 + t)^{\frac{n}{2}}} \\
&\leq c\frac{t^{-1/2}e^{\frac{c\min\{y_n,z_n\}^2}{t}}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + \min\{y_n, z_n\})^2 + t)^{\frac{n}{2}}} \\
&\leq c\frac{t^{-1/2}e^{-\frac{cy_n^2}{t}}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + y_n)^2 + t)^{-\frac{n}{2}}} + c\frac{t^{-1/2}e^{-\frac{cz_n^2}{t}}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + z_n)^2 + t)^{-\frac{n}{2}}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&|G^2_{ij}(\mathbf{x}, \bar{\mathbf{y}}, y_n, t) - G^2_{ij}(\mathbf{x}, \bar{\mathbf{y}}, z_n, t)| \\
&\leq C|y_n - z_n|\frac{t^{-1/2}e^{-\frac{cy_n^2}{t}}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + y_n)^2 + t)^{-\frac{n}{2}}} \\
&\quad + C|y_n - z_n|\frac{t^{-1/2}e^{-\frac{cz_n^2}{t}}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + z_n)^2 + t)^{-\frac{n}{2}}}.
\end{aligned}$$

Note that if $|z_n - y_n| \leq \frac{1}{2}|x_n - z_n|$, then $|z_n - y_n| \leq |x_n - y_n|$. Hence

$$\begin{aligned}
I \leq &\int_{|z_n-y_n|\leq\frac{1}{2}|x_n-z_n|} c\frac{t^{-1/2}|y_n - z_n|e^{-\frac{cy_n^2}{t}}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + y_n)^2 + t)^{\frac{n}{2}}}|b_i(\bar{\mathbf{x}}, x_n) - b_i(\bar{\mathbf{y}}, x_n)|dx \\
&+ \int_{|z_n-y_n|\leq|x_n-y_n|} c\frac{t^{-1/2}|y_n - z_n|e^{-\frac{cz_n^2}{t}}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + z_n)^2 + t)^{\frac{n}{2}}}|b_i(\bar{\mathbf{x}}, x_n) - b_i(\bar{\mathbf{y}}, x_n)|dx.
\end{aligned}$$

If we integrate $\frac{I}{|y_n-z_n|^{1+\alpha}}$ over $(0,\infty)\times(0,\infty)$, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{I}{|y_n - z_n|^{1+\alpha}}dy_n dz_n \\
\leq &\int_{\mathbb{R}^n_+}\int_{|z_n-y_n|\leq\frac{1}{2}|x_n-z_n|} c\frac{t^{-1/2}e^{-\frac{cy_n^2}{t}}|b_i(\bar{\mathbf{x}}, x_n) - b_i(\bar{\mathbf{y}}, x_n)|}{|y_n - z_n|^\alpha(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + y_n)^2 + t)^{\frac{n}{2}}}dy_n dz_n dx \\
&+ \int_{\mathbb{R}^n_+}\int_{|z_n-y_n|\leq|x_n-y_n|} c\frac{t^{-1/2}e^{-\frac{cz_n^2}{t}}|b_i(\bar{\mathbf{x}}, x_n) - b_i(\bar{\mathbf{y}}, x_n)|}{|y_n - z_n|^\alpha(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + z_n)^2 + t)^{\frac{n}{2}}}dy_n dz_n dx.
\end{aligned}$$

Note that

$$\int_{|z_n-y_n|\leq\frac{1}{2}|x_n-z_n|} \frac{t^{-1/2}e^{-\frac{cy_n^2}{t}}}{|y_n - z_n|^\alpha(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + (x_n + y_n)^2 + t)^{\frac{n}{2}}}dy_n dz_n$$

$$\leq \int_0^\infty \Big(\int_{|z_n-y_n|\leq|x_n-y_n|} c\frac{1}{|y_n-z_n|^\alpha}dz_n\Big)c\frac{t^{-1/2}e^{-\frac{cy_n^2}{t}}}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+y_n)^2+t)^{\frac{n}{2}}}dy_n$$

$$\leq \int_0^\infty c\frac{t^{-1/2}|x_n-y_n|^{1-\alpha}e^{-\frac{cy_n^2}{t}}}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+y_n)^2+t)^{\frac{n}{2}}}dy_n \leq C\frac{1}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^{n-1+\alpha}}\int_0^\infty t^{-1/2}e^{-\frac{cy_n^2}{t}}$$

$$\leq C\frac{1}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^{n-1+\alpha}}.$$

Likewise, we have

$$\int_{|z_n-y_n|\leq|x_n-y_n|}\frac{t^{-1/2}e^{-\frac{cz_n^2}{t}}}{|y_n-z_n|^\alpha(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+z_n)^2+t)^{\frac{n}{2}}}dy_ndz_n$$

$$\leq C\frac{1}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^{n-1+\alpha}}.$$

Hence if we again integrate $\int_0^\infty\int_0^\infty \frac{I}{|y_n-z_n|^{1+\alpha}}dz_ndy_n$ over $\mathbb{R}^{n-1}$, then we have

$$\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_0^\infty \frac{I}{|y_n-z_n|^{1+\alpha}}dz_ndy_nd\bar{y}$$

$$\leq \int_0^\infty\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{|b_i(\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{y}},x_n)|}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^{n-1+\alpha}}d\bar{x}d\bar{y}dx_n \leq C\|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}. \tag{29}$$

2) Second, we estimate $\int_0^\infty\int_0^\infty\int_{\mathbb{R}^{n-1}}\frac{II}{|y_n-z_n|^{1+\alpha}}dy_ndz_nd\bar{y}$. Since

$$|G^2_{ij}(\mathbf{x},\bar{\mathbf{y}},y_n,t)| \leq c\frac{e^{-\frac{cy_n^2}{t}}}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+y_n)^2+t)^{\frac{n}{2}}}$$

and

$$|G^2_{ij}(\mathbf{x},\bar{\mathbf{y}},z_n,t)| \leq c\frac{e^{-\frac{cz_n^2}{t}}}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+z_n)^2+t)^{\frac{n}{2}}},$$

and if $2|z_n-y_n|\geq|x_n-z_n|$, then $|x_n-y_n|\leq 3|y_n-z_n|$, we have

$$II \leq \int_{|z_n-y_n|\geq\frac{1}{2}|x_n-z_n|}\frac{e^{-\frac{cz_n^2}{t}}|b_i(\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{y}},x_n)|}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+z_n)^2+t)^{\frac{n}{2}}}dx$$

$$+\int_{|z_n-y_n|\geq\frac{1}{3}|x_n-y_n|}\frac{e^{-\frac{cy_n^2}{t}}|b_i(\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{y}},x_n)|}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+y_n)^2+t)^{\frac{n}{2}}}dx.$$

If we integrate $\frac{II}{|y_n-z_n|^{1+\alpha}}$ over $(0,\infty)\times(0,\infty)$, we have

$$\int_0^\infty\int_0^\infty\frac{II}{|y_n-z_n|^{1+\alpha}}dy_ndz_n$$

$$\leq \int_{\mathbb{R}^n_+}[\int_{|z_n-y_n|\geq\frac{1}{2}|x_n-z_n|}\frac{e^{-\frac{cz_n^2}{t}}|b_i(\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{y}},x_n)|}{|y_n-z_n|^{1+\alpha}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+z_n)^2+t)^{\frac{n}{2}}}dy_ndz_n]dx$$

$$+\int_{\mathbb{R}^n_+}[\int_{|z_n-y_n|\geq\frac{1}{3}|x_n-y_n|}\frac{e^{-\frac{cy_n^2}{t}}|b_i(\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{y}},x_n)|}{|y_n-z_n|^{1+\alpha}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+y_n)^2+t)^{\frac{n}{2}}}dy_ndz_n]dx.$$

Note that

$$\begin{aligned}
&\int_{|z_n-y_n|\geq\frac{1}{3}|x_n-y_n|}\frac{e^{-\frac{cy_n^2}{t}}}{|y_n-z_n|^{1+\alpha}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+y_n)^2+t)^{\frac{n}{2}}}dy_ndz_n\\
\leq&\int_0^\infty[\int_{|z_n-y_n|\geq\frac{1}{3}|x_n-y_n|}\frac{1}{|y_n-z_n|^{1+\alpha}}dz_n]\frac{e^{-\frac{cy_n^2}{t}}}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+y_n)^2+t)^{\frac{n}{2}}}dy_n\\
\leq&\int_0^\infty\frac{1}{|x_n-y_n|^\alpha}\frac{e^{-\frac{cy_n^2}{t}}}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+y_n)^2+t)^{\frac{n}{2}}}dy_n\\
\leq& C\frac{1}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^{n-1+\alpha}}\int_0^\infty\frac{1}{|x_n-y_n|^\alpha}\frac{e^{-\frac{cy_n^2}{t}}}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+y_n)^2+t)^{\frac{1-\alpha}{2}}}dy_n\\
\leq& C\frac{1}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^{n-1+\alpha}}
\end{aligned}$$

since

$$\begin{aligned}
&\int_0^\infty\frac{1}{|x_n-y_n|^\alpha}\frac{e^{-\frac{cy_n^2}{t}}}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+y_n)^2+t)^{\frac{1-\alpha}{2}}}dy_n\\
\leq&\int_0^\infty\frac{1}{|x_n-y_n|^\alpha}\frac{e^{-\frac{cy_n^2}{t}}}{y_n^{1-\alpha}}dy_n\\
\leq&\int_0^{x_n}\frac{e^{-\frac{cy_n^2}{t}}}{(x_n-y_n)^\alpha y_n^{1-\alpha}}dy_n+\int_{x_n}^{2x_n}\frac{e^{-\frac{cy_n^2}{t}}}{(y_n-x_n)^\alpha y_n^{1-\alpha}}dy_n+\int_{x_n}^\infty\frac{e^{-\frac{cy_n^2}{t}}}{(y_n-x_n)^\alpha y_n^{1-\alpha}}dy_n\\
\leq&\int_0^{x_n}\frac{1}{(x_n-y_n)^\alpha y_n^{1-\alpha}}dy_n+\int_{x_n}^{2x_n}\frac{1}{(y_n-x_n)^\alpha y_n^{1-\alpha}}dy_n+\int_{x_n}^\infty\frac{e^{-\frac{cy_n^2}{t}}}{y_n}dy_n\\
\leq& C.
\end{aligned}$$

By the same reasoning,

$$\begin{aligned}
&\int_{|z_n-y_n|\geq\frac{1}{2}|x_n-z_n|}\frac{e^{-\frac{cz_n^2}{t}}}{|y_n-z_n|^{1+\alpha}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^2+(x_n+z_n)^2+t)^{\frac{n}{2}}}dy_ndz_n\\
\leq& C\frac{1}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^{n-1+\alpha}}.
\end{aligned}$$

Hence

$$\int_0^\infty\int_0^\infty\frac{II}{|y_n-z_n|^{1+\alpha}}dy_ndz_n\leq\int_{\mathbb{R}^n_+}\frac{|b_i(\bar{\mathbf{x}},x_n)-b_i(\bar{\mathbf{y}},x_n)|}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|^{n-1+\alpha}}dx.$$

If we again integrate $\int_0^\infty \int_0^\infty \frac{II}{|y_n - z_n|^{1+\alpha}} dz_n dy_n$ over $\mathbb{R}^{n-1}$, then we have

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \int_0^\infty \int_0^\infty \frac{II}{|y_n - z_n|^{1+\alpha}} dz_n dy_n d\bar{y} \\ (30) \qquad &\leq \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|b_i(\bar{\mathbf{x}}, x_n) - b_i(\bar{\mathbf{y}}, x_n)|}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^{n-1+\alpha}} d\bar{x} d\bar{y} dx_n \leq C \|\mathbf{b}\|_{B^\alpha_{1,1}(\mathbb{R}^n_+)}. \end{aligned}$$

□

4. Estimate of bilinear operator

It is well-known that

$$\|e^{-tA}\mathbb{P}\mathrm{div}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-\frac{1}{2}}\|\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)} \tag{31}$$

(The above estimates are used in [22] in constructing mild solution of Navier-Stokes equations in the half space with nondecaying initial data).

Theorem 4.1. *Let $0 < \alpha < 1$ and $1 < q \leq \infty$. Then for $0 < t \leq 1$, we have*

(32)

$$\|e^{-tA}\mathbb{P}div(\mathbf{u}\otimes\mathbf{v})\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)} \leq C\|\mathbf{u}\|^{1+\alpha}_{L^\infty}\|\mathbf{v}\|^{1-\alpha}_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)} + C\|\mathbf{v}\|^{1+\alpha}_{L^\infty}\|\mathbf{u}\|^{1-\alpha}_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}$$

and

$$\begin{aligned} &\|e^{-tA}\mathbb{P}div(\mathbf{u}\otimes\mathbf{v})\|_{B^{\beta}_{\infty,r}(\mathbb{R}^n_+)} \\ (33) \qquad &\leq Ct^{-\frac{\beta+\alpha}{2}}[\|\mathbf{u}\|^{1+\alpha}_{L^\infty}\|\mathbf{v}\|^{1-\alpha}_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)} + \|\mathbf{v}\|^{1+\alpha}_{L^\infty}\|\mathbf{u}\|^{1-\alpha}_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}] \end{aligned}$$

for $0 < \beta, 1 \leq r \leq \infty$.

The proof of Theorem 4.1 will be given at the end of this section making use of the subsequent lemmas.

The following is useful for the proof of Theorem 4.1.

Proposition 4.2. *Let $n \geq 1, 0 < \alpha$ and $s \in \mathbb{R}$. Then there exists a positive constant $C = C(n, \alpha, s)$ such that*

$$\|(-\Delta)^{\alpha/2}R_i f\|_{B^s_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{B^{s+\alpha}_{p,q}(\mathbb{R}^n)} \tag{34}$$

for any $1 \leq p, q \leq \infty$, $i = 1, \ldots, n$ and $f \in B^{s+\alpha}_{\infty,q}$. Here R_i's are Riesz operator.

(In [19], the proof of the above proposition is given for $p = \infty$ and $0 < \alpha < 1$. However, the proof can also be applied analogously for $1 \leq p \leq \infty$ and $\alpha > 0$.)

Lemma 4.3. *Let $0 < \alpha < 1$ and $1 < q \leq \infty$. Then*

$$\|\mathbb{P}div\mathcal{F}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)} \leq \|\mathcal{F}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}. \tag{35}$$

Proof. Let $(\mathbb{P}\mathrm{div})^*$ be the dual operator of $\mathbb{P}\mathrm{div}$. Since $C^\infty_0(\mathbb{R}^n_+)$ is dense in $B^\alpha_{1,q'}(\mathbb{R}^n_+)$ for $0 < \alpha < 1$ and $1 < q \leq \infty$, we have

$$\|\mathbb{P}\mathrm{div}\mathcal{F}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)} = \sup_{\phi\in C^\infty_0(\mathbb{R}^n_+)} \frac{\langle\mathcal{F}, (\mathbb{P}\mathrm{div})^*\phi\rangle}{\|\phi\|_{B^\alpha_{1,q'}(\mathbb{R}^n_+)}}.$$

Our aim is to show that

$$\|(\mathbb{P}\mathrm{div})^*\phi\|_{B^{-1+\alpha}_{1,q'}(\mathbb{R}^n_+)} \le C\|\phi\|_{B^{\alpha}_{1,q'}(\mathbb{R}^n_+)}.$$

$(\mathbb{P}\mathrm{div})^*\phi$ can be written as

$$(\mathbb{P}\mathrm{div})^*_{jk}\phi(\mathbf{y}) = -\frac{\partial}{\partial y_k}\phi_j(\mathbf{y}) + \frac{\partial}{\partial y_k}\int_{\mathbb{R}^n_+}\frac{\partial}{\partial y_j}G(\mathbf{x},\mathbf{y})\frac{\partial \phi_i}{\partial x_i}(\mathbf{x})dx \tag{36}$$

since

$$\begin{aligned}
&\int_{\mathbb{R}^n_+}[\mathbb{P}\mathrm{div}\mathcal{F}](\mathbf{x})\cdot\phi(\mathbf{x})dx\\
&= \int_{\mathbb{R}^n_+}[\frac{\partial F_{ik}}{\partial x_k}(\mathbf{x}) + \frac{\partial}{\partial x_i}\int_{\mathbb{R}^n_+}\frac{\partial}{\partial y_j}G(\mathbf{x},\mathbf{y})\frac{\partial F_{jk}}{\partial y_k}(\mathbf{y})dy]\phi_i(\mathbf{x})dx\\
&= \int_{\mathbb{R}^n_+}F_{jk}(\mathbf{y})[-\frac{\partial}{\partial y_k}\phi_j(\mathbf{y}) + \frac{\partial}{\partial y_k}\int_{\mathbb{R}^n_+}\frac{\partial}{\partial y_j}G(\mathbf{x},\mathbf{y})\frac{\partial \phi_j}{\partial x_i}(\mathbf{x})dx]dy
\end{aligned}$$

for $\mathcal{F} = (F_{jk})^n_{i,j=1}, \phi = (\phi)^n_{i=1} \in C^\infty_0(\mathbb{R}^n_+)$.

Let $\widetilde{\phi}$ be the zero extension of ϕ and ϕ^* be the reflection of ϕ. For $0 < \alpha < 1$ and $1 < q \le \infty$, we have the inequality

$$\|\tilde{\phi}\|_{B^{\alpha}_{1,q'}(\mathbb{R}^n)} \le C\|\phi\|_{B^{\alpha}_{1,q'}(\mathbb{R}^n_+)}.$$

Then $(\mathbb{P}\mathrm{div})^*_{jk}\phi$ can be rewritten as

$$\begin{aligned}
(\mathbb{P}\mathrm{div})^*_{jk}\phi(\mathbf{y}) &= -\frac{\partial}{\partial y_k}\tilde{\phi}_j(\mathbf{y}) + \frac{\partial}{\partial y_k}\int_{\mathbb{R}^n}\frac{\partial}{\partial y_j}G(\mathbf{x},\mathbf{y})\frac{\partial \tilde{\phi}_j}{\partial x_i}(\mathbf{x})dx\\
&= -\frac{\partial}{\partial y_k}\tilde{\phi}_j(\mathbf{y}) + \frac{\partial}{\partial y_k}\int_{\mathbb{R}^n}\frac{\partial}{\partial y_j}N(\mathbf{x}-\mathbf{y})\frac{\partial \tilde{\phi}_j}{\partial x_i}(\mathbf{x})dx\\
&\quad + \frac{\partial}{\partial y_k}\int_{\mathbb{R}^n}\frac{\partial}{\partial y_j}N(\mathbf{x}-\mathbf{y})\frac{\partial \tilde{\phi}^*_j}{\partial x_i}(\mathbf{x})dx.
\end{aligned}\tag{37}$$

Note that $(\mathbb{P}\mathrm{div})^*\phi$ is a linear sum of $\nabla R_jR_k\widetilde{\phi}$ and $\nabla R_jR_k\widetilde{\phi^*}$. By Proposition 4.2, the right-hand side of the above identity is bounded from $B^{-\alpha}_{1,q'}(\mathbb{R}^n)$ to $B^{1-\alpha}_{1,q'}(\mathbb{R}^n)$ so that

$$\|(\mathbb{P}\mathrm{div})^*\phi\|_{B^{-1+\alpha}_{1,q'}(\mathbb{R}^n_+)} \le C\|\tilde{\phi}\|_{B^{\alpha}_{1,q'}(\mathbb{R}^n)} \le C\|\phi\|_{B^{\alpha}_{1,q'}(\mathbb{R}^n_+)}.$$ □

Lemma 4.4. *Let $0 < \alpha < 1$ and $1 < q \le \infty$. Then for $0 < t \le 1$, we have*

$$\|e^{-tA}\mathbb{P}div\mathcal{F}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)} \le C\|\mathcal{F}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)} \tag{38}$$

and

$$\|e^{-tA}\mathbb{P}div\mathcal{F}\|_{B^{\beta}_{\infty,r}(\mathbb{R}^n_+)} \le Ct^{-\frac{\beta+\alpha}{2}}\|\mathcal{F}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)},\ 0<\beta,\ 1\le r\le\infty. \tag{39}$$

Proof. By (13) and Lemma 4.3,

$$\begin{aligned}\|e^{-tA}\mathbb{P}\mathrm{div}\mathcal{F}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)} &\leq \|\mathbb{P}\mathrm{div}\mathcal{F}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)}\\ &\leq \|\mathcal{F}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}. \qquad (40)\end{aligned}$$

By (15) and (38),

$$\begin{aligned}\|e^{-tA}\mathbb{P}\mathrm{div}\mathcal{F}\|_{B^{\beta}_{\infty,r}(\mathbb{R}^n_+)} &\leq Ct^{-\frac{\beta+\alpha}{2}}\|e^{-\frac{t}{2}A}\mathbb{P}\mathrm{div}\mathcal{F}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)}\\ &\leq Ct^{-\frac{\beta+\alpha}{2}}\|\mathcal{F}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}. \qquad (41)\end{aligned}$$

□

Proposition 4.5. *Let* $0 < \alpha < 1$ *and* $1 \leq q \leq \infty$. *Then*

$$\|fg\|_{B^{\alpha}_{\infty,q}(\mathbb{R}^n_+)} \leq C(\|f\|_{L^\infty(\mathbb{R}^n_+)}\|g\|_{B^{\alpha}_{\infty,q}(\mathbb{R}^n_+)} + \|g\|_{L^\infty(\mathbb{R}^n_+)}\|f\|_{B^{\alpha}_{\infty,q}(\mathbb{R}^n_+)}). \qquad (42)$$

Proof. Note that

$$\begin{aligned}|f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{y})g(\mathbf{y})| &\leq C|f(\mathbf{x}) - f(\mathbf{y})||g(\mathbf{x})| + |f(\mathbf{y})||g(\mathbf{x}) - g(\mathbf{y})|\\ &\leq |f(\mathbf{x}) - f(\mathbf{y})|\|g\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)}|g(\mathbf{x}) - g(\mathbf{y})|.\end{aligned}$$

Hence for $1 \leq q < \infty$, we have the inequality

$$\begin{aligned}&(\int_{\mathbb{R}^n_+}(\sup_{x\in\mathbb{R}^n_+}\frac{|f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{y})g(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\frac{n}{q}+\alpha}})^q dy)^{1/q}\\ &\leq C(\int_{\mathbb{R}^n_+}(\sup_{x\in\mathbb{R}^n_+}\frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\frac{n}{q}+\alpha}})^q dy)^{1/q}\|g\|_{L^\infty(\mathbb{R}^n_+)}\\ &\quad + C\|f\|_{L^\infty(\mathbb{R}^n_+)}(\int_{\mathbb{R}^n_+}(\sup_{x\in\mathbb{R}^n_+}\frac{|g(\mathbf{x}) - g(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\frac{n}{q}+\alpha}})^q dy)^{1/q}\\ &\leq C(\|f\|_{L^\infty(\mathbb{R}^n_+)}\|g\|_{B^{\alpha,\infty}_q(\mathbb{R}^n_+)} + \|g\|_{L^\infty(\Omega)}\|f\|_{B^{\alpha,\infty}_q(\mathbb{R}^n_+)}),\end{aligned}$$

and for $q = \infty$, we have

$$\begin{aligned}&\sup_{\mathbf{x}\neq\mathbf{y}\in\mathbb{R}^n_+}\frac{|f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{y})g(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}\\ &\leq C\sup_{\mathbf{x}\neq\mathbf{y}\in\mathbb{R}^n_+}\frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}dx\|g\|_{L^\infty(\mathbb{R}^n_+)} + C\|f\|_{L^\infty(\mathbb{R}^n_+)}\sup_{\mathbf{x}\neq\mathbf{y}\in\Omega}\frac{|g(\mathbf{x}) - g(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}\\ &\leq C(\|f\|_{L^\infty(\mathbb{R}^n_+)}\|g\|_{B^{\alpha,\infty}_\infty(\mathbb{R}^n_+)} + \|g\|_{L^\infty(\mathbb{R}^n_+)}\|f\|_{B^{\alpha,\infty}_\infty(\mathbb{R}^n_+)}).\end{aligned}$$

This completes the proof of the proposition. □

Now we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1. By (4.5),

$$\begin{aligned}&\|\mathbf{u}\otimes\mathbf{v}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}\\ &\leq C(\|\mathbf{u}\|_{L^\infty(\mathbb{R}^n_+)}\|\mathbf{v}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)} + \|\mathbf{v}\|_{L^\infty(\mathbb{R}^n_+)}\|\mathbf{u}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}).\end{aligned}$$

It is well-known that $B^{1-\alpha}_{\infty,1}(\mathbb{R}^n_+) \hookrightarrow B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+) \hookrightarrow B^{1-\alpha}_{\infty,\infty}(\mathbb{R}^n_+)$. Hence $\|\mathbf{u}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)} \le C\|\mathbf{u}\|_{B^{1-\alpha}_{\infty,1}(\mathbb{R}^n_+)}$ and $\|\mathbf{v}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)} \le C\|\mathbf{v}\|_{B^{1-\alpha}_{\infty,1}(\mathbb{R}^n_+)}$. By interpolation theorem, $\|\mathbf{u}\|_{B^{1-\alpha}_{\infty,1}(\mathbb{R}^n_+)} \le C\|\mathbf{u}\|^{\alpha}_{L^\infty(\mathbb{R}^n_+)}\|\mathbf{u}\|^{1-\alpha}_{B^1_{\infty,1}(\mathbb{R}^n_+)}$ and $\|\mathbf{v}\|_{B^{1-\alpha}_{\infty,1}} \le C\|\mathbf{v}\|^{\alpha}_{L^\infty(\mathbb{R}^n_+)}\|\mathbf{v}\|^{1-\alpha}_{B^1_{\infty,1}(\mathbb{R}^n_+)}$.

Combining all the above estimates, we have that

$$\|\mathbf{u}\otimes\mathbf{v}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)} \le C\|\mathbf{u}\|^{1+\alpha}_{L^\infty}\|\mathbf{v}\|^{1-\alpha}_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)} + C\|\mathbf{v}\|^{1+\alpha}_{L^\infty}\|\mathbf{u}\|^{1-\alpha}_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}.$$

Applying the above estimate to the right-hand side of (38) and (39), we conclude that

$$\begin{aligned}&\|e^{-tA}\mathbb{P}\mathrm{div}(\mathbf{u}\otimes\mathbf{v})\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)} \le C\|\mathbf{u}\otimes\mathbf{v}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}\\ &\le C\|\mathbf{u}\|^{1+\alpha}_{L^\infty}\|\mathbf{v}\|^{1-\alpha}_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)} + C\|\mathbf{v}\|^{1+\alpha}_{L^\infty}\|\mathbf{u}\|^{1-\alpha}_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}\end{aligned}$$

and

$$\begin{aligned}&\|e^{-tA}\mathbb{P}\mathrm{div}(\mathbf{u}\otimes\mathbf{v})\|_{B^{\beta}_{\infty,r}(\mathbb{R}^n_+)}\\ &\le Ct^{-\frac{\alpha+\beta}{2}}\|\mathbf{u}\otimes\mathbf{v}\|_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}\\ &\le Ct^{-\frac{\alpha+\beta}{2}}\|\mathbf{u}\|^{1+\alpha}_{L^\infty}\|\mathbf{v}\|^{1-\alpha}_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)} + Ct^{-\frac{\alpha+\beta}{2}}\|\mathbf{v}\|^{1+\alpha}_{L^\infty}\|\mathbf{u}\|^{1-\alpha}_{B^{1-\alpha}_{\infty,q}(\mathbb{R}^n_+)}.\end{aligned}$$ □

5. Proof of the main theorem

5.1. Existence of mild solution

Construct a successive approximation by

$$(43)\quad \begin{cases}\mathbf{u}_0(t) = e^{-tA}\mathbf{a},\\ \mathbf{u}_{m+1}(t) = \mathbf{u}_0(t) - \int_0^t e^{(t-\tau)A}\mathbb{P}\mathrm{div}(\mathbf{u}_m\otimes\mathbf{u}_m)(\tau)d\tau,\ m=0,1,\ldots.\end{cases}$$

Set $\|\mathbf{a}\|_{B^{-\epsilon}_{\infty,q}(\mathbb{R}^n_+)} = N_0$. Assume that for $k=1,\ldots,m$,

$$\sup_{0<t<T_0} t^{\frac{\epsilon}{2}}\|\mathbf{u}_k(t)\|_{L^\infty(\mathbb{R}^n_+)} \le M_0,$$

$$\sup_{0<t<T_0} t^{\frac{1+\epsilon}{2}}\|\mathbf{u}_k(t)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)} \le M_1.$$

Step 1. First, we will show the uniform boundedness of the sequences $\{t^{\frac{\epsilon}{2}}\mathbf{u}_m : m=1,\ldots\}$ and $\{t^{\frac{1+\epsilon}{2}}\mathbf{u}_m : m=1,\ldots\}$ in $L^\infty((0,T_0)\times\mathbb{R}^n_+)$ and $L^\infty(0,T_0;B^1_{\infty,\infty}(\mathbb{R}^n_+))$, respectively.

i) By (12) and (31), we have

$$\begin{aligned}&\|\mathbf{u}_{m+1}(t)\|_{L^\infty(\mathbb{R}^n_+)}\\ &\le \|e^{-tA}\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)} + \int_0^t \|e^{-(t-\tau)A}\mathbb{P}\mathrm{div}(\mathbf{u}_m\otimes\mathbf{u}_m)(\tau)\|_{L^\infty}d\tau\\ &\le Ct^{-\frac{\epsilon}{2}}\|\mathbf{a}\|_{B^{-\alpha}_{\infty,q}(\mathbb{R}^n_+)} + C\int_0^t (t-\tau)^{-\frac12}\|\mathbf{u}_m\otimes\mathbf{u}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)}d\tau\end{aligned}$$

$$\leq Ct^{-\frac{\epsilon}{2}}N_0 + C\int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\epsilon}d\tau M_0^2 \tag{44}$$

for $0 < t < T_0$. Here the term of integration is finite with

$$\int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\epsilon}d\tau = Ct^{\frac{1}{2}-\epsilon}$$

since $\epsilon < 1$. Hence (44) implies the inequality

$$\|\mathbf{u}_{m+1}(t)\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-\frac{\epsilon}{2}}N_0 + Ct^{\frac{1}{2}-\epsilon}M_0^2 \quad \text{for } 0 < t < T_0. \tag{45}$$

This again implies that

$$\sup_{0<t<T_0} t^{\frac{\epsilon}{2}}\|\mathbf{u}_{m+1}(t)\|_{L^\infty(\mathbb{R}^n_+)} \leq CN_0 + CT_0^{\frac{1}{2}-\frac{\epsilon}{2}}M_0^2. \tag{46}$$

ii) By (15) and (33), we have

(47)

$$\begin{aligned}
&\|\mathbf{u}_{m+1}\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}\\
&\leq \|e^{-tA}\mathbf{a}\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)} + \int_0^t \|e^{-(t-s)A}\mathbb{P}\mathrm{div}(\mathbf{u}_m\otimes\mathbf{u}_m)(s)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}ds\\
&\leq Ct^{-\frac{\epsilon+1}{2}}\|\mathbf{a}\|_{B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)} + C\int_0^t (t-s)^{-\frac{1+\epsilon}{2}}\|\mathbf{u}_m(s)\|^{1+\epsilon}_{L^\infty(\mathbb{R}^n_+)}\|\mathbf{u}_m\|^{1-\epsilon}_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}ds\\
&\leq Ct^{-\frac{\epsilon+1}{2}}N_0 + CM_0^{1+\epsilon}M_1^{1-\epsilon}\int_0^t (t-s)^{-\frac{1+\epsilon}{2}}s^{-\frac{1+\epsilon}{2}}ds.
\end{aligned}$$

Here the term of integration is finite with $\int_0^t (t-s)^{-\frac{1+\epsilon}{2}}s^{-\frac{1+\epsilon}{2}}ds = Ct^{-\epsilon}$ since $\epsilon < 1$. Hence (47) implies the inequality

$$\|\mathbf{u}_{m+1}\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)} \leq Ct^{-\frac{\epsilon+1}{2}}N_0 + CM_0^{1+\alpha}M_1^{1-\epsilon}t^{-\epsilon},\ 0 < t < T_0.$$

This again implies the inequality

$$\sup_{0<t<T_0} t^{\frac{\epsilon+1}{2}}\|\mathbf{u}_{m+1}\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)} \leq CN_0 + CM_0^{1+\epsilon}M_1^{1-\epsilon}T_0^{\frac{1-\epsilon}{2}}. \tag{48}$$

iii) Take M_0 and M_1 sufficiently large so that

$$cN_0 \leq \frac{M_0}{2},\ CN_0 \leq \frac{M_1}{2}, \tag{49}$$

and take T_0 sufficiently small so that

$$CT_0^{\frac{1}{2}-\frac{\epsilon}{2}}M_0^2 \leq \frac{M_0}{2},\ CM_0^{1+\epsilon}M_1^{1-\alpha}T_0^{\frac{1-\epsilon}{2}} \leq \frac{M_1}{2}. \tag{50}$$

Then (47) and(48) imply that

$$\sup_{0<t<T_0} t^{\frac{\epsilon}{2}}\|\mathbf{u}_{m+1}(t)\|_{L^\infty(\mathbb{R}^n_+)} \leq M_0 \tag{51}$$

and

$$\sup_{0<t<T_0} t^{\frac{\epsilon+1}{2}} \|\mathbf{u}_{m+1}\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)} \le M_1. \tag{52}$$

Step 2. Second, we show the uniform convergence of $\{t^{\frac{\epsilon}{2}}\mathbf{u}_m : m = 1, 2, \ldots\}$ and $\{t^{\frac{1+\epsilon}{2}}\mathbf{u}_m : m = 1, 2, \ldots\}$ in $L^\infty(0, T_0; L^\infty(\mathbb{R}^n_+))$ and $L^\infty(0, T_0; B^1_{\infty,\infty}(\mathbb{R}^n_+))$, respectively.

Set $\mathbf{w}_m = \mathbf{u}_{m+1} - \mathbf{u}_m$. Note that

$$\mathbf{w}_m = \int_0^t e^{-tA}\mathbb{P}\mathrm{div}(\mathbf{w}_{m-1} \otimes u_m + \mathbf{u}_{m-1} \otimes \mathbf{w}_{m-1})(\tau)d\tau.$$

Since $\mathbf{w}_0 = \mathbf{w}_1 - \mathbf{u}_0 = \int_0^t e^{-(t-s)A}\mathbb{P}\mathrm{div}(\mathbf{u}_0 \otimes \mathbf{u}_0)(s)ds$,

$$\begin{aligned} \|\mathbf{w}_0\|_{L^\infty(\mathbb{R}^n_+)} &\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|\mathbf{u}_0(\tau)\|^2_{L^\infty(\mathbb{R}^n_+)}d\tau \\ &\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\epsilon}d\tau N_0^2 = Ct^{\frac{1}{2}-\epsilon}N_0^2. \end{aligned} \tag{53}$$

i) Recall (51). Then $\sup_{0<\tau<t}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}(\tau)\|_{L^\infty(\mathbb{R}^n_+)} < +\infty$ for $t < T_0$ and by (31), we have that

(54)

$$\begin{aligned} &\|\mathbf{w}_m(t)\|_{L^\infty(\mathbb{R}^n_+)} \\ \le\ & \int_0^t \|e^{-(t-\tau)A}\mathbb{P}\mathrm{div}(\mathbf{w}_{m-1} \otimes \mathbf{u}_m + \mathbf{u}_{m-1} \otimes \mathbf{w}_{m-1})(\tau)\|_{L^\infty(\mathbb{R}^n_+)}d\tau \\ \le\ & C\int_0^t (t-\tau)^{-\frac{1}{2}}(\|\mathbf{u}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)} + \|\mathbf{u}_{m-1}(\tau)\|_{L^\infty(\mathbb{R}^n_+)})\|\mathbf{w}_{m-1}(\tau)\|_{L^\infty(\mathbb{R}^n_+)}d\tau \\ \le\ & CM_0\int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}(\tau)\|_{L^\infty(\mathbb{R}^n_+)}d\tau \\ \le\ & CM_0\int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\epsilon}d\tau \sup_{0<\tau<t}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}(\tau)\|_{L^\infty(\mathbb{R}^n_+)} \\ =\ & CM_0 t^{\frac{1}{2}-\epsilon}\sup_{0<\tau<t}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}(\tau)\|_{L^\infty(\mathbb{R}^n_+)},\ 0 < t < T_0. \end{aligned}$$

Hence we have the recursive inequality

$$\sup_{0<\tau<t}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)} \le CM_0 t^{\frac{1}{2}-\frac{\epsilon}{2}}\sup_{0<\tau<t}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}(\tau)\|_{L^\infty(\mathbb{R}^n_+)} \tag{55}$$

for $0 < t < T_0$.

ii) Recall (52). Then $\sup_{0<\tau<t}\tau^{\frac{\epsilon+1}{2}}\|\mathbf{u}_{m+1}(\tau)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)} < +\infty$ for all $0 < t < T_0$ and by (33), we have that

$$\begin{aligned} &\|\mathbf{w}_m(t)\|_{B^1_{\infty,\infty}} \\ \le\ & \int_0^t \|e^{-(t-s)A}\mathbb{P}\mathrm{div}(\mathbf{u}_m \otimes \mathbf{w}_{m-1} + \mathbf{w}_{m-1} \otimes \mathbf{u}_{m-1})(s)\|_{B^{1-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)}ds \end{aligned}$$

$$\le C\int_0^t (t-s)^{-\frac{1+\epsilon}{2}}[\|\mathbf{u}_m\|^{1+\epsilon}_{L^\infty(\mathbb{R}^n_+)}\|\mathbf{w}_{m-1}(s)\|^{1-\epsilon}_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}$$
$$+\|\mathbf{w}_{m-1}\|^{1+\epsilon}_{L^\infty(\mathbb{R}^n_+)}\|\mathbf{u}_m(s)\|^{1-\epsilon}_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}]ds$$
$$\le CM_0^{1+\epsilon}\int_0^t (t-s)^{-\frac{1+\epsilon}{2}}s^{-\frac{1+\epsilon}{2}}ds[\sup_{0<s<t} s^{\frac{1+\epsilon}{2}}\|\mathbf{w}_{m-1}(s)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}]^{1-\epsilon}$$
$$+CM_1^{1-\epsilon}\int_0^t (t-s)^{-\frac{1+\epsilon}{2}}s^{-\frac{1+\epsilon}{2}}ds[\sup_{0<s<t} s^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}\|_{L^\infty(\mathbb{R}^n_+)}]^{1+\epsilon}$$
$$\le CM_0^{1+\epsilon}t^{-\epsilon}[\sup_{0<s<t} s^{\frac{1+\epsilon}{2}}\|\mathbf{w}_{m-1}(s)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}]^{1-\epsilon}$$
$$+CM_1^{1-\epsilon}t^{-\epsilon}[\sup_{0<s<t} s^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}\|_{L^\infty(\mathbb{R}^n_+)}]^{1+\epsilon},\ 0<t<T_0.$$

Hence we have the recursive inequality

$$\sup_{0<\tau<t}\tau^{\frac{1+\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)} \le CM_0^{1+\epsilon}t^{\frac{1-\epsilon}{2}}[\sup_{0<s<t} s^{\frac{1+\epsilon}{2}}\|\mathbf{w}_{m-1}(s)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}]^{1-\epsilon}$$
$$+CM_1^{1-\epsilon}t^{\frac{1-\epsilon}{2}}[\sup_{0<s<t} s^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}\|_{L^\infty(\mathbb{R}^n_+)}]^{1+\epsilon} \tag{56}$$

for $0<t<T_0$.

iii) Take $T_1\le T_0$ sufficiently small so that

$$\max\{2CM_0T_1^{\frac{1}{2}-\frac{\epsilon}{2}}, CM_0^{1+\epsilon}T_1^{\frac{1-\epsilon}{2}}, 2CM_1^{1-\epsilon}T_1^{\frac{1-\epsilon}{2}}\}\le\frac{1}{2}.$$

Then (55) and (56) imply that

$$\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)}\le\frac{1}{4}\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}(\tau)\|_{L^\infty(\mathbb{R}^n_+)} \tag{57}$$

and

$$\sup_{0<\tau<T_1}\tau^{\frac{1+\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}\le\frac{1}{2}[\sup_{0<s<T_1} s^{\frac{1+\epsilon}{2}}\|\mathbf{w}_{m-1}(s)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}]^{1-\epsilon}$$
$$+\frac{1}{4}[\sup_{0<s<T_1} s^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}\|_{L^\infty(\mathbb{R}^n_+)}]^{1+\epsilon}. \tag{58}$$

iv) Considering the inequality (57) recursively, we have that

(59)
$$\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)}\le(\frac{1}{4})^m\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_0\|_{L^\infty(\mathbb{R}^n_+)}\le CT_1^{\frac{1}{2}-\frac{\epsilon}{2}}N_0^2(\frac{1}{4})^m.$$

This implies that the series $\sum_{m=1}^\infty t^{\frac{\epsilon}{2}}\mathbf{w}_m$ converges to some $\mathbf{v}$ in

$$L^\infty(0,T_1;L^\infty(\mathbb{R}^n_+)).$$

This again implies that $\mathbf{t}^{\frac{\epsilon}{2}}u_m = t^{\frac{\epsilon}{2}}\mathbf{u}_0+\sum_{k=1}^m t^{\frac{\epsilon}{2}}\mathbf{w}_m$ converges to $t^{\frac{\epsilon}{2}}\mathbf{u}_0+\mathbf{v}$ in $L^\infty(0,T_1;L^\infty(\mathbb{R}^n_+))$.

v) Since the series $\sum_{m=1}^{\infty}\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)}$ converges, there is a positive integer N_0 such that for all $m\geq N_0$, $\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)}<1$. This implies that for $m\geq N_0$,

$$\begin{aligned}[\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)}]^{1+\epsilon} &\leq \sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)}\\ &\leq [\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)}]^{1-\epsilon}.\end{aligned}$$

Hence, by adding (57) and (58), we have that for $m\geq N_0$,

(60)

$$\begin{aligned}&\sup_{0<\tau<T_1}\tau^{\frac{1+\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}+\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)}\\ &\leq \frac{1}{2}[\sup_{0<s<T_1}s^{\frac{1+\epsilon}{2}}\|\mathbf{w}_{m-1}(s)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}]^{1-\epsilon}\\ &\quad+\frac{1}{4}[\sup_{0<s<T_1}s^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}\|_{L^\infty(\mathbb{R}^n_+)}]^{1+\epsilon}+\frac{1}{4}\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}(\tau)\|_{L^\infty(\mathbb{R}^n_+)}\\ &\leq \frac{1}{2}[\sup_{0<s<T_1}s^{\frac{1+\epsilon}{2}}\|\mathbf{w}_{m-1}(s)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}]^{1-\epsilon}+\frac{1}{2}[\sup_{0<s<T_1}s^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}\|_{L^\infty(\mathbb{R}^n_+)}]^{1-\epsilon}\\ &\leq [\frac{1}{2}(\sup_{0<s<T_1}s^{\frac{1+\epsilon}{2}}\|\mathbf{w}_{m-1}(s)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}+\sup_{0<s<T_1}s^{\frac{\epsilon}{2}}\|\mathbf{w}_{m-1}\|_{L^\infty(\mathbb{R}^n_+)})]^{1-\epsilon}.\end{aligned}$$

Here we used the fact that $\frac{1}{2}(x^{1-\epsilon}+y^{1-\epsilon})\leq(\frac{x+y}{2})^{1-\epsilon}$ for $x,y>0$.

vi) Considering (60) recursively, for $m\geq N_0$,

$$\begin{aligned}&\sup_{0<\tau<T_1}\tau^{\frac{1+\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}+\sup_{0<\tau<T_1}\tau^{\frac{\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{L^\infty(\mathbb{R}^n_+)}\\ &\leq\Big(\frac{1}{2}[\sup_{0<s<T_1}s^{\frac{1+\epsilon}{2}}\|\mathbf{w}_{N_0}(s)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}+\sup_{0<s<T_1}s^{\frac{\epsilon}{2}}\|\mathbf{w}_{N_0}\|_{L^\infty(\mathbb{R}^n_+)}]\Big)^{(1-\epsilon)^{m-N_0}}.\end{aligned}$$

This implies

$$\sum_{m\geq N_0}\sup_{0<\tau<T_1}\tau^{\frac{1+\epsilon}{2}}\|\mathbf{w}_m(\tau)\|_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}<\infty.$$

Therefore, $t^{\frac{1+\epsilon}{2}}\mathbf{u}_m=t^{\frac{1+\epsilon}{2}}\mathbf{u}_0+\sum_{k=1}^{N_0-1}t^{\frac{1+\epsilon}{2}}\mathbf{w}_k+\sum_{k=N_0}^{m}t^{\frac{1+\epsilon}{2}}\mathbf{w}_k$ converges to some $t^{\frac{1+\epsilon}{2}}\mathbf{u}_0+\tilde{\mathbf{v}}$ in $L^\infty(0,T_1;B^1_{\infty,\infty}(\mathbb{R}^n_+))$.

It is trivial that $t^{\frac{1}{2}}\mathbf{v}=\tilde{\mathbf{v}}$. Set $\mathbf{u}=\mathbf{u}_0+t^{-\frac{\epsilon}{2}}\mathbf{v}$. Then $\mathbf{u}$ is the limit of $\mathbf{u}_m$ for each $t>0$. It is easy to show that $\mathbf{u}$ satisfies the integral equation (3).

5.2. The estimates in Besov space $B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)$

Let $\mathbf{u}$ be the mild solution of (1) in time interval $(0,T_0)$ satisfying that

$$t^{\frac{\epsilon}{2}}\mathbf{u}(t)\in L^\infty((0,T_0)\times\mathbb{R}^n_+),\ t^{\frac{1+\epsilon}{2}}\mathbf{u}(t)\in B^1_{\infty,\infty}(\mathbb{R}^n_+),\ 0<t<T_0.$$

Let

$$\sup_{0<t<T_0} t^{\frac{\epsilon}{2}}\|\mathbf{u}(t)\|_{L^\infty(\mathbb{R}^n_+)} = M_0, \ \sup_{0<t<T_0} t^{\frac{1+\epsilon}{2}}\|\mathbf{u}(t)\|_{L^\infty(\mathbb{R}^n_+)} = M_1.$$

By (13) and (38), we have

$$\|\mathbf{u}\|_{B^{-\epsilon}_{\infty,\infty}} \le C\|\mathbf{a}\|_{B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)} + C\int_0^t \|(\mathbf{u}\otimes\mathbf{u})(s)\|_{B^{1-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)}ds.$$

By interpolation theory, we have

$$\|(\mathbf{u}\otimes\mathbf{u})(s)\|_{B^{1-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)} \le C\|\mathbf{u}(s)\|^{1+\epsilon}_{L^\infty(\mathbb{R}^n_+)}\|\mathbf{u}\|^{1-\epsilon}_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}, \tag{61}$$

and hence

$$\begin{aligned}\|\mathbf{u}\|_{B^{-\epsilon}_{\infty,\infty}} &\le C\|\mathbf{a}\|_{B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)} + C\int_0^t \|\mathbf{u}(s)\|^{1+\epsilon}_{L^\infty(\mathbb{R}^n_+)}\|\mathbf{u}\|^{1-\epsilon}_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}ds\\ &\le CN_0 + CM_0^{1+\epsilon}M_1^{1-\epsilon}\int_0^t s^{-\frac{\epsilon+1}{2}}ds = CN_0 + CM_0^{1+\epsilon}M_1^{1-\epsilon}t^{\frac{1-\epsilon}{2}}.\end{aligned}$$

This implies the inequality

$$\sup_{0<t<T_0}\|\mathbf{u}\|_{B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)} \le CN_0 + CM_0^{1+\alpha}M_1^{1-\epsilon}T_0^{\frac{1-\epsilon}{2}}.$$

Set $M_2 = CN_0 + CM_0^{1+\epsilon}M_1^{1-\epsilon}T_0^{\frac{1-\epsilon}{2}}$. Then we have

$$\sup_{0<t<T_0}\|\mathbf{u}\|_{B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)} \le M_2. \tag{62}$$

Moreover,

$$\begin{aligned}\|\mathbf{u} - e^{-tA}\mathbf{a}\|_{B^{-\epsilon}_{\infty,\infty}(\mathbb{R}^n_+)} &\le C\int_0^t \|\mathbf{u}(s)\|^{1+\epsilon}_{L^\infty(\mathbb{R}^n_+)}\|\mathbf{u}\|^{1-\epsilon}_{B^1_{\infty,\infty}(\mathbb{R}^n_+)}ds\\ &\le CM_0^{1+\epsilon}M_1^{1-\epsilon}\int_0^t s^{-\frac{\epsilon+1}{2}}ds = CM_0^{1+\epsilon}M_1^{1-\epsilon}t^{\frac{1-\epsilon}{2}}.\end{aligned}$$

Therefore,

$$\mathbf{u}(t)\to e^{-tA}\mathbf{a}\in L^\infty(0,T_0;B^{-\epsilon}_{\infty,\infty}) \text{ as } t\to 0+.$$

5.3. Uniqueness of mild solution

Let $\mathbf{u}$ and $\mathbf{v}$ be the mild solution of the Navier-Stokes equations in the class $L^\infty_{\text{loc}}(0,T;L^\infty(\mathbb{R}^n_+))$ with the property

$$\sup_{0<t<T} t^{\frac{\alpha}{2}}\|\mathbf{u}(t)\|_{L^\infty(\mathbb{R}^n_+)} = M_1 \text{ and } \sup_{0<t<T} t^{\frac{\alpha}{2}}\|\mathbf{v}(t)\|_{L^\infty(\mathbb{R}^n_+)} = M_2,$$

where $0<\alpha<1$.

Set $\mathbf{w} = \mathbf{u}-\mathbf{v}$. Then $\mathbf{w}$ satisfies the integral equation

$$\begin{aligned}\mathbf{w}(t) &= -\int_0^t e^{-A(t-s)}\mathbb{P}\text{div}(\mathbf{u}\otimes\mathbf{u})(s)ds + \int_0^t e^{-A(t-s)}\mathbb{P}\text{div}(\mathbf{v}\otimes\mathbf{v})(s)ds\\ &= -\int_0^t e^{-A(t-s)}\mathbb{P}\text{div}[(\mathbf{w}\otimes\mathbf{v})(s) + (\mathbf{u}\otimes\mathbf{w})(s)]ds.\end{aligned}$$

Apply (31) for the estimate of the bilinear term. Then we obtain the following Gronwall inequality

$$\begin{aligned}
&\|\mathbf{w}(t)\|_{L^\infty(\mathbb{R}^n_+)} \\
&\le C_1 \int_0^t (t-s)^{-\frac{1}{2}} \big(\|\mathbf{u}(s)\|_{L^\infty(\mathbb{R}^n_+)} + \|\mathbf{v}(s)\|_{L^\infty(\mathbb{R}^n_+)}\big)\|\mathbf{w}(s)\|_{L^\infty(\mathbb{R}^n_+)} ds \\
&\le C_1(M_1+M_2)\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{\alpha}{2}} \|\mathbf{w}(s)\|_{L^\infty(\mathbb{R}^n_+)} ds. \qquad (63)
\end{aligned}$$

Set $X(t) = \|\mathbf{w}(t)\|_{L^\infty(\mathbb{R}^n_+)}$. Then X satisfies the inequality

$$X(t) \le C_1(M_1+M_2)\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{\alpha}{2}} X(s) ds. \qquad (64)$$

Since $\int_0^t (t-s)^{-\frac{1}{2}} s^{-\alpha} ds = C_2 t^{\frac{1}{2}-\alpha}$ when $0 \le \alpha < 1$, we have that

$$\sup_{0<t\le t_0} t^{\frac{\alpha}{2}} X(t) \le C_1C_2(M_1+M_2) t_0^{\frac{1}{2}-\frac{\alpha}{2}} \Big(\sup_{0<s\le t_0} s^{\frac{\alpha}{2}} X(s)\Big). \qquad (65)$$

Choose t_0 small enough so that $C_1C_2(M_1+M_2)t_0^{\frac{1}{2}-\frac{\alpha}{2}} < 1$. Then (65) implies that

$$\sup_{0<t\le t_0} t^{\frac{\alpha}{2}} X(t) = 0.$$

This again implies that $X(t) = 0$ for all $0 < t \le t_0$. Then from (64), we have

$$\begin{aligned}
X(t) &\le C_1(M_1+M_2)\Big[\int_{t_0}^t (t-s)^{-\frac{1}{2}} s^{-\frac{\alpha}{2}} X(s) ds\Big] \\
&\le C_1(M_1+M_2) t_0^{-\frac{\alpha}{2}} \int_{t_0}^t (t-s)^{-\frac{1}{2}} X(s) ds. \qquad (66)
\end{aligned}$$

Iterating the estimate (66), we obtain

$$\begin{aligned}
X(t) &\le C_1^2(M_1+M_2)^2 t_0^{-\alpha} \int_{t_0}^t (t-s)^{-\frac{1}{2}} \Big[\int_{t_0}^s (s-\tau)^{-\frac{1}{2}} X(\tau) d\tau\Big] ds \\
&= C_1^2(M_1+M_2)^2 t_0^{-\alpha} \int_{t_0}^t \Big[\int_\tau^t (t-s)^{-\frac{1}{2}} (s-\tau)^{-\frac{1}{2}} ds\Big] X(\tau) d\tau \\
&\le C_1^2 C_3 (M_1+M_2)^2 t_0^{-\alpha} \int_{t_0}^t X(\tau) d\tau. \qquad (67)
\end{aligned}$$

Here, we note that $\int_\tau^t (t-s)^{-\frac{1}{2}}(s-\tau)^{-\frac{1}{2}} ds = C_3 < \infty$.

Set $Y(t) = \int_{t_0}^t X(\tau)d\tau$. Then from (67), we have the Gronwall inequality

$$Y'(t) \le C_1^2 C_3 (M_1+M_2)^2 t_0^{-\alpha} Y(t) \ \text{ and } \ Y(t_0) = 0.$$

Solving the above Gronwall inequality, we conclude that $Y(t) \equiv 0$ for $t_0 < t < T$. Applying this result to (67), we have $X(t) \le C_1^2 C_3 (M_1+M_2)^2 t_0^{-\alpha} Y(t) = 0$ for $t_0 < t < T$. Therefore, we conclude that $\|w(t)\|_{L^\infty(\mathbb{R}^n_+)} = X(t) = 0$ for all $0 < t < T$, that is, $\mathbf{u}(t) = \mathbf{v}(t)$ for all $0 < t < T$.

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