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NEWTON'S METHOD FOR SYMMETRIC AND BISYMMETRIC SOLVENTS OF THE NONLINEAR MATRIX EQUATIONS

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Abstract. One of the interesting nonlinear matrix equations is the quadratic matrix equation defined by

$$Q(X) = AX^2 + BX + C = 0,$$

where X is a $n \times n$ unknown real matrix, and A, B and C are $n \times n$ given matrices with real elements. Another one is the matrix polynomial

$$P(X) = A_0X^m + A_1X^{m-1} + \cdots + A_m = 0, \quad X, A_i \in \mathbb{R}^{n\times n}.$$

Newton's method is used to find the symmetric and bisymmetric solvents of the nonlinear matrix equations $Q(X)$ and $P(X)$. The method does not depend on the singularity of the Fréchet derivative. Finally, we give some numerical examples.

1. Introduction

We consider two kinds of nonlinear matrix equations, namely, the quadratic matrix equation

$$Q(X) = AX^2 + BX + C = 0, \qquad A, B, C \text{ and } X \in \mathbb{R}^{n\times n}, \tag{1}$$

and the matrix polynomial

$$P(X) = A_0X^m + A_1X^{m-1} + \cdots + A_m = 0, \quad A_i, X \in \mathbb{R}^{n\times n}. \tag{2}$$

Solving nonlinear matrix equations addresses many problems which occur in many applications and in modeling of scientific problems.

Newton's method is a natural approach in solving nonlinear matrix equations. For the quadratic case (1), Davis [2, 3] considered Newton's method and Higham and Kim [5, 6] incorporated the exact line searches into Newton's method, which reduced the number of iterations required for the most part. For solving the matrix polynomial (2), Newton's method was considered by Kratz and Stickel [8].

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However, we need to overcome the following challenges when solving matrix equations by Newton's method.

(i) The method only works well when the Fréchet derivative is nonsingular.
(ii) How to guarantee the convergence of a particular starting matrix.

Guo and Laub [4] considered the nonsymmetric algebraic Riccati equation

$$R(X) = XEX + XG + HX + F = 0,$$

which arises from the transport theory. They proposed an algorithm for Newton's method with a special starting matrix to find the elementwise minimal positive solvent. Kim [7] presented that the elementwise minimal positive definite solvent for some different types of quadratic matrix equations can be found by Newton's method with a zero starting matrix.

In this paper, we introduce two iterative algorithms for solving the Newton step with the symmetric and bisymmetric solutions. Then we apply Newton's method with iterative algorithms to solve the quadratic matrix equation (1) and the matrix polynomial (2). We show that for the symmetric (bisymmetric) starting matrix X_0, our Newton's method converges to a solvent which has the same properties as X_0. Finally, we give some numerical experiments that confirm our Newton's method is efficient for solving the case when the Fréchet derivative is singular. The definition of a bisymmetric matrix is as follows.

Definition 1.1 ([12])**.** A matrix $A \in \mathbb{R}^{n\times n}$ is called a bisymmetric (BS) matrix if its elements a_{ij} satisfy the properties

$$a_{ij} \;=\; a_{ji} \text{ and } a_{ij} = a_{n-j+1,n-i+1} \quad \text{for } 1 \le i,\ j \le n.$$

In order to construct an iterative method for finding a bisymmetric solution of the Newton step, the following basic properties of bisymmetric matrices are needed.

Lemma 1.2 ([10])**.** *A matrix B is bisymmetric if and only if $B = B^T = S_nBS_n$, where $S_n = [e_n, e_{n-1}, \ldots, e_1]$ and e_i denotes the elementary standard vector of $\mathbb{R}^n$.*

Lemma 1.3 ([11])**.** *If the matrix $X \in \mathbb{R}^{n\times n}$ is a symmetric matrix, then $X + S_nXS_n$ is a bisymmetric matrix.*

2. Newton's method

In this section, let us review Newton's method for the general nonlinear matrix equation $G : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ such that

$$G(X) = 0. \tag{3}$$

Let the matrix S be a solvent of equation (3) and write

$$X_k = S + H_k. \tag{4}$$

Then by Taylor's Theorem, we have

$$\begin{aligned} G(X_k) &= G(S + H_k) \\ &= G(S) + G^{'}(S)H_k + O(H_k^2) \\ &= G^{'}(S)H_k + O(H_k^2), \end{aligned} \tag{5}$$

where $G^{'} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is the Fréchet derivative as $G(X)$ at X. If H_k is the value that we know, then the matrix equation (3) can be automatically solved from (4). Moreover, if we evaluate the Fréchet derivative $G^{'}$ at X_k, replace H_k in (5) by $X_k - X_{k+1}$ rather than $X_k - S$ and ignore the second order terms, then we get

$$G(X_k) = G^{'}(X_k)(X_k - X_{k+1}). \tag{6}$$

So if $G^{'}(X_k)$ is nonsingular, then from (6), we obtain the next approximation X_{k+1} as

$$X_{k+1} = X_k - [G^{'}(X_k)]^{-1}G(X_k),$$

which is called Newton's method [1, 9].

In the nonsingular Fréchet derivative case, the Kantorovich theorem gives information on the convergence of Newton's method for solving the nonlinear matrix equation (3) [2].

Theorem 2.1 (Kantorovich). *If there exists K such that*

$$\|G^{'}(X) - G^{'}(Y)\| \le K\|X - Y\| \quad \textit{for all} \quad X, Y \in R^{n\times n}$$

in some closed ball $\bar{U}(X_0, r)$ and $h_0 = B_0\eta_0 K \le \frac{1}{2}$ with $\|[G^{'}(X_0)]^{-1}\| \le B_0$ and $\|X_1 - X_0\| \le \eta_0$, then the Newton sequence starting from X_0 will converge to a solvent S of $G(X)$ which exists in $\bar{U}(X_0, r)$, provided that

$$r \ge r_0 = \frac{1 - \sqrt{1 - 2h_0}}{h_0}\eta_0.$$

However, Theorem 2.1 cannot affect the settlement for the weak points of Newton's method.

If we define E_k as the solution of the linear equation $G(X_k)+G^{'}(X_k)E_k = 0$, then Newton's method for the nonlinear matrix equations (3) with the given starting matrix X_0 can be written in the iteration form

$$\begin{cases} G^{'}(X_k)E_k = -G(X_k), \\ \mathrm{X}_{k+1} = X_k + E_k, \end{cases} \quad \text{where} \quad k = 0, 1, \ldots. \tag{7}$$

Thus each step of Newton's method requires finding of the solution E of the linear equation

$$G^{'}(X)E = -G(X). \tag{8}$$

Now, we should derive the Fréchet derivatives of the quadratic matrix equations (1) and the matrix polynomial (2) to solve them using Newton's method.

From the definition of the quadratic matrix equation (1), we easily obtain

$$Q^{'}(X)[E] = (AX + B)E + AEX,$$

which is the Fréchet derivative of equation (1) at X in the direction E. The Fréchet derivative of the matrix polynomial (2) is

$$P^{'}(X)[H] = \left(\sum_{\nu=0}^{m-1} A_\nu X^{(m-1)-\nu}\right) H + \left(\sum_{\nu=0}^{m-2} A_\nu X^{(m-2)-\nu}\right) HX + \cdots + A_0 H X^{m-1}.$$

Therefore, each step of Newton's method for equations (1) and (2) involves finding the solution E and the solution H of

$$(AX + B)E + AEX = -Q(X) \tag{9}$$

and

$$\left(\sum_{\nu=0}^{m-1} A_\nu X^{(m-1)-\nu}\right) H + \left(\sum_{\nu=0}^{m-2} A_\nu X^{(m-2)-\nu}\right) HX + \cdots + A_0 H X^{m-1} = -P(X), \tag{10}$$

respectively.

3. The symmetric solvents of $Q(X)$ and $P(X)$

In Section 2, we have already seen that for solving $Q(X)$ and $P(X)$ using Newton's method, we need to solve the linear equations (9) and (10), respectively. So we first give an iterative method to find a symmetric solution of (9), then extend it to solve equation (10). Then we consider the convergence of our Newton's method. From here, $\|\cdot\|$ denotes the Euclidean norm of matrices.

3.1. An iterative method for solving equation (9)

The following algorithm is to find a symmetric solution of the qth Newton step (9).

Algorithm 3.1. Let $A, B, C \in \mathbb{R}^{n\times n}$ and a symmetric matrix $X_q \in \mathbb{R}^{n\times n}$ be given. Choose a symmetric starting matrix $E_{q_0} \in \mathbb{R}^{n\times n}$.

$k = 0$;
$R_0 = -Q(X_q) - [(AX_q + B)E_{q_0} + AE_{q_0}X_q]$
$Z_0 = (AX_q + B)^{\mathrm{T}} R_0 + A^{\mathrm{T}} R_0 (X_q)^{\mathrm{T}}$
$P_0 = \frac{1}{2}(Z_0 + Z_0^{\mathrm{T}})$
$\alpha_0 = \frac{\|R_0\|^2}{\|P_0\|^2}$

while $R_k \neq 0$ or $P_k \neq 0$
$\alpha_k = \frac{\|R_k\|^2}{\|P_k\|^2}$
$E_{q_{k+1}} = E_{q_k} + \alpha_k P_k$
$R_{k+1} = -Q(X_q) - \left[(AX_q + B)E_{q_{k+1}} + AE_{q_{k+1}}X_q\right]$
$Z_{k+1} = (AX_q + B)^{\mathrm{T}} R_{k+1} + A^{\mathrm{T}} R_{k+1} (X_q)^{\mathrm{T}}$
$\beta_k = \frac{\|R_{k+1}\|^2}{\|R_k\|^2}$

$$P_{k+1} = \frac{1}{2}(Z_{k+1} + Z_{k+1}^{\mathrm{T}}) + \beta_k P_k$$

end

Remark 3.2. In Algorithm 3.1, the matrices P_k and E_{q_k} are symmetric matrices for all $k = 0, 1, 2, \ldots$.

From Algorithm 3.1, we have some basic properties.

Lemma 3.3. *Let E_q be a symmetric solution of the qth Newton step* (9)*, and the sequences $\{Z_k\}, \{R_k\}, \{E_{q_k}\}$ be generated by Algorithm 3.1. Then the following statement holds.*

$$\mathrm{tr}\left[Z_k^{\mathrm{T}}(E_q - E_{q_k})\right] = \|R_k\|^2 \quad \textit{for all } k = 0, 1, \ldots.$$

Proof. From Algorithm 3.1, for any k, we have that

$$\begin{aligned}\mathrm{tr}\left[Z_k^{\mathrm{T}}(E_q - E_{q_k})\right] &= \mathrm{tr}\left\{\left[(AX_q + B)^{\mathrm{T}} R_k + A^{\mathrm{T}} R_k (X_q)^{\mathrm{T}}\right]^{\mathrm{T}} (E_q - E_{q_k})\right\} \\ &= \mathrm{tr}\left\{R_k^{\mathrm{T}}\left[(AX_q + B)(E_q - E_{q_k}) + A(E_q - E_{q_k})X_q\right]\right\} \\ &= \mathrm{tr}\left\{R_k^{\mathrm{T}}\left[-Q(X_q) - (AX_q + B)E_{q_k} - AE_{q_k}X_q\right]\right\} \\ &= \|R_k\|^2.\end{aligned}$$

□

Lemma 3.4. *Suppose E_q is a symmetric solution of equation* (9)*. Then*

$$\mathrm{tr}\left[P_k^{\mathrm{T}}(E_q - E_{q_k})\right] = \|R_k\|^2 \quad \textit{for all } k = 0, 1, \ldots. \tag{11}$$

Proof. We prove the conclusion (11) by induction.

When $k = 0$, from Algorithm 3.1 and Lemma 3.3, we have

$$\mathrm{tr}\left[P_0^{\mathrm{T}}(E_q - E_{q_0})\right] = \mathrm{tr}\left[Z_0^{\mathrm{T}}(E_q - E_{q_0})\right] = \|R_0\|^2.$$

Assume that the conclusion (11) holds for $k = l$. Then when $k = l + 1$,

$$\begin{aligned}\mathrm{tr}\left[P_{l+1}^{\mathrm{T}}(E_q - E_{q_{l+1}})\right] &= \mathrm{tr}\left[Z_{l+1}^{\mathrm{T}}(E_q - E_{q_{l+1}})\right] + \beta_l \mathrm{tr}\left[P_l^{\mathrm{T}}(E_q - E_{q_{l+1}})\right] \\ &= \|R_{l+1}\|^2\end{aligned}$$

by Lemma 3.3 since

$$\begin{aligned}\mathrm{tr}\left[P_l^{\mathrm{T}}(E_q - E_{q_{l+1}})\right] &= \mathrm{tr}\left[P_l^{\mathrm{T}}(E_q - E_{q_l} - \alpha_l P_l)\right] \\ &= \mathrm{tr}\left[P_l^{\mathrm{T}}(E_q - E_{q_l})\right] - \alpha_l \mathrm{tr}\left(P_l^{\mathrm{T}} P_l\right) = 0.\end{aligned}$$

□

Remark 3.5. Lemma 3.4 implies that if the qth Newton step (9) has a symmetric solution and $R_k \neq 0$ for some integer k, then $P_k \neq 0$ must hold for k.

Lemma 3.6. *For the sequences $\{R_i\}$ and $\{P_i\}$ generated by Algorithm 3.1, we have that*

$$\mathrm{tr}\left(R_i^{\mathrm{T}} R_j\right) = 0 \quad \textit{and} \quad \mathrm{tr}\left(P_i^{\mathrm{T}} P_j\right) = 0 \quad \textit{for} \quad i > j = 0, 1, \ldots, k, \ k \geq 1. \tag{12}$$

Proof. We prove (12) by induction.

Step 1. When $k = 1$,

$$\begin{aligned}\operatorname{tr}\left(R_1^{\mathrm{T}} R_0\right) &= \operatorname{tr}\left\{\left[R_0 - \alpha_0(AX_q + B)P_0 - \alpha_0 AP_0 X_q\right]^{\mathrm{T}} R_0\right\} \\ &= \|R_0\|^2 - \alpha_0 \operatorname{tr}\left\{P_0^{\mathrm{T}}\left[(AX_q + B)^{\mathrm{T}} R_0 + A^{\mathrm{T}} R_0 (X_q)^{\mathrm{T}}\right]\right\} \\ &= \|R_0\|^2 - \alpha_0 \operatorname{tr}\left(P_0^{\mathrm{T}} P_0\right) \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}\left(P_1^{\mathrm{T}} P_0\right) &= \operatorname{tr}(Z_1^{\mathrm{T}} P_0) + \beta_0 \operatorname{tr}(P_0^{\mathrm{T}} P_0) \\ &= \operatorname{tr}\left\{R^{\mathrm{T}}\left[\left(AX_q + B\right)P_0 + AR_0 X_q\right]\right\} + \frac{\|R_1\|^2\|P_0\|^2}{\|R_0\|^2} \\ &= -\frac{1}{\alpha_0}\operatorname{tr}\left(R_1^{\mathrm{T}} R_1\right) + \frac{\|R_1\|^2\|P_0\|^2}{\|R_0\|^2} \\ &= 0.\end{aligned}$$

Assume the statement (12) holds for $k = l$, i.e., $\operatorname{tr}(R_l^{\mathrm{T}} R_{l-1}) = 0$ and $\operatorname{tr}(P_l^{\mathrm{T}} P_{l-1}) = 0$. Then

$$\begin{aligned}\operatorname{tr}\left(R_{l+1}^{\mathrm{T}} R_l\right) &= \operatorname{tr}(R_l^{\mathrm{T}} R_l) - \alpha_l \operatorname{tr}\left\{\left[(AX_q + B)P_l + AP_l X_q\right]^{\mathrm{T}} R_l\right\} \\ &= \|R_l\|^2 - \alpha_l \operatorname{tr}\left(P_l^{\mathrm{T}} Z_l\right) \\ &= \|R_l\|^2 - \alpha_l \operatorname{tr}\left(P_l^{\mathrm{T}} P_l\right) - \alpha_l \beta_{l-1} \operatorname{tr}\left(P_l^{\mathrm{T}} P_{l-1}\right) \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}\left(P_{l+1}^{\mathrm{T}} P_l\right) &= \operatorname{tr}\left(Z_{l+1}^{\mathrm{T}} P_l\right) + \beta_l \operatorname{tr}\left(P_l^{\mathrm{T}} P_l\right) \\ &= \operatorname{tr}\left\{R_{l+1}^{\mathrm{T}}\left[\left(AX_q + B\right)P_l + AP_l X_q\right]\right\} + \frac{\|R_{l+1}\|^2\|P_l\|^2}{\|R_l\|^2} \\ &= -\frac{1}{\alpha_l}\operatorname{tr}\left(R_{l+1}^{\mathrm{T}} R_{l+1}\right) + \frac{\|R_{l+1}\|^2\|P_l\|^2}{\|R_l\|^2} \\ &= 0.\end{aligned}$$

Step 2. Suppose that $\operatorname{tr}\left(R_l^{\mathrm{T}} R_j\right) = 0$ and $\operatorname{tr}\left(P_l^{\mathrm{T}} P_j\right) = 0$ for all $j = 0, 1, \ldots, l-1$, i.e., $\operatorname{tr}\left(P_l^{\mathrm{T}} P_{j-1}\right) = 0$. Now we show that $\operatorname{tr}\left(R_{l+1}^{\mathrm{T}} R_j\right) = 0$ and $\operatorname{tr}\left(P_{l+1}^{\mathrm{T}} P_j\right) = 0$ for $j = 0, 1, \ldots, l-1$.

By Algorithm 3.1 and the accompanying assumptions, we have

$$\begin{aligned}\operatorname{tr}\left(R_{l+1}^{\mathrm{T}} R_j\right) &= \operatorname{tr}\left(R_l^{\mathrm{T}} R_j\right) - \alpha_l \operatorname{tr}\left\{\left[(AX_q + B)P_l + AP_l X_q\right]^T R_j\right\} \\ &= -\alpha_l \operatorname{tr}\left(P_l^{\mathrm{T}} Z_j\right) \\ &= -\alpha_l \operatorname{tr}\left[P_l^{\mathrm{T}}\left(P_j - \beta_{j-1} P_{j-1}\right)\right] \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}\left(P_{l+1}^{\mathrm{T}} P_j\right) &= \operatorname{tr}\left(Z_{l+1}^{\mathrm{T}} P_j\right) + \beta_l \operatorname{tr}\left(P_l^{\mathrm{T}} P_j\right)\\ &= \operatorname{tr}\left\{R_{l+1}^{\mathrm{T}}\left[(AX_q + B)P_j + AP_j X_q\right]\right\}\\ &= \frac{1}{\alpha_j}\operatorname{tr}\left[R_{l+1}^{\mathrm{T}}(R_j - R_{j+1})\right]\\ &= 0.\end{aligned}$$

Hence the statement (12) holds for $k = l + 1$. Therefore, from Steps 1 and 2, we complete the proof. □

Theorem 3.7. *Assume the qth Newton step* (9) *has a symmetric solution. Then for any symmetric starting matrix E_{q_0}, its symmetric solution can be obtained, at most, in n^2 steps.*

Proof. Suppose that $R_k \neq 0$ for $k = 0, 1, \ldots, n^2-1$. Then from Lemma 3.6, the set $\{R_0, R_1, \ldots, R_{n^2-1}\}$ is an orthogonal basis of the matrix space $\mathbb{R}^{n\times n}$. Since the qth Newton step (9) has a symmetric solution, $P_k \neq 0$ for $k = 0, 1, \ldots, n^2-1$ by Lemma 3.6. Therefore, we can evaluate $E_{q_{n^2}}$ and R_{n^2} in Algorithm 3.1, and $\operatorname{tr}\left(R_{n^2}^{\mathrm{T}} R_k\right) = 0$ for $k = 0, 1, \ldots, n^2 - 1$ by Lemma 3.6. But $\operatorname{tr}\left(R_{n^2}^{\mathrm{T}} R_k\right) = 0$ holds only when $R_{n^2} = 0$, which implies that $E_{q_{n^2}}$ is a solution of equation (9). □

From Newton's method and Theorem 3.7, we have the following main theorem.

Theorem 3.8. *Suppose that the quadratic matrix equation* (1) *has a symmetric solvent and each Newton step is consistent for a symmetric starting matrix X_0. The sequence $\{X_k\}$ is generated by Newton's method with X_0 such that*

$$\lim_{k\to\infty} X_k = S,$$

and if the matrix S satisfies $Q(S) = 0$, then S is a symmetric solvent.

Proof. Let E_0 be a symmetric solution of the first Newton step

$$(AX_0 + B)E_0 + AE_0X_0 = -Q(X_0) = -AX_0^2 - BX_0 - C$$

with the symmetric starting matrix X_0. Then according to Newton's method and Theorem 3.7, we obtain the symmetric matrix

$$\begin{aligned}X_{k+1} &= X_k + E_k\\ &= X_0 + E_0 + \cdots + E_k\end{aligned}$$

for all $k = 0, 1, \ldots$ with starting matrix X_0. Since the matrix X_0 guarantees

$$\lim_{k\to\infty} X_{k+1} = \lim_{k\to\infty}(X_0 + E_0 + \cdots + E_k) = S,$$

the matrix S is a symmetric matrix. □

3.2. An iterative method for solving (10)

We now propose an iterative method for solving the qth Newton step (10) of a matrix polynomial.

Algorithm 3.9. Input $n \times n$ real matrices $A_0, A_1, \ldots, A_m$ and a symmetric matrix $X_q \in \mathbb{R}^{n\times n}$. Choose a symmetric starting matrix $H_{q_0} \in \mathbb{R}^{n\times n}$.

$k = 0$; $\quad R_0 = -P(X_q) - \left(\sum_{\nu=0}^{m-1} A_\nu X^{(m-1)-\nu}\right) H_{q_0} - \cdots - A_0 H_{q_0} X_q^{m-1}$

$\quad Y_0 = \left(\sum_{\nu=0}^{m-1} A_\nu X^{(m-1)-\nu}\right)^{\mathrm{T}} R_0 + \cdots + A_0^{\mathrm{T}} R_0 (X_q^{m-1})^{\mathrm{T}}$

$\quad Q_0 = \frac{1}{2}(Y_0 + Y_0^{\mathrm{T}})$

while $R_k \neq 0$ or $Q_k \neq 0$

$\quad H_{q_{k+1}} = H_{q_k} + \frac{\|R_k\|^2}{\|Q_k\|^2} Q_k$

$\quad R_{k+1} = -P(X_q) - \left(\sum_{\nu=0}^{m-1} A_\nu X^{(m-1)-\nu}\right) H_{q_{k+1}} - \cdots - A_0 H_{q_{k+1}} X_q^{m-1}$

$\quad Y_{k+1} = \left(\sum_{\nu=0}^{m-1} A_\nu X^{(m-1)-\nu}\right)^{\mathrm{T}} R_{k+1} + \cdots + A_0^{\mathrm{T}} R_{k+1} (X_q^{m-1})^{\mathrm{T}}$

$\quad Q_{k+1} = \frac{1}{2}(Y_{k+1} + Y_{k+1}^{\mathrm{T}}) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} Q_k$

end

Regarding Algorithm 3.9, we have the following basic properties.

Lemma 3.10. *Suppose H_q is a symmetric solution of the qth Newton step* (10)*, and the sequences $\{R_k\}, \{Y_k\}$ and $\{H_{q_k}\}$ are generated by Algorithm 3.9. Then we have*

$$\operatorname{tr}\left[Y_k^{\mathrm{T}}(H_q - H_{q_k})\right] = \|R_k\|^2 \quad \text{for all } k = 0, 1, \ldots.$$

Lemma 3.11. *Let H_q be a symmetric solution of equation* (10)*. Then for any starting symmetric matrix H_{q_0}, we have*

$$\operatorname{tr}\left[Q_k^{\mathrm{T}}(H_q - H_{q_k})\right] = \|R_k\|^2 \quad \text{for } k = 0, 1, \ldots.$$

Remark 3.12. Lemma 3.11 implies that if there exists an integer k such that $Q_k = 0$ but $R_k \neq 0$, then the matrix equation (10) is inconsistent over symmetric matrices.

Lemma 3.13. *Suppose that the sequences $\{R_i\}$ and $\{Q_i\}$ are generated by Algorithm 3.9. Then we have*

$$\operatorname{tr}\left(R_i^{\mathrm{T}} R_j\right) = 0 \quad \text{and} \quad \operatorname{tr}\left(Q_i^{\mathrm{T}} Q_j\right) = 0 \quad \text{for } i > j = 0, 1, \ldots, k,\ k \geq 1.$$

Similar to Theorem 3.7, we can prove the following theorem by using Lemmas 3.11, 3.13, and Remark 3.12.

Theorem 3.14. *Suppose that the qth Newton step* (10) *is consistent. Then for any symmetric starting matrix H_{q_0}, its symmetric solution can be obtained by Algorithm 3.9.*

From Newton's method and the above theorem, we can easily prove the following result.

Theorem 3.15. *Suppose that the matrix polynomial* (2) *has a symmetric solvent and each Newton step is consistent for a symmetric starting matrix* X_0*. The sequence* $\{X_k\}$ *is generated by Newton's method with* X_0 *such that*

$$\lim_{k\to\infty} X_k = S,$$

and if the matrix S *satisfies* $P(S) = 0$*, then* S *is a symmetric solvent.*

4. The BS solvents of $Q(X)$ and $P(X)$

In this section, we consider Newton's method for finding the BS solvents of the quadratic matrix equation (1) and matrix polynomial (2).

4.1. An iterative method for solving (9) over BS matrices

Before proposing the iterative method for finding the BS solution of (9), we give the following well-known results.

Lemma 4.1 ([11])**.** *Assume that* X *is BS. Then for any* $n \times n$ *real matrix* Y,

$$\text{tr}\left\{\frac{1}{4}\left[\left(Y+Y^{\mathrm{T}}\right)+S_n\left(Y+Y^{\mathrm{T}}\right)S_n\right]^{\mathrm{T}} X\right\} = \text{tr}\left(Y^{\mathrm{T}}X\right). \tag{13}$$

Algorithm 4.2. The matrices $A, B, C, X_q \in \mathbb{R}^{n\times n}$ are given, where $X_q \in \mathbb{R}^{n\times n}$ is BS. Choose a BS starting matrix $E_{q_0} \in \mathbb{R}^{n\times n}$.

$k = 0$; $\quad R_0 = -Q(X_q) - (AX_q + B)E_{q_0} - AE_{q_0}X_q$

$Z_0 = (AX_q + B)^{\mathrm{T}}R_0 + A^{\mathrm{T}}R_0(X_q)^{\mathrm{T}}$

$P_0 = \frac{1}{4}\left[\left(Z_0 + Z_0^{\mathrm{T}}\right) + S_n\left(Z_0 + Z_0^{\mathrm{T}}\right)S_n\right]$

$\alpha_0 = \frac{\|R_0\|^2}{\|P_0\|^2}$

while $R_k \neq 0$ or $P_k \neq 0$

$\alpha_k = \frac{\|R_k\|^2}{\|P_k\|^2}$

$E_{q_{k+1}} = E_{q_k} + \alpha_k P_k$

$R_{k+1} = -Q(X_q) - (AX_q + B)E_{q_{k+1}} - AE_{q_{k+1}}X_q$

$Z_{k+1} = (AX_q + B)^{\mathrm{T}}R_{k+1} + A^{\mathrm{T}}R_{k+1}(X_q)^{\mathrm{T}}$

$\beta_k = \frac{\text{tr}\left(Z_{k+1}^{\mathrm{T}}P_k\right)}{\|P_k\|^2}$

$P_{k+1} = \frac{1}{4}\left[\left(Z_{k+1} + Z_{k+1}^{\mathrm{T}}\right) + S_n\left(Z_{k+1} + Z_{k+1}^{\mathrm{T}}\right)S_n\right] - \beta_k P_k$

end

Note that the matrices P_k and E_{q_k} are BS matrices in Algorithm 4.2.

Lemma 4.3. *If the matrix* E_q *is a BS solution of the* q*th Newton step* (9) *and the sequences* $\{Z_k\}, \{R_k\}$ *and* $\{E_{q_k}\}$ *are generated by Algorithm 4.2, then*

$$\text{tr}\left[Z_k^{\mathrm{T}}\left(E_q - E_{q_k}\right)\right] = \|R_k\|^2 \quad \textit{for all } k = 0, 1, \ldots. \tag{14}$$

Similarly as in Lemma 3.3, from Algorithm 4.2, we can prove the conclusion (14). By Lemma 4.3, we easily prove the following property.

Lemma 4.4. *Assume that E_q is a BS solution of* (9)*. Then*

$$\mathrm{tr}\left[P_k^{\mathrm{T}}\left(E_q - E_{q_k}\right)\right] = \|R_k\|^2 \quad \textit{for all } k = 0, 1, \ldots. \tag{15}$$

Proof. When $k = 0$, from Algorithm 4.2, Lemmas 4.1 and 4.3, we have

$$\mathrm{tr}\left[P_0^{\mathrm{T}}\left(E_q - E_{q_0}\right)\right] = \mathrm{tr}\left[Z_0^{\mathrm{T}}\left(E_q - E_{q_0}\right)\right] = \|R_0\|^2.$$

Assume that the conclusion (15) holds for $k = l$. Since $\mathrm{tr}\left[P_l^{\mathrm{T}}\left(E_q - E_{q_{l+1}}\right)\right] = \mathrm{tr}\left[P_l^{\mathrm{T}}\left(E_q - E_{q_l} - \alpha_l P_l\right)\right] = \|R_l\|^2 - \alpha_l\|P_l\|^2 = 0$ and by Lemma 4.1 and 4.3, we have

$$\begin{aligned}\mathrm{tr}\left[P_{l+1}^{\mathrm{T}}\left(E_q - E_{q_{l+1}}\right)\right] &= \mathrm{tr}\left[Z_{l+1}^{\mathrm{T}}\left(E_q - E_{q_{l+1}}\right)\right] - \beta_l \mathrm{tr}\left[P_l^{\mathrm{T}}\left(E_q - E_{q_{l+1}}\right)\right] \\ &= \|R_{l+1}\|^2.\end{aligned}$$

Hence the conclusion (15) holds for $k = 0, 1, \ldots$ by the principle of induction. □

Lemma 4.5. *For the sequences $\{R_i\}$ and $\{P_i\}$ generated by Algorithm 3.1, we have*

$$\mathrm{tr}\left(R_i^{\mathrm{T}} R_j\right) = 0 \quad \textit{and} \quad \mathrm{tr}\left(P_i^{\mathrm{T}} P_j\right) = 0 \quad \textit{for } i > j = 0, 1, \ldots, k, \ k \geq 1. \tag{16}$$

Proof. We prove (16) by induction.

Step 1. When $k = 1$, we have

$$\begin{aligned}\mathrm{tr}\left(R_1^{\mathrm{T}} R_0\right) &= \mathrm{tr}\left\{\left[R_0 - \alpha_0\left(AX_q + B\right)P_0 - \alpha_0 A P_0 X_q\right]^{\mathrm{T}} R_0\right\} \\ &= \mathrm{tr}\left(R_0^{\mathrm{T}} R_0\right) - \alpha_0\left\{P_0^{\mathrm{T}}\left[\left(AX_q + B\right)^{\mathrm{T}} R_0 + A^{\mathrm{T}} R_0 \left(X_q\right)^{\mathrm{T}}\right]\right\} \\ &= \|R_0\|^2 - \alpha_0 \mathrm{tr}\left(P_0^{\mathrm{T}} Z_0\right) \\ &= \|R_0\|^2 - \alpha_0 \mathrm{tr}\left(P_0^{\mathrm{T}} P_0\right) = 0,\end{aligned}$$

and

$$\mathrm{tr}\left(P_1^{\mathrm{T}} P_0\right) = \mathrm{tr}\left(Z_1^{\mathrm{T}} P_0\right) - \beta_0 \mathrm{tr}\left(P_0^{\mathrm{T}} P_0\right) = 0.$$

Suppose that (16) holds for $k = l$. Then we have

$$\begin{aligned}\mathrm{tr}\left(R_{l+1}^{\mathrm{T}} R_l\right) &= \mathrm{tr}\left\{\left[R_l - \alpha_l\left(AX_q + B\right)P_l - \alpha_l A P_0 X_q\right]^{\mathrm{T}} R_l\right\} \\ &= \mathrm{tr}\left(R_l^{\mathrm{T}} R_l\right) - \alpha_l \mathrm{tr}\left\{P_l^{\mathrm{T}}\left[\left(AX_q + B\right)^{\mathrm{T}} R_l + A^{\mathrm{T}} R_l \left(X_q\right)^{\mathrm{T}}\right]\right\} \\ &= \mathrm{tr}\left(R_l^{\mathrm{T}} R_l\right) - \alpha_l \mathrm{tr}\left(P_l^{\mathrm{T}} Z_l\right) \\ &= \mathrm{tr}\left(R_l^{\mathrm{T}} R_l\right) - \alpha_l \mathrm{tr}\left[P_l^{\mathrm{T}}\left(P_l + \beta_l P_{l-1}\right)\right] \\ &= \|R_l\|^2 - \alpha_l \mathrm{tr}\left(P_l^{\mathrm{T}} P_l\right) \\ &= 0,\end{aligned}$$

and

$$\mathrm{tr}\left(P_{l+1}^{\mathrm{T}} P_l\right) = \mathrm{tr}\left(Z_{l+1}^{\mathrm{T}} P_l\right) - \beta_l \mathrm{tr}\left(P_l^{\mathrm{T}} P_l\right) = 0.$$

Step 2. Assume that $\operatorname{tr}\left(R_l^{\mathrm{T}} R_j\right) = 0$ and $\operatorname{tr}\left(P_l^{\mathrm{T}} P_j\right) = 0$ for all $j = 0, 1, \ldots, l-1$. Then from Algorithm 4.2 and the above assumptions, we have

$$\begin{aligned}
\operatorname{tr}\left(R_{l+1}^{\mathrm{T}} R_j\right) &= \operatorname{tr}\left\{\left[R_l - \alpha_l \left(AX_q + B\right) P_l - \alpha_l A P_l X_q\right]^{\mathrm{T}} R_j\right\} \\
&= \operatorname{tr}\left(R_l^{\mathrm{T}} R_j\right) - \alpha_l \operatorname{tr}\left\{P_l^{\mathrm{T}}\left[\left(AX_q + B\right)^{\mathrm{T}} R_j + A^{\mathrm{T}} R_j \left(X_q\right)^{\mathrm{T}}\right]\right\} \\
&= -\alpha_l \operatorname{tr}\left(P_l^{rmT} Z_j\right) \\
&= \alpha_l \operatorname{tr}\left[P_l^{\mathrm{T}}\left(P_j + \beta_j P_{j-1}\right)\right] \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{tr}\left(P_{l+1}^{\mathrm{T}} P_j\right) &= \operatorname{tr}\left(Z_{l+1}^{\mathrm{T}} P_j\right) - \beta_j \operatorname{tr}\left(P_l^{\mathrm{T}} P_j\right) \\
&= \operatorname{tr}\left\{R_{l+1}^{\mathrm{T}}\left[\left(AX_q + B\right) P_j + A P_j X_q\right]\right\} \\
&= \frac{1}{\alpha_j} \operatorname{tr}\left[R_{l+1}^{\mathrm{T}}\left(R_j - R_{j+1}\right)\right] \\
&= 0.
\end{aligned}$$

Thus we complete the proof by Steps 1 and 2. □

Theorem 4.6. *Suppose that the qth Newton step* (9) *has a bisymmetric solution. Then for any bisymmetric starting matrix E_{q_0}, its symmetric solution can be obtained, at most, in n^2 steps.*

Proof. This proof is similar to that of Theorem 3.7. □

Theorem 4.7. *Suppose that the quadratic matrix equation* (1) *has a bisymmetric solvent and each Newton step is consistent for a bisymmetric starting matrix X_0. The sequence $\{X_k\}$ is generated by Newton's method with X_0 such that*

$$\lim_{k\to\infty} X_k = S,$$

and if the matrix S satisfies $Q(S) = 0$, then S is a bisymmetric solvent.

Proof. Similar to Theorem 3.8, we can complete the proof by Newton's method and Theorem 4.7. □

4.2. An iterative method for finding the BS solution of equation (10)

Algorithm 4.8. Input $n \times n$ real matrices $A_0, A_1, \ldots, A_m$ and BS matrix $X_q \in \mathbb{R}^{n\times n}$. Choose a BS starting matrix $H_{q_0} \in \mathbb{R}^{n\times n}$.

$k = 0;\quad R_0 = -P(X_q) - \left(\sum_{\nu=0}^{m-1} A_\nu X^{(m-1)-\nu}\right) H_{q_0} - \cdots - A_0 H_{q_0} X_q^{m-1}$

$Y_0 = \left(\sum_{\nu=0}^{m-1} A_\nu X^{(m-1)-\nu}\right)^{\mathrm{T}} R_0 + \cdots + A_0^{\mathrm{T}} R_0 (X_q^{m-1})^{\mathrm{T}}$

$Q_0 = \frac{1}{4}\left[\left(Y_0 + Y_0^{\mathrm{T}}\right) + S_n\left(Y_0 + Y_0^{\mathrm{T}}\right) S_n\right]$

while $R_k \neq 0$ or $Q_k \neq 0$

$H_{q_{k+1}} = H_{q_k} + \frac{\|R_k\|^2}{\|Q_k\|^2} Q_k$
$R_{k+1} = -P(X_q) - \left(\sum_{\nu=0}^{m-1} A_\nu X^{(m-1)-\nu}\right) H_{q_{k+1}} - \cdots - A_0 H_{q_{k+1}} X_q^{m-1}$
$Y_{k+1} = \left(\sum_{\nu=0}^{m-1} A_\nu X^{(m-1)-\nu}\right)^{\mathrm{T}} R_{k+1} + \cdots + A_0^{\mathrm{T}} R_{k+1} (X_q^{m-1})^{\mathrm{T}}$
$Q_{k+1} = \frac{1}{2}\left[\left(Y_{k+1} + Y_{k+1}^{\mathrm{T}}\right) + S_n \left(Y_{k+1} + Y_{k+1}^{\mathrm{T}}\right) S_n\right] - \frac{\mathrm{tr}\left(Y_{k+1}^{\mathrm{T}} Q_k\right)}{\|Q_k\|^2} Q_k$
end

Regarding Algorithm 4.8, we have the following basic properties.

Lemma 4.9. *Suppose H_q is a BS solution of the qth Newton step* (10)*, and the sequences $\{R_k\}, \{Y_k\}$ and $\{H_{q_k}\}$ are generated by Algorithm 4.8. Then we have*

$$\mathrm{tr}\left[Y_k^{\mathrm{T}}(H_q - H_{q_k})\right] = \|R_k\|^2 \quad \text{for all } k = 0, 1, \ldots.$$

Lemma 4.10. *Let H_q be a BS solution of equation* (10)*. Then for any starting BS matrix H_{q_0}, we have*

$$\mathrm{tr}\left[Q_k^{\mathrm{T}}(H_q - H_{q_k})\right] = \|R_k\|^2 \quad \text{for } k = 0, 1, \ldots.$$

Lemma 4.11. *Suppose that the sequences $\{R_i\}$ and $\{Q_i\}$ are generated by Algorithm 4.8. Then we have*

$$\mathrm{tr}\left(R_i^{\mathrm{T}} R_j\right) = 0 \quad \text{and} \quad \mathrm{tr}\left(Q_i^{\mathrm{T}} Q_j\right) = 0 \quad \text{for } i > j = 0, 1, \ldots, k,\ k \geq 1.$$

Similar to Theorem 4.6, we can prove the following theorem by using Lemmas 4.10 and 4.11.

Theorem 4.12. *Suppose that the qth Newton step* (10) *is consistent. Then for any BS starting matrix H_{q_0}, its BS solution can be obtained by Algorithm 4.8.*

From Newton's method and the above theorem, we easily prove the following main theorem.

Theorem 4.13. *Suppose that the matrix polynomial* (2) *has a BS solvent and each Newton step is consistent for a BS starting matrix X_0. The sequence $\{X_k\}$ is generated by Newton's method with X_0 such that*

$$\lim_{k\to\infty} X_k = S,$$

and if the matrix S satisfies $P(S) = 0$, then S is a bisymmetric solvent.

5. Numerical examples

In this section, we give some numerical experiments to present the convergence of our Newton's method. Computations were done in MATLAB 7.1 and we regard the relative residuals $\rho_Q(X_k)$ and $\rho_P(X_k)$ as zeros if

$$\begin{aligned}
\rho_Q(X_k) &= \frac{\|fl(Q(X_k))\|}{\|A\|\|X_k\|^2 + \|B\|\|X_k\| + \|C\|} \leq n\mu,\\
\rho_P(X_k) &= \frac{\|fl(P(X_k))\|}{\|A_0\|\|X_k\|^m + \|A_1\|\|X_k\|^{m-1} + \cdots + \|A_m\|} \leq n\mu,
\end{aligned}$$

where n is the maximum size of A and A_0, and $\mu = 2^{-53} \simeq 1.1102e-016$ is the unit round off. In Algorithms 3.1, 3.9, 4.2, and 4.8, the iteration will be terminated whenever $\|R_k\| < \epsilon = 1.0e-016$.

Now, we give four numerical examples with Fréchet derivatives that are all singular according to their respective starting matrices.

Example 5.1. Let the coefficients of the quadratic matrix equation $Q_1(X)$ be

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 \\ -2 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \tag{17}$$

Starting Newton's method with Algorithm 3.1 with a symmetric matrix $X_0 =$

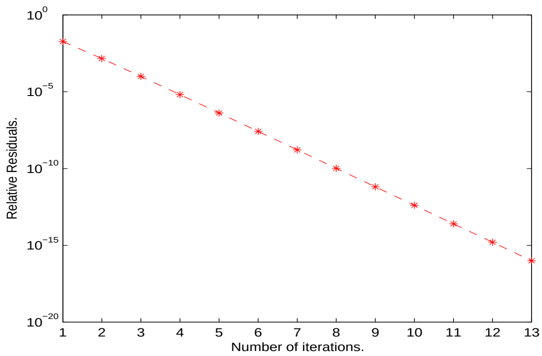

FIGURE 1. Convergence of problem (17).

1_2, where 1_2 denote the 2×2 identity matrix, we obtain the symmetric solvent of problem (17), that is, $X_{13} = [\begin{smallmatrix} 1 & 0 \\ 0 & 16384 \end{smallmatrix}]$. In this case, $\rho_{Q_1}(X_{13}) = 5.55e-017 < 2\mu$. The convergence results are presented in Figure 1, which confirms the conclusion of Theorem 3.8.

Example 5.2. For convenience, we consider a simple matrix polynomial of degree 3 given as

$$P_1(X) = Q_1(X)X = 0. \tag{18}$$

Similarly as in the first example, the symmetric starting matrix is chosen to be $X_0 = 1_2$. Then, for this starting matrix X_0, the Fréchet derivative of problem (18)

$$\begin{aligned} P_1'(X_0) &= 1_2 \otimes (AX^2 + BX + C) + X^{\mathrm{T}} \otimes (AX + B) + \left(X^2\right)^{\mathrm{T}} \otimes A \\ &= 0_4, \end{aligned}$$

where 0_4 is the 4×4 zero matrix, is singular. So Kratz and Stickel's method cannot solve problem (18). But by using Newton's method with Algorithm 3.9

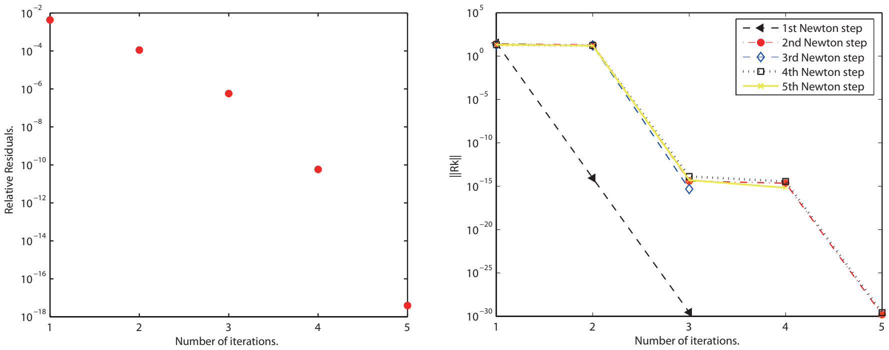

FIGURE 2. Convergence of problem (20).

and 13 iterative steps, we can obtain the symmetric solvent of equation (18) as follows:

$$X_{13} = \begin{bmatrix} 1 & 0 \\ 0 & 4096 \end{bmatrix}$$

with

$$\rho_{P_1}(X_{13}) = 1.16e - 017 < 2\mu.$$

This is a very simple example that verifies Theorem 3.15.

Example 5.3. We consider the quadratic matrix equation

$$(19) \qquad Q_2(X) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} X^2 + \begin{bmatrix} -4 & 0 & -4 \\ -4 & 0 & -4 \\ -4 & 0 & -4 \end{bmatrix} X + \begin{bmatrix} 12 & 0 & 3 \\ 12 & 0 & 3 \\ 12 & 0 & 3 \end{bmatrix} = 0.$$

Choose the BS starting matrix $X_0 = \left[\begin{smallmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{smallmatrix}\right]$. Applying Newton's method with Algorithm 4.2, we obtain the BS solvent $X_5 = \left[\begin{smallmatrix} 1 & 0 & 4 \\ 0 & 36 & 0 \\ 4 & 0 & 1 \end{smallmatrix}\right]$ with the corresponding relative residual $\rho_{Q_2}(X_5) = 4.01e{-}018 < 3\mu$. The obtained convergence results from our Newton's method are shown in the first figure of Figure 2. By using Algorithm 4.2, we can get BS solutions of five Newton steps within 9 iterations. This is illustrated in the second figure of Figure 2. In this figure, we can see that using iterations 3, 5, 3, 5, and 4 yields the BS solutions of the 1st, 2nd, 3rd, 4th, and 5th Newton steps, respectively. This demonstrates the conclusion of Theorem 4.6 for problem (19).

Example 5.4. We consider a matrix polynomial of degree 3 given as follows:

$$(20) \qquad P_2(X) = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} X^3 + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} X^2 + \begin{bmatrix} 6 & -1 \\ 6 & -1 \end{bmatrix} X + \begin{bmatrix} 14 & 14 \\ 14 & 14 \end{bmatrix} = 0.$$

By applying Newton's method with Algorithm 4.8 for the BS starting matrix $X_0 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, the BS solvent $X_{10} = \left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]$ of $P_2(X)$ can be obtained. The results

Table 1. Comparison of the relative residuals from the Kratz and Stickel's method with those from our Newton's method for problem (20).

	$\rho_P(X_k)$	
No.ite	Kratz and Stickel's method	Our Newton's method
1	fail	$2.00e+000$
2		$2.66e-001$
3		$2.30e-001$
4		$1.69e-001$
5		$8.71e-002$
6		$2.13e-002$
7		$1.20e-003$
8		$3.70e-016$
9		$3.53e-011$
10		$6.18e-017$

are provided in Table 1. Here, Kratz and Stickel's method also fails to find the BS solvent since the Fréchet derivative for the starting matrix X_0 is singular.

Example 5.5. Our final example is

$$Q_3(X) = AX^2 + BX + C,$$

where

$$A = I_5, \quad B = \begin{bmatrix} 20 & -10 & 0 & 0 & 0 \\ -10 & 30 & -10 & 0 & 0 \\ 0 & -10 & 30 & -10 & 0 \\ 0 & 0 & -10 & 30 & -10 \\ 0 & 0 & 0 & -10 & 20 \end{bmatrix},$$

$$C = \begin{bmatrix} -15 & -9 & -12 & -14 & -15 \\ -19 & -47 & -43 & -47 & -49 \\ -22 & -43 & -72 & -68 & -71 \\ -24 & -47 & -68 & -96 & -90 \\ -25 & -49 & -71 & -90 & -115 \end{bmatrix}.$$

The equation $Q_3(X)$ has a symmetric solvent S, where $S_{ij} = \min\{i, j\}$. Our iteration converges to S with the starting symmetric matrices I_5 as well as with a matrix whose entries are all 1.

6. Conclusion

In this work, four iterative methods are introduced for solving Newton steps (9) and (10) over symmetric and BS matrices, respectively. Then we incorporated the iterative methods into Newton's method to find the symmetric and

BS solvents of the quadratic matrix equation and the matrix polynomial. The contributions of this paper are as follows:

1. Our Newton's method can solve $Q(X)$ and $P(X)$ even when the Fréchet derivative is singular;
2. Our Newton's method can find the symmetric and BS solvents for any given symmetric and BS starting matrices, respectively.

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