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SECOND-ORDER UNIVEX FUNCTIONS AND GENERALIZED DUALITY MODELS FOR MULTIOBJECTIVE PROGRAMMING PROBLEMS CONTAINING ARBITRARY NORMS

G. J. Zalmai

Abstract. In this paper, we introduce three new broad classes of second-order generalized convex functions, namely, $(\mathcal{F}, b, \phi, \rho, \theta)$-sounivex functions, $(\mathcal{F}, b, \phi, \rho, \theta)$-pseudosounivex functions, and $(\mathcal{F}, b, \phi, \rho, \theta)$-quasisounivex functions; formulate eight general second-order duality models; and prove appropriate duality theorems under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-sounivexity assumptions for a multiobjective programming problem containing arbitrary norms.

1. Introduction

In this paper, we present a fairly large number of second-order duality results under a variety of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-sounivexity conditions for the following multiobjective programming problem involving nondifferentiable functions:

$$(P) \qquad \text{Minimize } \big(F_1(x) + \|A_1x\|_{a(1)}, \ldots, F_p(x) + \|A_px\|_{a(p)}\big)$$

subject to

$$G_j(x) + \|B_jx\|_{b(j)} \leqq 0,\ j \in \underline{q}, \quad H_k(x) = 0,\ k \in \underline{r}, \quad x \in X,$$

where X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), F_i, $i \in \underline{p} \equiv \{1, 2, \ldots, p\}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are real-valued functions defined on X, for each $i \in \underline{p}$ and each $j \in \underline{q}$, A_i and B_j are, respectively, $m_i \times n$ and $n_j \times n$ matrices, and $\|\cdot\|_{a(i)}$ and $\|\cdot\|_{b(j)}$ are arbitrary norms on $\mathbb{R}^{m_i}$ and $\mathbb{R}^{n_j}$, respectively.

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Second-order duality for a conventional nonlinear programming problem of the form

$$(P_0) \qquad \text{Minimize } f(x) \text{ subject to } g_i(x) \leqq 0, \ i \in \underline{m}, \ x \in \mathbb{R}^n,$$

where f and g_i, $i \in \underline{m}$, are real-valued functions defined on $\mathbb{R}^n$, was initially considered by Mangasarian [24]. The idea underlying his approach to constructing a second-order dual problem was based on taking linear and quadratic approximations of the objective and constraint functions about an arbitrary but fixed point, forming the Wolfe dual of the approximated problem, and then letting the fixed point to vary. More specifically, he formulated the following second-order dual problem for (P_0):

$$(D_0) \qquad \text{Maximize } f(y) + \sum_{i=1}^{m} u_i g_i(y) - \frac{1}{2}\Big\langle z, \Big[\nabla^2 f(y) + \sum_{i=1}^{m} u_i \nabla^2 g_i(y)\Big] z\Big\rangle$$

subject to

$$\nabla f(y) + \sum_{i=1}^{m} u_i \nabla g_i(y) + \Big[\nabla^2 f(y) + \sum_{i=1}^{m} u_i \nabla^2 g_i(y)\Big] z = 0,$$

$$y \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ u \geqq 0, \ z \in \mathbb{R}^n,$$

where $\nabla F(y)$ and $\nabla^2 F(y)$ are, respectively, the gradient and Hessian of the function $F : \mathbb{R}^n \to \mathbb{R}$ evaluated at y, and $\langle a, b\rangle$ denotes the inner (scalar) product of the ν-dimensional vectors a and b, that is, $\langle a, b\rangle = \sum_{i=1}^{\nu} a_i b_i$, where a_i and b_i are the ith components of a and b, respectively. Imposing somewhat complicated conditions on f, g_i, $i \in \underline{m}$, and z, he proved weak, strong, and converse duality theorems for (P_0) and (D_0). Reconsidering Mangasarian's second-order problem, Mond [28] established some duality results under relatively simpler conditions involving a certain second-order generalization of the concept of convexity, pointed out some possible computational advantages of second-order duality results, and also studied a pair of second-order symmetric dual problems. Subsequently, Mond's original notion of second-order convexity was generalized by other authors in different ways and utilized for establishing various second-order duality results for several classes of nonlinear programming problems. For brief accounts of the evolution of these generalized second-order convexity concepts, the reader is referred to [3, 18, 31, 32], and for more information about second- and higher-order duality results, the reader may consult [1-11, 13-18, 20-29, 31-34, 36-39, 42-44].

In this paper, we propose and discuss substantial improvements and extensions of the problem models as well as the related second-order duality results presented earlier in several of the above-mentioned publications. In particular, our results generalize those obtained previously by Aghezzaf [1]. More specifically, we consider a much more general multiobjective optimization problem than the one studied by Aghezzaf in that our problem includes both equality and inequality constraints, and contains arbitrary norms in the objective functions and inequality constraints, which, in turn, subsumes multiobjective nonlinear programming problems involving square roots of positive semidefinite quadratic forms; we formulate and discuss eight general second-order duality models, whereas Aghezzaf considers only one dual problem; the convexity conditions (generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-sounivexity) under which duality theorems are proved in our paper are much more general than those (generalized (F, ρ)-convexity) used in his paper; we present a great variety of second-order duality results most of which are not even mentioned in his paper; and our paper contains a much more complete list of references dealing with various second-order duality issues.

The rest of this paper is organized as follows. In Section 2, we present a few definitions and auxiliary results which will be needed in the sequel. Utilizing a partitioning scheme, in Section 3 we formulate four general second-order duality models and prove weak, strong, and strict converse duality theorems under a great variety of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-sounivexity hypotheses. We continue our discussion of duality in Section 4 where we use another partitioning method and construct four additional generalized duality models and obtain several duality results under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-sounivexity assumptions. In fact, each one of the eight duality models discussed in Sections 3 and 4 is a family of dual problems for (P) whose members can easily be identified by appropriate choices of certain sets and functions. Finally, in Section 5 we summarize our main results and also point out some further research opportunities arising from certain modifications of the principal problem model studied in this paper.

It is evident that all the second-order duality results obtained for (P) are also applicable, when appropriately specialized, to the following five classes of problems with single and multiple objective functions, which are particular cases of (P):

$$(P1) \qquad \underset{x \in \mathbb{F}}{\text{Minimize}} \;\; F_1(x) + \|A_1 x\|_{a(1)},$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in X : G_j(x) + \|B_j x\|_{b(j)} \leqq 0, \; j \in \underline{q}, \quad H_k(x) = 0, \;\; k \in \underline{r}\};$$

$$(P2) \qquad \underset{x \in \mathbb{G}}{\text{Minimize}} \;\; \Big(F_1(x) + \langle x, R_1 x\rangle^{1/2}, \ldots, F_p(x) + \langle x, R_p x\rangle^{1/2}\Big);$$

subject to

$$G_j(x) + \langle x, S_j x\rangle^{1/2} \leqq 0, \;\; j \in \underline{q}, \quad H_k(x) = 0, \;\; k \in \underline{r}, \quad x \in X,$$

where $R_i, \; i \in \underline{p}$, and $S_j, \; j \in \underline{q}$, are $n \times n$ symmetric positive semidefinite matrices;

$$(P3) \qquad \underset{x \in \mathbb{G}}{\text{Minimize}} \;\; F_1(x) + \langle x, R_1 x\rangle^{1/2},$$

where $\mathbb{G}$ is the feasible set of $(P2)$, that is,

$$\mathbb{G} = \{x \in X : G_j(x) + \langle x, S_j x\rangle^{1/2} \leqq 0,\ \ j \in \underline{q}, \quad H_k(x) = 0,\ \ k \in \underline{r}\};$$

$$(P4) \qquad \underset{x\in\mathbb{H}}{\text{Minimize}}\ \big(F_1(x), \ldots, F_p(x)\big);$$

$$(P5) \qquad \underset{x\in\mathbb{H}}{\text{Minimize}}\ F_1(x),$$

where $\mathbb{H} = \{x \in X : G_j(x) \leqq 0,\ j \in \underline{q},\ \ H_k(x) = 0,\ k \in \underline{r}\}$.

The problems $(P2)$ and $(P3)$ are special cases of (P) and $(P1)$, respectively, which are obtained by choosing $\|\cdot\|_{a(i)}$, $i \in \underline{p}$, and $\|\cdot\|_{b(j)}$, $j \in \underline{q}$, to be the ℓ_2-norm $\|\cdot\|_2$, and defining $R_i = A_i^T A_i$, $i \in \underline{p}$, and $S_j = B_j^T B_j$, $j \in \underline{q}$.

Since in most cases the duality results established for (P) can easily be modified and restated for each one of the above five problems, we shall not state them explicitly.

Optimization problems containing norms arise naturally in many areas of the decision sciences, applied mathematics, and engineering. They are encountered most frequently in facility location problems, approximation theory, and engineering design. A number of these problems have already been investigated in the related literature. Likewise, optimization problems involving square roots of positive semidefinite quadratic forms have arisen in stochastic programming, multifacility location problems, and portfolio selection problems, among others. A fairly extensive list of references pertaining to several aspects of these two classes of problems is given in [40].

2. Preliminaries

In this section, we define some new classes of generalized **s**econd-**o**rder **univex** functions, called **sounivex** for short, which will be utilized for formulating and proving our duality theorems in the sequel. They contain as special cases a fairly large number of generalized convex functions, including, of course, univex functions [12], proposed previously and used for establishing various duality results for several types of nonlinear programming problems. In particular, they may be viewed as further extensions of the second-order generalized convex functions defined in [3] where the reader will also find numerous references dealing with many kinds of generalized convex functions.

Let $x^* \in \mathbb{R}^n$ and assume that the function $f : X \to \mathbb{R}$ is twice differentiable at x^*.

Definition 2.1. The function f is said to be (*strictly*) $(\mathcal{F}, b, \phi, \rho, \theta)$-*sounivex* at x^* if there exist functions $b : X \times X \to \mathbb{R}_+\backslash\{0\} \equiv (0, \infty)$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X (x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\phi\big(f(x) - f(x^*) + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle\big) (>) \geqq \mathcal{F}\big(x, x^*; b(x, x^*)[\nabla f(x^*) + \nabla^2 f(x^*) z]\big)$$

$$+ \rho(x, x^*)\|\theta(x, x^*)\|^2,$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

Definition 2.2. The function f is said to be (*strictly*) $(\mathcal{F}, b, \phi, \rho, \theta)$-*pseudosounivex* at x^* if there exist functions $b : X \times X \to \mathbb{R}_+\backslash\{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X (x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\mathcal{F}\big(x, x^*; b(x, x^*)[\nabla f(x^*) + \nabla^2 f(x^*)z]\big) \geqq -\rho(x, x^*)\|\theta(x, x^*)\|^2$$
$$\Rightarrow \phi\big(f(x) - f(x^*) + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle\big)(>) \geqq 0.$$

Definition 2.3. The function f is said to be (*prestrictly*) $(\mathcal{F}, b, \phi, \rho, \theta)$-*quasisounivex* at x^* if there exist functions $b : X \times X \to \mathbb{R}_+\backslash\{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\phi\big(f(x) - f(x^*) + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle\big)(<) \leqq 0$$
$$\Rightarrow \mathcal{F}\big(x, x^*; b(x, x^*)[\nabla f(x^*) + \nabla^2 f(x^*)z]\big) \leqq -\rho(x, x^*)\|\theta(x, x^*)\|^2.$$

From the above definitions it is clear that if f is $(\mathcal{F}, b, \phi, \rho, \theta)$-sounivex at x^*, then it is both $(\mathcal{F}, b, \phi, \rho, \theta)$-pseudosounivex and $(\mathcal{F}, b, \phi, \rho, \theta)$-quasisounivex at x^*, if f is $(\mathcal{F}, b, \phi, \rho, \theta)$-quasisounivex at x^*, then it is prestrictly $(\mathcal{F}, b, \phi, \rho, \theta)$-quasisounivex at x^*, and if f is strictly $(\mathcal{F}, b, \phi, \rho, \theta)$-pseudosounivex at x^*, then it is $(\mathcal{F}, b, \phi, \rho, \theta)$-quasisounivex at x^*.

In the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\mathcal{F}, b, \phi, \rho, \theta)$-quasisounivexity can be defined in the following equivalent way: f is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$-quasisounivex at x^* if there exist functions $b : X \times X \to \mathbb{R}_+\backslash\{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\mathcal{F}\big(x, x^*; b(x, x^*)[\nabla f(x^*) + \nabla^2 f(x^*)z]\big) > -\rho(x, x^*)\|\theta(x, x^*)\|^2$$
$$\Rightarrow \phi\big(f(x) - f(x^*) + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle\big) > 0.$$

Needless to say that the new classes of generalized convex functions specified in Definitions 2.1-2.3 contain a great variety of special cases that can easily be identified by appropriate choices of the functions $\mathcal{F}, b, \phi, \rho$, and θ.

In the remainder of this section, we recall a set of necessary efficiency conditions for (P) which will play an important role in the construction and analysis of the dual problems that will be discussed in this paper. We begin by introducing a consistent notation for vector inequalities. For $a, b \in \mathbb{R}^m$, the following order notation will be used: $a \geqq b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$; $a \geqslant b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$, but $a \neq b$; $a > b$ if and only if $a_i > b_i$ for all $i \in \underline{m}$; and $a \ngeqslant b$ is the negation of $a \geqslant b$.

Consider the multiobjective problem

$$(P^*) \qquad \underset{x\in\mathcal{X}}{\text{Minimize}}\ f(x)=(f_1(x),\dots,f_p(x)),$$

where $f_i,\ i\in \underline{p}$, are real-valued functions defined on the set $\mathcal{X}$.

An element $x^\circ\in\mathcal{X}$ is said to be an *efficient* (*Pareto optimal, nondominated, noninferior*) solution of (P^*) if there exists no $x\in\mathcal{X}$ such that $f(x)\leqslant f(x^\circ)$.

The following necessary efficiency result will be needed in the sequel for proving strong duality theorems.

Theorem 2.1 ([41]). *Let x^* be a normal efficient solution of (P), let $\lambda_i^*=\varphi_i(x^*),\ i\in\underline{p}$, and assume that the functions $f_i,\ g_i,\ i\in\underline{p},\ G_j,\ j\in\underline{q}$, and $H_k,\ k\in\underline{r}$ are continuously differentiable at x^*. Then there exist $u^*\in U,\ v^*\in\mathbb{R}^q_+,\ w^*\in\mathbb{R}^r,\ \alpha^{*i}\in\mathbb{R}^{m_i},\ i\in\underline{p}$, and $\beta^{*j}\in\mathbb{R}^{n_j},\ j\in\underline{q}$, such that*

$$\sum_{i=1}^p u_i^*[\nabla F_i(x^*)+A_i^T\alpha^{*i}]+\sum_{j=1}^q v_j^*[\nabla G_j(x^*)+B_j^T\beta^{*j}]+\sum_{k=1}^r w_k^*\nabla H_k(x^*)=0,$$

$$v_j^*[G_j(x^*)+\|B_jx^*\|_{b(j)}]=0,\ \ j\in\underline{q},$$

$$\|\alpha^{*i}\|^*_{a(i)}\leqq 1,\ \ i\in\underline{p},$$

$$\|\beta^{*j}\|^*_{b(j)}\leqq 1,\ \ j\in\underline{q},$$

$$\langle\alpha^{*i},A_ix^*\rangle=\|A_ix^*\|_{a(i)},\ \ i\in\underline{p},$$

$$\langle\beta^{*j},B_jx^*\rangle=\|B_jx^*\|_{b(j)},\ \ j\in\underline{q},$$

where $\mathbb{R}^q_+=\{v\in\mathbb{R}^q: v\geqq 0\}$, $U=\{u\in\mathbb{R}^p: u>0,\ \sum_{i=1}^p u_i=1\}$, and $\|\cdot\|^*_a$ is the dual to the norm $\|\cdot\|_a$, that is, $\|\delta\|^*_a=\max_{\|\xi\|_a=1}|\langle\delta,\xi\rangle|$.

In the remainder of this paper, we shall assume that the functions F_i, $i\in\underline{p}$, G_j, $j\in\underline{q}$, and H_k, $k\in\underline{r}$, are twice continuously differentiable on the open set X.

3. Duality model I

In this section, we discuss four families of second-order duality models and establish appropriate duality results under various generalized $(\mathcal{F},b,\phi,\rho,\theta)$-sounivexity hypotheses imposed on certain combinations of the problem functions. This is accomplished by employing a certain partitioning scheme which was originally proposed in [30] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let $\{J_0,J_1,\dots,J_m\}$ and $\{K_0,K_1,\dots,K_m\}$ be partitions of the index sets $\underline{q}$ and $\underline{r}$, respectively; thus, $J_\mu\subseteq\underline{q}$ for each $\mu\in\underline{m}\cup\{0\}$, $J_\mu\cap J_\nu=\emptyset$ for each $\mu,\nu\in\underline{m}\cup\{0\}$ with $\mu\neq\nu$, and $\cup_{\mu=0}^m J_\mu=\underline{q}$. Obviously, similar properties hold for $\{K_0,K_1,\dots,K_m\}$. Moreover, if m_1 and m_2 are the numbers of the partitioning sets of $\underline{q}$ and $\underline{r}$, respectively, then $m = \max\{m_1, m_2\}$ and $J_\mu = \emptyset$ or $K_\mu = \emptyset$ for $\mu > \min\{m_1, m_2\}$

In addition, we use the real-valued functions $\Phi_i(\cdot, v, w, \alpha, \beta)$, $i \in \underline{p}$, $\Phi(\cdot, u, v, w, \alpha, \beta)$, and $\Lambda_t(\cdot, v, w)$ defined, for fixed $u, v, w, \alpha = (\alpha^1, \ldots, \alpha^p)$, and $\beta = (\beta^1, \ldots, \beta^q)$, on X as follows:

$$\begin{aligned}
\Phi_i(x, v, w, \alpha, \beta) &= F_i(x) + \langle \alpha^i, A_i x\rangle + \sum_{j \in J_0} v_j [G_j(x) + \langle \beta^j, B_j x\rangle] \\
&\quad + \sum_{k \in K_0} w_k H_k(x), \quad i \in \underline{p}, \\
\Phi(x, u, v, w, \alpha, \beta) &= \sum_{i=1}^{p} u_i [F_i(x) + \langle \alpha^i, A_i x\rangle] + \sum_{j \in J_0} v_j [G_j(x) \\
&\quad + \langle \beta^j, B_j x\rangle] + \sum_{k \in K_0} w_k H_k(x), \\
\Lambda_t(x, v, w, \beta) &= \sum_{j \in J_t} v_j [G_j(x) + \langle \beta^j, B_j x\rangle] + \sum_{k \in K_t} w_k H_k(x), \quad t \in \underline{m}.
\end{aligned}$$

Making use of the sets and functions defined above, we can state our general second-order duality models as follows:

$$(\mathbf{CI}) \quad \text{Maximize } \xi^I(y, z, u, v, w, \alpha, \beta) = \big(\xi_1^I(y, z, u, v, w, \alpha, \beta), \ldots, \xi_p^I(y, z, u, v, w, \alpha, \beta)\big)$$

subject to

$$\begin{aligned}
&\sum_{i=1}^{p} u_i [\nabla F_i(y) + A_i^T \alpha^i] + \sum_{j=1}^{q} v_j [\nabla G_j(y) + B_j^T \beta^j] + \sum_{k=1}^{r} w_k \nabla H_k(y) \\
&+ \Big[\sum_{i=1}^{p} u_i \nabla^2 F_i(y) + \sum_{j=1}^{q} v_j \nabla^2 G_j(y) + \sum_{k=1}^{r} w_k \nabla^2 H_k(y)\Big] z = 0,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\sum_{j \in J_t} v_j [G_j(y) + \|B_j y\|_{b(j)}] + \sum_{k \in K_t} w_k H_k(y) \\
&- \frac{1}{2}\Big\langle z, \Big[\sum_{j \in J_t} v_j \nabla^2 G_j(y) + \sum_{k \in K_t} w_k \nabla^2 H_k(y)\Big] z\Big\rangle \geqq 0, \quad t \in \underline{m},
\end{aligned} \tag{3.2}$$

$$\|\alpha^i\|_{a(i)}^* \leqq 1, \quad i \in \underline{p}, \tag{3.3}$$

$$\|\beta^j\|_{b(j)}^* \leqq 1, \quad j \in \underline{q}, \tag{3.4}$$

$$\langle \alpha^i, A_i y\rangle = \|A_i y\|_{a(i)}, \quad i \in \underline{p}, \tag{3.5}$$

$$\langle \beta^j, B_j y\rangle = \|B_j y\|_{b(j)}, \quad j \in \underline{q}, \tag{3.6}$$

(3.7)

$$y \in X,\ z \in \mathbb{R}^n,\ u \in U,\ v \in \mathbb{R}^q_+,\ w \in \mathbb{R}^r,\ \alpha^i \in \mathbb{R}^{m_i},\ i \in \underline{p},\ \beta^j \in \mathbb{R}^{n_j},\ j \in \underline{q},$$

where

$$\begin{aligned}&\xi_i^I(y,z,u,v,w,\alpha,\beta)\\ &= F_i(y) + \|A_i y\|_{a(i)} + \sum_{j\in J_0} v_j[G_j(y) + \|B_j y\|_{b(j)}] + \sum_{k\in K_0} w_k H_k(y)\\ &\quad - \frac{1}{2}\Big\langle z, \Big[\nabla^2 F_i(y) + \sum_{j\in J_0} v_j \nabla^2 G_j(y) + \sum_{k\in K_0} w_k \nabla^2 H_k(y)\Big] z\Big\rangle,\ \ i \in \underline{p};\end{aligned}$$

($\tilde{\mathbf{C}}\mathbf{I}$) Maximize $\xi^I(y,z,u,v,w,\alpha,\beta) = \big(\xi_1^I(y,z,u,v,w,\alpha,\beta),\dots,\xi_p^I(y,z,u,v,w,\alpha,\beta)\big)$

subject to (3.2)-(3.7) and

(3.8)

$$\begin{aligned}&\mathcal{F}\Big(x,y;\sum_{i=1}^p u_i[\nabla F_i(y) + A_i^T\alpha^i] + \sum_{j=1}^q v_j[\nabla G_j(y) + B_j^T\beta^j]\\ &+ \sum_{k=1}^r w_k \nabla H_k(y) + \Big[\sum_{i=1}^p u_i \nabla^2 F_i(y) + \sum_{j=1}^q v_j \nabla^2 G_j(y) + \sum_{k=1}^r w_k \nabla^2 H_k(y)\Big] z\Big)\\ &\geqq 0 \quad \text{for all } x \in \mathbb{F},\end{aligned}$$

where $\mathcal{F}(x,y;\cdot)$ is a sublinear function from $\mathbb{R}^n$ to $\mathbb{R}$;

(**DI**) Maximize $\psi^I(y,z,u,v,w,\alpha,\beta) = \big(\psi_1^I(y,z,u,v,w,\alpha,\beta),\dots,\psi_p^I(y,z,u,v,w,\alpha,\beta)\big)$

subject to

$$\begin{aligned}&\sum_{i=1}^p u_i[\nabla F_i(y) + A_i^T\alpha^i] + \sum_{j=1}^q v_j[\nabla G_j(y) + B_j^T\beta^j] + \sum_{k=1}^r w_k \nabla H_k(y)\\ &+ \Big[\sum_{i=1}^p u_i \nabla^2 F_i(y) + \sum_{j=1}^q v_j \nabla^2 G_j(y) + \sum_{k=1}^r w_k \nabla^2 H_k(y)\Big] z = 0,\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\sum_{j\in J_t} v_j[G_j(y) + \langle\beta^j, B_j y\rangle] + \sum_{k\in K_t} w_k H_k(y)\\ &- \frac{1}{2}\Big\langle z, \Big[\sum_{j\in J_t} v_j \nabla^2 G_j(y) + \sum_{k\in K_t} w_k \nabla^2 H_k(y)\Big] z\Big\rangle \geqq 0, \quad t \in \underline{m},\end{aligned} \tag{3.10}$$

$$\|\alpha^i\|^*_{a(i)} \leqq 1, \quad i \in \underline{p}, \tag{3.11}$$

$$\|\beta^j\|^*_{b(j)} \leqq 1, \quad j \in \underline{q}, \tag{3.12}$$

(3.13)

$$y \in X,\ z \in \mathbb{R}^n,\ u \in U,\ v \in \mathbb{R}^q_+,\ w \in \mathbb{R}^r,\ \alpha^i \in \mathbb{R}^{m_i},\ i \in \underline{p},\ \beta^j \in \mathbb{R}^{n_j},\ j \in \underline{q},$$

where

$$\begin{aligned}\psi_i^I&(y,z,u,v,w,\alpha,\beta)\\ &= F_i(y) + \langle \alpha^i, A_i y\rangle + \sum_{j\in J_0} v_j[G_j(y) + \langle \beta^j, B_j y\rangle] + \sum_{k\in K_0} w_k H_k(y)\\ &\quad - \frac{1}{2}\Big\langle z, \Big[\nabla^2 F_i(y) + \sum_{j\in J_0} v_j \nabla^2 G_j(y) + \sum_{k\in K_0} w_k \nabla^2 H_k(y)\Big] z\Big\rangle,\ \ i \in \underline{p};\end{aligned}$$

($\tilde{\mathbf{DI}}$) Maximize $\psi^I(y,z,u,v,w,\alpha,\beta) = \big(\psi_1^I(y,z,u,v,w,\alpha,\beta),\ldots, \psi_p^I(y,z,u,v,w,\alpha,\beta)\big)$

subject to (3.8) and (3.10)-(3.13).

A careful examination of the structures and properties of (CI) and $(\tilde{C}I)$ (as well as the proofs of the weak and strong duality theorems for (P) - (DI) given below), will readily reveal the fact that the constraints (3.5) and (3.6) are superfluous and their omission will not invalidate the duality relations between (P) and (CI), and (P) and $(\tilde{C}I)$. More specifically, if (3.5) and (3.6) are deleted and the remaining constraints of (CI) and $(\tilde{C}I)$ are modified accordingly, then one obtains the reduced versions (DI) and $(\tilde{D}I)$.

Comparing (DI) and $(\tilde{D}I)$, we see that $(\tilde{D}I)$ is relatively more general than (DI) in the sense that any feasible solution of (DI) is also feasible for $(\tilde{D}I)$, but the converse is not necessarily true. Furthermore, we observe that (3.9) is a system of n equations, whereas (3.8) is a single inequality. Clearly, from a computational point of view, (DI) is preferable to $(\tilde{D}I)$ because of the dependence of (3.8) on the feasible set of (P).

Despite these apparent differences, it turns out that the statements and proofs of all the duality theorems for (P) - (DI) and (P) - $(\tilde{D}I)$ are almost identical and, therefore, we shall consider only the pair (P) - (DI). Similarly, it is easily seen that all of the duality theorems established for (P) - (DI) can readily be altered and restated for (P) - (CI) and (P) - $(\tilde{C}I)$.

In the sequel, we shall make frequent use of the following well-known generalized Cauchy inequality.

Lemma 3.1 ([19]). *For each $a, b \in \mathbb{R}^m$, $\langle a, b\rangle \leqq \|a\|^*\|b\|$.*

The next two theorems show that (DI) is a dual problem for (P).

Theorem 3.1 (Weak Duality). *Let x and $s \equiv (y,z,u,v,w,\alpha,\beta)$ be arbitrary feasible solutions of (P) and (DI), respectively, and assume that any one of the following four sets of hypotheses is satisfied:*

(a) (i) *$\Phi(\cdot,u,v,w,\alpha,\beta)$ is $(\mathcal{F},b,\bar{\phi},\bar{\rho},\theta)$-pseudosounivex at y and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*

(ii) *for each* $t \in \underline{m}$, $\Lambda_t(\cdot, v, w, \beta)$ *is* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-quasisounivex at* y, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$;

(iii) $\bar{\rho}(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) \geqq 0$;

(b) (i) $\Phi(\cdot, u, v, w, \alpha, \beta)$ *is prestrictly* $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$*-quasisounivex at* y *and* $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;

(ii) *for each* $t \in \underline{m}$, $\Lambda_t(\cdot, v, w, \beta)$ *is* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-quasisounivex at* y, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$;

(iii) $\bar{\rho}(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) > 0$;

(c) (i) $\Phi(\cdot, u, v, w, \alpha, \beta)$ *is prestrictly* $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$*-quasisounivex at* y, $\bar{\phi}$ *is strictly increasing, and* $\bar{\phi}(0) = 0$;

(ii) *for each* $t \in \underline{m}$, $\Lambda_t(\cdot, v, w, \beta)$ *is strictly* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-pseudosounivex at* y, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$;

(iii) $\bar{\rho}(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) \geqq 0$;

(d) (i) $\Phi(\cdot, u, v, w, \alpha, \beta)$ *is prestrictly* $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$*-quasisounivex at* y, $\bar{\phi}$ *is strictly increasing, and* $\bar{\phi}(0) = 0$;

(ii) *for each* $t \in \underline{m_1}$, $\Lambda_t(\cdot, v, w, \beta)$ *is* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-quasisounivex at* y, *for each* $t \in \underline{m_2} \neq \emptyset$, $\Lambda_t(\cdot, v, w, \beta)$ *is strictly* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-pseudosounivex at* y, *and for each* $t \in \underline{m}$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0) = 0$, *where* $\{\underline{m_1}, \underline{m_2}\}$ *is a partition of* $\underline{m}$;

(iii) $\bar{\rho}(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) \geqq 0$.

Then $\varphi(x) \nleq \psi^I(s)$.

Proof. (a) It is clear that (3.9) can be expressed as follows:

$$
\begin{aligned}
(3.14) \quad & \sum_{i=1}^{p} u_i[\nabla F_i(y) + A_i^T \alpha^i] + \sum_{j \in J_0} v_j[\nabla G_j(y) + B_j^T \beta^j] + \sum_{k \in K_0} w_k \nabla H_k(y) \\
& + \Big[\sum_{i=1}^{p} u_i \nabla^2 F_i(y) + \sum_{j \in J_0} v_j \nabla^2 G_j(y) + \sum_{k \in K_0} w_k \nabla^2 H_k(y) \Big] z \\
& + \sum_{t=1}^{m} \Big\{ \sum_{j \in J_t} v_j[\nabla G_j(y) + B_j^T \beta^j] + \sum_{k \in K_t} w_k \nabla H_k(y) \\
& + \Big[\sum_{j \in J_t} v_j \nabla^2 G_j(y) + \sum_{k \in K_t} w_k \nabla^2 H_k(y) \Big] z \Big\} = 0.
\end{aligned}
$$

Since for each $t \in \underline{m}$,

$$
\begin{aligned}
\Lambda_t(x, v, w, \beta) = & \sum_{j \in J_t} v_j[G_j(x) + \langle \beta^j, B_j x \rangle] + \sum_{k \in K_t} w_k H_k(x) \\
\leqq & \sum_{j \in J_t} v_j[G_j(x) + \|\beta^j\|_{b(j)}^* \|B_j x\|_{b(j)}] + \sum_{k \in K_t} w_k H_k(x) \\
& \text{(by Lemma 3.1 and the nonnegativity of } v)
\end{aligned}
$$

$$
\begin{aligned}
&\leqq \sum_{j\in J_t} v_j[G_j(x)+\|B_jx\|_{b(j)}]+\sum_{k\in K_t} w_kH_k(x) \quad \text{(by (3.12))}\\
&\leqq 0 \quad \text{(by the primal feasibility of } x\text{)}\\
&\leqq \sum_{j\in J_t} v_j[G_j(y)+\langle\beta^j, B_jy\rangle]+\sum_{k\in K_t} w_kH_k(y)\\
&\quad -\frac{1}{2}\Big\langle z, \Big[\sum_{j\in J_t} v_j\nabla^2G_j(y)+\sum_{k\in K_t} w_k\nabla^2H_k(y)\Big]z\Big\rangle \quad \text{(by (3.10))}\\
&= \Lambda_t(y,v,w,\beta)-\frac{1}{2}\big\langle z, \nabla^2\Lambda_t(y,v,w,\beta)z\big\rangle,
\end{aligned}
$$

and so

$$
\tilde{\phi}_t\Big(\Lambda_t(x,v,w,\beta)-\Lambda_t(y,v,w,\beta)+\frac{1}{2}\big\langle z,\nabla^2\Lambda_t(y,v,w,\beta)z\big\rangle\Big)\leqq 0,
$$

it follows from (ii) that

$$
\begin{aligned}
&\mathcal{F}\Big(x,y;b(x,y)\Big\{\sum_{j\in J_t} v_j[\nabla G_j(y)+B_j^T\beta^j]+\sum_{k\in K_t} w_k\nabla H_k(y)\\
&+\Big[\sum_{j\in J_t} v_j\nabla^2G_j(y)+\sum_{k\in K_t} w_k\nabla^2H_k(y)\Big]z\Big\}\Big)\leqq -\tilde{\rho}_t(x,y)\|\theta(x,y)\|^2.
\end{aligned}
$$

Summing over $t\in \underline{m}$ and using the sublinearity of $\mathcal{F}(x,y;\cdot)$, we obtain

$$
\begin{aligned}
(3.15)\qquad &\mathcal{F}\Big(x,y;b(x,y)\sum_{t=1}^{m}\Big\{\sum_{j\in J_t} v_j[\nabla G_j(y)+B_j^T\beta^j]+\sum_{k\in K_t} w_k\nabla H_k(y)\\
&+\Big[\sum_{j\in J_t} v_j\nabla^2G_j(y)+\sum_{k\in K_t} w_k\nabla^2H_k(y)\Big]z\Big\}\Big)\\
&< -\sum_{t=1}^{m}\tilde{\rho}_t(x,y)\|\theta(x,y)\|^2.
\end{aligned}
$$

Combining (3.14) and (3.15), and using (iii) we get

$$
\begin{aligned}
&\mathcal{F}\Big(x,y;b(x,y)\Big\{\sum_{i=1}^{p} u_i[\nabla F_i(y)+A_i^T\alpha^i]+\sum_{j\in J_0} v_j[\nabla G_j(y)+B_j^T\beta^j]\\
&+\sum_{k\in K_0} w_k\nabla H_k(y)\\
&+\Big[\sum_{i=1}^{p} u_i\nabla^2F_i(y)+\sum_{j\in J_0} v_j\nabla^2G_j(y)+\sum_{k\in K_0} w_k\nabla^2H_k(y)\Big]z\Big\}\Big)\\
&\geqq \sum_{t=1}^{m}\tilde{\rho}_t(x,y)\|\theta(x,y)\|^2\geqq -\bar{\rho}(x,y)\|\theta(x,y)\|^2,
\end{aligned}
$$

which by virtue of (i) implies that

$$\bar{\phi}\Big(\Phi(x,u,v,w,\alpha,\beta)-\Phi(y,\lambda,u,v,w,\alpha,\beta)+\frac{1}{2}\big\langle z,\nabla^2\Phi(y,u,v,w,\alpha,\beta)z\big\rangle\Big)\geqq 0.$$

But $\bar{\phi}(a)\geqq 0 \;\Rightarrow\; a\geqq 0$, and hence

$$\begin{aligned}\Phi(x,u,v,w,\alpha,\beta)&\geqq \Phi(y,u,v,w,\alpha,\beta)-\frac{1}{2}\big\langle z,\nabla^2\Phi(y,u,v,w,\alpha,\beta)z\big\rangle\\&=\sum_{i=1}^{p}u_i\psi_i^I(s)\ \ (\text{since } u>0 \text{ and } \textstyle\sum_{i=1}^{p}u_i=1).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\sum_{i=1}^{p}u_i\psi_i^I(s)\\&\leqq \sum_{i=1}^{p}u_i[F_i(x)+\|\alpha^i\|_{a(i)}^*\|A_ix\|_{a(i)}]+\sum_{j\in J_0}v_j[G_j(x)+\|\beta^j\|_{b(j)}^*\|B_jx\|_{b(j)}]\\&\quad\text{(by Lemma 3.1 and the primal feasibility of } x)\\&\leqq \sum_{i=1}^{p}u_i[F_i(x)+\|A_ix\|_{a(i)}]+\sum_{j\in J_0}v_j[G_j(x)+\|B_jx\|_{b(j)}]\\&\quad\text{(by (3.11) and (3.12))}\\&\leqq \sum_{i=1}^{p}u_i[F_i(x)+\|A_ix\|_{a(i)}]\ \ \text{(by the primal feasibility of } x)\\&=\sum_{i=1}^{p}u_i\varphi_i(x).\end{aligned}$$

Since $u>0$ the above inequality implies that $\varphi(x)\nleq\psi^I(s)$.

(b) The proof is similar to that of part (a).

(c) Suppose to the contrary that $\varphi(x)\leqslant\psi^I(s)$. This implies that for each $i\in \underline{p}$,

$$\begin{aligned}&F_i(x)+\|A_ix\|_{a(i)}\\&\leqq F_i(y)+\langle\alpha^i,A_iy\rangle+\sum_{j\in J_0}v_j[G_j(y)+\langle\beta^j,B_jy\rangle]+\sum_{k\in K_0}w_kH_k(y)\\&\quad-\frac{1}{2}\Big\langle z,\Big[\nabla^2F_i(y)+\sum_{j\in J_0}v_j\nabla^2G_j(y)+\sum_{k\in K_0}w_k\nabla^2H_k(y)\Big]z\Big\rangle,\end{aligned}$$

with strict inequality holding for at least one index $\ell\in\underline{p}$. Since $u>0$ and $\sum_{i=1}^{p}u_i=1$, the above inequalities yield

$$\sum_{i=1}^{p}u_i[F_i(x)+\|A_ix\|_{a(i)}] \tag{3.16}$$

$$< \sum_{i=1}^{p} u_i[F_i(y) + \langle \alpha^i, A_i y\rangle] + \sum_{j\in J_0} v_j[G_j(y) + \langle \beta^j, B_j y\rangle] + \sum_{k\in K_0} w_k H_k(y) - \frac{1}{2}\Big\langle z, \Big[\sum_{i=1}^{p} u_i\nabla^2 F_i(y) + \sum_{j\in J_0} v_j\nabla^2 G_j(y) + \sum_{k\in K_0} w_k\nabla^2 H_k(y)\Big] z\Big\rangle.$$

Keeping in mind that $v \geqq 0$, we see that

$$\begin{aligned}
&\Phi(x, u, v, w, \alpha, \beta)\\
&= \sum_{i=1}^{p} u_i[F_i(x) + \langle \alpha^i, A_i x\rangle] + \sum_{j\in J_0} v_j[G_j(x) + \langle \beta^j, B_j x\rangle] + \sum_{k\in K_0} w_k H_k(x)\\
&\leqq \sum_{i=1}^{p} u_i[F_i(x) + \|\alpha^i\|_{a(i)}^{*}\|A_i x\|_{a(i)}] + \sum_{j\in J_0} v_j[G_j(x) + \|\beta^j\|_{b(j)}^{*}\|B_j x\|_{b(j)}]\\
&\quad \text{(by Lemma 3.1 and the primal feasibility of } x)\\
&\leqq \sum_{i=1}^{p} u_i[F_i(x) + \|A_i x\|_{a(i)}] + \sum_{j\in J_0} v_j[G_j(x) + \|B_j x\|_{b(j)}]\\
&\quad \text{(by (3.11) and (3.12))}\\
&\leqq \sum_{i=1}^{p} u_i[F_i(x) + \|A_i x\|_{a(i)}] \ \text{ (by the primal feasibility of } x)\\
&< \sum_{i=1}^{p} u_i[F_i(y) + \langle \alpha^i, A_i y\rangle] + \sum_{j\in J_0} v_j[G_j(y) + \langle \beta^j, B_j y\rangle] + \sum_{k\in K_0} w_k H_k(y)\\
&\quad - \frac{1}{2}\Big\langle z, \Big[\sum_{i=1}^{p} u_i\nabla^2 F_i(y) + \sum_{j\in J_0} v_j\nabla^2 G_j(y) + \sum_{k\in K_0} w_k\nabla^2 H_k(y)\Big] z\Big\rangle\\
&\quad \text{(by (3.16))}\\
&= \Phi(y, u, v, w, \alpha, \beta) - \frac{1}{2}\big\langle z, \nabla^2\Phi(y, u, v, w, \alpha, \beta) z\big\rangle,
\end{aligned}$$

and so

$$\bar{\phi}\Big(\Phi(x, u, v, w, \alpha, \beta) - \Phi(y, u, v, w, \alpha, \beta) + \frac{1}{2}\big\langle z, \nabla^2\Phi(y, u, v, w, \alpha, \beta) z\big\rangle\Big) < 0,$$

which in view of (i) implies that

$$\mathcal{F}\Big(x, y; b(x, y)\Big\{\sum_{i=1}^{p} u_i[\nabla F_i(y) + A_i^T\alpha^i] + \sum_{j\in J_0} v_j[\nabla G_j(y) + B_j^T\beta^j] + \sum_{k\in K_0} w_k\nabla H_k(y) \tag{3.17}$$

$$+ \Big[\sum_{i=1}^{p} u_i \nabla^2 F_i(y) + \sum_{j \in J_0} v_j \nabla^2 G_j(y) + \sum_{k \in K_0} w_k \nabla^2 H_k(y) \Big] z \Big\} \Big)$$
$$\leqq \ - \bar{\rho}(x,y) \|\theta(x,y)\|^2.$$

Proceeding as in the proof of Theorem 3.1, we obtain

$$\tilde{\phi}_t \Big(\Lambda_t(x,v,w,\beta) - \Lambda_t(y,v,w,\beta) + \frac{1}{2} \big\langle z, \nabla^2 \Lambda_t(y,v,w,\beta) z \big\rangle \Big) \leqq 0,$$

which in view of (ii) implies that

$$\mathcal{F}\Big(x,y; b(x,y) \sum_{t=1}^{m} \Big\{ \sum_{j \in J_t} v_j [\nabla G_j(y) + B_j^T \beta^j] + \sum_{k \in K_t} w_k \nabla H_k(y)$$
$$+ \Big[\sum_{j \in J_t} v_j \nabla^2 G_j(y) + \sum_{k \in K_t} w_k \nabla^2 H_k(y) \Big] z \Big\} \Big)$$
$$< \ - \sum_{t=1}^{m} \tilde{\rho}_t(x,y) \|\theta(x,y)\|^2,$$

which when combined with (3.14), results in

$$\mathcal{F}\Big(x,y; b(x,y) \Big\{ \sum_{i=1}^{p} u_i [\nabla F_i(y) + A_i^T \alpha^i]$$
$$+ \sum_{j \in J_0} v_j [\nabla G_j(y) + B_j^T \beta^j] + \sum_{k \in K_0} w_k \nabla H_k(y)$$
$$+ \Big[\sum_{i=1}^{p} u_i \nabla^2 F_i(y) + \sum_{j \in J_0} v_j \nabla^2 G_j(y) + \sum_{k \in K_0} w_k \nabla^2 H_k(y) \Big] z \Big\} \Big)$$
$$> \sum_{t=1}^{m} \tilde{\rho}_t(x,y) \|\theta(x,y)\|^2.$$

In view of (iii), this inequality contradicts (3.17). Hence, $\varphi(x) \nleq \psi^I(s)$.

(d) The proof is similar to that of part (c). □

Theorem 3.2 (Strong Duality)**.** *Let x^* be a normal efficient solution of (P) and assume that any one of the four sets of conditions set forth in Theorem 3.1 is satisfied for all feasible solutions of (DI). Then there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\beta^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that $s^* \equiv (x^*, z^* = 0, u^*, v^*, w^*, \alpha^*, \beta^*)$ is an efficient solution of (DI) and $\varphi(x^*) = \psi^I(s^*)$.*

Proof. (a) Since x^* is a normal efficient solution of (P), by Theorem 2.1, there exist $u^*, v^*, w^*, \alpha^{*i}$, and β^{*j}, as specified above, such that s^*, is a feasible solution of (DI). If it were not efficient, then there would exist a feasible solution $\hat{s} \equiv (\hat{x}, \hat{z}, \hat{u}, \hat{v}, \hat{w}, \hat{\alpha}, \hat{\beta})$ of (DI) such that $\psi^I(\hat{s}) \geqslant \psi^I(s^*)$. But $\psi^I(s^*) = \varphi(x^*)$ and hence $\psi^I(\hat{s}) \geqslant \varphi(x^*)$, which contradicts Theorem 3.1. Therefore, we conclude that s^* is an efficient solution of (DI).

(b)-(d): The proofs are similar to that of part (a). □

We also have the following converse duality result for (P) and (DI).

Theorem 3.3 (Strict Converse Duality). *Let x^* and $\tilde{s} \equiv (\tilde{x}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta})$ be arbitrary feasible solutions of (P) and (DI), respectively, such that*

$$\sum_{i=1}^{p} \tilde{u}_i \varphi_i(x^*) \leqq \sum_{i=1}^{p} \tilde{u}_i \psi_i^I(\tilde{s}). \tag{3.18}$$

Furthermore, assume that any one of the following four sets of conditions holds:

(a) *The assumptions specified in part* (a) *of Theorem 3.1 are satisfied for the feasible solution $\tilde{s}$ of* (DI)*, the function $\Phi(\cdot, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta})$ is strictly $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-pseudosounivex at $\tilde{x}$, and $\bar{\phi}(a) > 0 \Rightarrow a > 0$.*

(b) *The assumptions specified in part* (b) *of Theorem 3.1 are satisfied for the feasible solution $\tilde{s}$ of* (DI)*, the function $\Phi(\cdot, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta})$ is $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-quasisounivex at $\tilde{x}$, and $\bar{\phi}(a) > 0 \Rightarrow a > 0$.*

(c) *The assumptions specified in part* (c) *of Theorem 3.1 are satisfied for the feasible solution $\tilde{s}$ of* (DI) *and the function $\Phi(\cdot, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta})$ is $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-quasisounivex at $\tilde{x}$.*

(d) *The assumptions specified in part* (d) *of Theorem 3.1 are satisfied for the feasible solution $\tilde{s}$ of* (DI) *and the function $\Phi(\cdot, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta})$ is $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-quasisounivex at $\tilde{x}$. Then $\tilde{x} = x^*$ and $\varphi(x^*) = \psi^I(\tilde{s})$.*

Proof. (a) Suppose to the contrary that $\tilde{x} \neq x^*$. Now proceeding as in the proof of part (a) of Theorem 3.1 (with x replaced by x^* and s by $\tilde{s}$), we arrive at the inequality

$$\mathcal{F}\Big(x^*, \tilde{x}; b(x^*, \tilde{x})\Big\{ \sum_{i=1}^{p} \tilde{u}_i [\nabla F_i(\tilde{x}) + A_i^T \tilde{\alpha}^i] + \sum_{j \in J_0} \tilde{v}_j [\nabla G_j(\tilde{x}) + B_j^T \tilde{\beta}^j]$$
$$+ \sum_{k \in K_0} \tilde{w}_k \nabla H_k(\tilde{x}) + \Big[\sum_{i=1}^{p} \tilde{u}_i \nabla^2 F_i(\tilde{x}) + \sum_{j \in J_0} \tilde{v}_j \nabla^2 G_j(\tilde{x}) + \sum_{k \in K_0} \tilde{w}_k \nabla^2 H_k(\tilde{x}) \Big] \tilde{z} \Big\}\Big)$$
$$\geqq - \bar{\rho}(x^*, \tilde{x}) \|\theta(x^*, \tilde{x})\|^2,$$

which by virtue of our strict $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-pseudosounivexity assumption implies that

$$\bar{\phi}\Big(\Phi(x^*, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}) - \Phi(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}) + \frac{1}{2}\big\langle \tilde{z}, \nabla^2 \Phi(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}) \tilde{z} \big\rangle\Big) > 0.$$

But $\bar{\phi}(a) > 0 \Rightarrow a > 0$, and hence we get

$$\Phi(x^*, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}) > \Phi(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}) - \frac{1}{2}\big\langle \tilde{z}, \nabla^2 \Phi(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}) \tilde{z} \big\rangle.$$

Following the steps used in the proof of part (a) of Theorem 3.1, it can be shown that this inequality yields

$$\sum_{i=1}^{p} \tilde{u}_i \varphi_i(x^*) > \sum_{i=1}^{p} \tilde{u}_i \psi_i^I(\tilde{s}),$$

which contradicts (3.18). Hence $\tilde{x} = x^*$ and $\varphi(x^*) = \psi^I(\tilde{s})$.

(b)-(d): The proofs are similar to that of part (a). □

Theorem 3.4 (Weak Duality). *Let x and $s \equiv (y, z, u, v, w, \alpha, \beta)$ be arbitrary feasible solutions of (P) and (DI), respectively, and assume that any one of the following seven sets of hypotheses is satisfied*:

(a) (i) *for each $i \in \underline{p}$, $\Phi_i(\cdot, v, w, \alpha, \beta)$ is strictly $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudosounivex at y, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v, w, \beta)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasisounivex at y, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) \geqq 0$;

(b) (i) *for each $i \in \underline{p}$, $\Phi_i(\cdot, v, w, \alpha, \beta)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasisounivex at y, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v, w, \beta)$ is strictly $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-pseudosounivex at y, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) \geqq 0$;

(c) (i) *for each $i \in \underline{p}$, $\Phi_i(\cdot, v, w, \alpha, \beta)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasisounivex at y, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v, w, \beta)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasisounivex at y, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) > 0$;

(d) (i) *for each $i \in \underline{p_1}$, $\Phi_i(\cdot, v, w, \alpha, \beta)$ is strictly $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudosounivex at y, for each $i \in \underline{p_2}$, $\Phi_i(\cdot, v, w, \alpha, \beta)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasisounivex at y, and for each $i \in \underline{p}$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0) = 0$, where $\{\underline{p_1}, \underline{p_2}\}$ is a partition of $\underline{p}$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v, w, \beta)$ is strictly $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-pseudosounivex at y, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) \geqq 0$;

(e) (i) *for each $i \in \underline{p_1} \neq \emptyset$, $\Phi_i(\cdot, v, w, \alpha, \beta)$ is strictly $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudosounivex at y, for each $i \in \underline{p_2}$, $\Phi_i(\cdot, v, w, \alpha, \beta)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasisounivex at y, and for each $i \in \underline{p}$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0) = 0$, where $\{\underline{p_1}, \underline{p_2}\}$ is a partition of $\underline{p}$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v, w, \beta)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasisounivex at y, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) \geqq 0$;

(f) (i) *for each $i \in \underline{p}$, $\Phi_i(\cdot, v, w, \alpha, \beta)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasisounivex at y, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each* $t \in \underline{m_1} \neq \emptyset$, $\Lambda_t(\cdot, v, w, \beta)$ *is strictly* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-*pseudosounivex at* y, *for each* $t \in \underline{m_2}$, $\Lambda_t(\cdot, v, w, \beta)$ *is* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-*quasisounivex at* y, *and for each* $t \in \underline{m}$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0) = 0$, *where* $\{\underline{m_1}, \underline{m_2}\}$ *is a partition of* $\underline{m}$;

(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) \geqq 0$;

(g) (i) *for each* $i \in \underline{p_1}$, $\Phi_i(\cdot, v, w, \alpha, \beta)$ *is strictly* $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-*pseudosounivex at* y, *for each* $i \in \underline{p_2}$, $\Phi_i(\cdot, v, w, \alpha, \beta)$ *is* $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-*quasisounivex at* y, *and for each* $i \in \underline{p}$, $\bar{\phi}_i$ *is increasing and* $\bar{\phi}_i(0) = 0$, *where* $\{\underline{p_1}, \underline{p_2}\}$ *is a partition of* $\underline{p}$;

(ii) *for each* $t \in \underline{m_1}$, $\Lambda_t(\cdot, v, w, \beta)$ *is strictly* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-*pseudosounivex at* y, *for each* $t \in \underline{m_2}$, $\Lambda_t(\cdot, v, w, \beta)$ *is* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-*quasisounivex at* y, *and for each* $t \in \underline{m}$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0) = 0$, *where* $\{\underline{m_1}, \underline{m_2}\}$ *is a partition of* $\underline{m}$;

(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) \geqq 0$;

(iv) $\underline{p_1} \neq \emptyset$, $\underline{m_1} \neq \emptyset$, *or* $\sum_{i=1}^{p} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{m} \tilde{\rho}_t(x, y) > 0$.

Then $\varphi(x) \nleqslant \psi^I(s)$.

Proof. (a) Suppose to the contrary that $\varphi(x) \leqslant \psi^I(s)$. This implies that for each $i \in \underline{p}$,

(3.19)

$$F_i(x) + \|A_i x\|_{a(i)} - \Big\{ F_i(y) + \langle \alpha^i, A_i y \rangle + \sum_{j \in J_0} v_j [G_j(y) + \langle \beta^j, B_j y \rangle] + \sum_{k \in K_0} w_k H_k(y) - \frac{1}{2} \Big\langle z, \Big[\nabla^2 F_i(y) + \sum_{j \in J_0} v_j \nabla^2 G_j(y) + \sum_{k \in K_0} w_k \nabla^2 H_k(y) \Big] z \Big\rangle \Big\} \leqq 0,$$

with strict inequality holding for at least one index $\ell \in \underline{p}$. Keeping in mind that $v \geqq 0$, we see that

$$\begin{aligned}
&\Phi_i(x, v, w, \alpha, \beta) \\
&= F_i(x) + \langle \alpha^i, A_i x \rangle + \sum_{j \in J_0} v_j [G_j(x) + \langle \beta^j, B_j x \rangle] + \sum_{k \in K_0} w_k H_k(x) \\
&\leqq F_i(x) + \|\alpha^i\|^*_{a(i)} \|A_i x\|_{a(i)} + \sum_{j \in J_0} v_j [G_j(x) + \|\beta^j\|^*_{b(j)} \|B_j x\|_{b(j)}] \\
&\quad \text{(by Lemma 3.1 and the primal feasibility of } x) \\
&\leqq F_i(x) + \|A_i x\|_{a(i)} + \sum_{j \in J_0} v_j [G_j(x) + \|B_j x\|_{b(j)}] \ \text{(by (3.11) and (3.12))} \\
&\leqq F_i(x) + \|A_i x\|_{a(i)} \ \text{(by the primal feasibility of } x) \\
&\leqq F_i(y) + \langle \alpha^i, A_i y \rangle + \sum_{j \in J_0} v_j [G_j(y) + \langle \beta^j, B_j y \rangle] + \sum_{k \in K_0} w_k H_k(y)
\end{aligned}$$

$$- \frac{1}{2}\Big\langle z, \Big[\nabla^2 F_i(y) + \sum_{j\in J_0} v_j \nabla^2 G_j(y) + \sum_{k\in K_0} w_k \nabla^2 H_k(y)\Big] z\Big\rangle \text{ (by (3.19))}$$
$$= \Phi_i(y,v,w,\alpha,\beta) - \frac{1}{2}\big\langle z, \nabla^2 \Phi_i(y,v,w,\alpha,\beta) z\big\rangle,$$

and so

$$\bar{\phi}_i\Big(\Phi_i(x,v,w,\alpha,\beta) - \Phi_i(y,v,w,\alpha,\beta) + \frac{1}{2}\big\langle z, \nabla^2 \Phi_i(y,v,w,\alpha,\beta) z\big\rangle\Big) \leqq 0,$$

which in view of (i) implies that for each $i \in \underline{p}$,

$$\mathcal{F}\Big(x,y;b(x,y)\Big\{\nabla F_i(y) + A_i^T \alpha^i + \sum_{j\in J_0} v_j[\nabla G_j(y) + B_j^T \beta^j]$$
$$+ \sum_{k\in K_0} w_k \nabla H_k(y) + \Big[\nabla^2 F_i(y) + \sum_{j\in J_0} v_j \nabla^2 G_j(y) + \sum_{k\in K_0} w_k \nabla^2 H_k(y)\Big] z\Big\}\Big)$$
$$< -\bar{\rho}_i(x,y)\|\theta(x,y)\|^2.$$

Since $u > 0$, $\sum_{i=1}^p u_i = 1$, and $\mathcal{F}(x,y;\cdot)$ is sublinear, the above inequalities yield

$$\mathcal{F}\Big(x,y;b(x,y)\Big\{\sum_{i=1}^p u_i[\nabla F_i(y) + A_i^T \alpha^i] \tag{3.20}$$
$$+ \sum_{j\in J_0} v_j[\nabla G_j(y) + B_j^T \beta^j] + \sum_{k\in K_0} w_k \nabla H_k(y)$$
$$+ \Big[\sum_{i=1}^p u_i \nabla^2 F_i(y) + \sum_{j\in J_0} v_j \nabla^2 G_j(y) + \sum_{k\in K_0} w_k \nabla^2 H_k(y)\Big] z\Big\}\Big)$$
$$< -\sum_{i=1}^p u_i \bar{\rho}_i(x,y)\|\theta(x,y)\|^2.$$

As seen in the proof of Theorem 3.1, our assumptions in (ii) lead to

$$\mathcal{F}\Big(x,y;b(x,y)\sum_{t=1}^m \Big\{\sum_{j\in J_t} v_j[\nabla G_j(y) + B_j^T \beta^j] + \sum_{k\in K_t} w_k \nabla H_k(y)$$
$$+ \Big[\sum_{j\in J_t} v_j \nabla^2 G_j(y) + \sum_{k\in K_t} w_k \nabla^2 H_k(y)\Big] z\Big\}\Big)$$
$$\leqq -\sum_{t=1}^m \tilde{\rho}_t(x,y)\|\theta(x,y)\|^2,$$

which when combined with (3.14), results in

$$\mathcal{F}\Big(x,y;b(x,y)\Big\{\sum_{i=1}^p u_i[\nabla F_i(y) + A_i^T \alpha^i]$$

$$+ \sum_{j\in J_0} v_j[\nabla G_j(y) + B_j^T \beta^j] + \sum_{k\in K_0} w_k \nabla H_k(y)$$
$$+ \Big[\sum_{i=1}^{p} u_i \nabla^2 F_i(y) + \sum_{j\in J_0} v_j \nabla^2 G_j(y) + \sum_{k\in K_0} w_k \nabla^2 H_k(y)\Big] z\Big\}\Big)$$
$$\geqq \sum_{t=1}^{m} \tilde{\rho}_t(x,y)\|\theta(x,y)\|^2.$$

In view of (iii), this inequality contradicts (3.20). Hence, $\varphi(x) \nleq \psi^I(s)$.

(b)-(g): The proofs are similar to that of part (a). □

Theorem 3.5 (Strong Duality)**.** *Let x^* be a normal efficient solution of (P) and assume that any one of the seven sets of conditions set forth in Theorem 3.4 is satisfied for all feasible solutions of (DI). Then there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{m_i}$, and $\beta^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that $s^* \equiv (x^*, z^* = 0, u^*, v^*, w^*, \alpha^*, \beta^*)$ is an efficient solution of (DI) and $\varphi(x^*) = \psi^I(s^*)$.*

Proof. The proof is similar to that of Theorem 3.2. □

4. Duality model II

In this section we discuss four additional second-order duality models for (P) which are different from those presented in the preceding section. These duality models may be viewed as extensions of the first-order dual problem considered previously in [35]. In these duality formulations we utilize a partition of $\underline{p}$ in addition to those of $\underline{q}$ and $\underline{r}$. In our duality theorems, we impose appropriate generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-sounivexity requirements on certain combinations of the problem functions.

Let $\{I_0, I_1, \ldots, I_\ell\}$ be a partition of $\underline{p}$ such that $L = \{0, 1, 2, \ldots, \ell\} \subseteq M = \{0, 1, \ldots, m\}$, and let the function $\Pi_t(\cdot, u, v, w, \alpha, \beta) : X \to \mathbb{R}$ be defined, for fixed u, v, w, α, and β, by

$$\Pi_t(x, u, v, w, \alpha, \beta) = \sum_{i\in I_t} u_i[f_i(x) + \langle \alpha^i, A_i x\rangle] + \sum_{j\in J_t} v_j[G_j(x) + \langle \beta^j, B_j x\rangle]$$
$$+ \sum_{k\in K_t} w_k H_k(x), \quad t \in L.$$

Consider the following dual problems:

$$\textbf{(CII)} \quad \text{Maximize} \quad \xi^{II}(y, z, u, v, w, \alpha, \beta) = (\xi_1^{II}(y, z, u, v, w, \alpha, \beta), \ldots, \xi_p^{II}(y, z, u, v, w, \alpha, \beta))$$

subject to (3.1), (3.3)-(3.7), and

(4.1)

$$\sum_{j\in J_t} v_j[G_j(y) + \|B_j y\|_{b(j)}] + \sum_{k\in K_t} w_k H_k(y)$$

$$-\frac{1}{2}\Big\langle z,\Big[\sum_{i\in I_t}u_i\nabla^2F_i(y)+\sum_{j\in J_t}v_j\nabla^2G_j(y)+\sum_{k\in K_t}w_k\nabla^2H_k(y)\Big]z\Big\rangle\geqq 0,\quad t\in L,$$

$$\sum_{j\in J_t}v_j[G_j(y)+\|B_jy\|_{b(j)}]+\sum_{k\in K_t}w_kH_k(y) -\frac{1}{2}\Big\langle z,\Big[\sum_{j\in J_t}v_j\nabla^2G_j(y)+\sum_{k\in K_t}w_k\nabla^2H_k(y)\Big]z\Big\rangle\geqq 0,\quad t\in M\backslash L, \tag{4.2}$$

where

$$\xi_i^{II}(y,z,u,v,w,\alpha,\beta)=F_i(y)+\|A_iy\|_{a(i)},\ \ i\in\underline{p};$$

($\tilde{\mathbf{CII}}$) Maximize $\xi^{II}(y,z,u,v,w,\alpha,\beta)=(\xi_1^{II}(y,z,u,v,w,\alpha,\beta),\dots,\xi_p^{II}(y,z,u,v,w,\alpha,\beta))$

subject to (3.3)-(3.8), (4.1), and (4.2);

(**DII**) Maximize $\psi^{II}(y,z,u,v,w,\alpha,\beta)=(\psi_1^{II}(y,z,u,v,w,\alpha,\beta),\dots,\psi_p^{II}(y,z,u,v,w,\alpha,\beta))$

subject to

$$\sum_{i=1}^p u_i[\nabla F_i(y)+A_i^T\alpha^i]+\sum_{j=1}^q v_j[\nabla G_j(y)+B_j^T\beta^j]+\sum_{k=1}^r w_k\nabla H_k(y) +\Big[\sum_{i=1}^p u_i\nabla^2F_i(y)+\sum_{j=1}^q v_j\nabla^2G_j(y)+\sum_{k=1}^r w_k\nabla^2H_k(y)\Big]z=0, \tag{4.3}$$

$$\sum_{j\in J_t}v_j[G_j(y)+\langle\beta^j,B_jy\rangle]+\sum_{k\in K_t}w_kH_k(y) -\frac{1}{2}\Big\langle z,\Big[\sum_{i\in I_t}u_i\nabla^2F_i(y)+\sum_{j\in J_t}v_j\nabla^2G_j(y)+\sum_{k\in K_t}w_k\nabla^2H_k(y)\Big]z\Big\rangle\geqq 0,\quad t\in L, \tag{4.4}$$

$$\sum_{j\in J_t}v_j[G_j(y)+\langle\beta^j,B_jy\rangle]+\sum_{k\in K_t}w_kH_k(y) -\frac{1}{2}\Big\langle z,\Big[\sum_{j\in J_t}v_j\nabla^2G_j(y)+\sum_{k\in K_t}w_k\nabla^2H_k(y)\Big]z\Big\rangle\geqq 0,\quad t\in M\backslash L, \tag{4.5}$$

$$\|\alpha^i\|_{a(i)}^*\leqq 1,\quad i\in\underline{p}, \tag{4.6}$$

$$\|\beta^j\|_{b(j)}^*\leqq 1,\quad j\in\underline{q}, \tag{4.7}$$

$$y\in X,\ z\in\mathbb{R}^n,\ u\in U,\ v\in\mathbb{R}_+^q,\ w\in\mathbb{R}^r,\ \alpha^i\in\mathbb{R}^{m_i},\ \beta^j\in\mathbb{R}^{n_j},\ j\in\underline{q}, \tag{4.8}$$

where

$$\psi_i^{II}(y,z,u,v,w,\alpha,\beta) = F_i(y) + \langle \alpha^i, A_i y\rangle, \quad i \in \underline{p};$$

$$(\tilde{\mathbf{DII}}) \quad \text{Maximize } \psi^{II}(y,z,u,v,w,\alpha,\beta) = (\psi_1^{II}(y,z,u,v,w,\alpha,\beta),\dots, \psi_p^{II}(y,z,u,v,w,\alpha,\beta))$$

subject to (3.8) and (4.4)-(4.8).

The remarks and observations made earlier about the relationships among (CI), $(\tilde{C}I)$, (DI), and $(\tilde{D}I)$ are, of course, also valid for (CII), $(\tilde{C}II)$, (DII), and $(\tilde{D}II)$. As in the preceding section, we shall work with the streamlined versions (DII) and $(\tilde{D}II)$, and, in particular, consider the pair (P) - (DII).

The next two theorems show that (DII) is a dual problem for (P).

Theorem 4.1 (Weak Duality)**.** *Let x and $s \equiv (y,z,u,v,w,\alpha,\beta)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following seven sets of hypotheses is satisfied*:

(a) (i) *for each $t \in L$, $\Pi_t(\cdot,y,u,v,w,\alpha,\beta)$ is strictly $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-pseudosounivex at y, ϕ_t is increasing, and $\phi_t(0)=0$;*
(ii) *for each $t \in M\setminus L$, $\Lambda_t(\cdot,v,w,\beta)$ is $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-quasisounivex at y, ϕ_t is increasing, and $\phi_t(0)=0$;*
(iii) $\sum_{t\in M}\rho_t(x,y) \geqq 0$;

(b) (i) *for each $t \in L$, $\Pi_t(\cdot,y,u,v,w,\alpha,\beta)$ is $(\mathcal{F},b, \phi_t,\rho_t,\theta)$-quasisounivex at y, ϕ_t is increasing, and $\phi_t(0)=0$;*
(ii) *for each $t \in M\setminus L$, $\Lambda_t(\cdot,v,w,\beta)$ is strictly $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-pseudosounivex at y, ϕ_t is increasing, and $\phi_t(0)=0$;*
(iii) $\sum_{t\in M}\rho_t(x,y) \geqq 0$;

(c) (i) *for each $t \in L$, $\Pi_t(\cdot,y,u,v,w,\alpha,\beta)$ is $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-quasisounivex at y, ϕ_t is increasing, and $\phi_t(0)=0$;*
(ii) *for each $t \in M\setminus L$, $\Lambda_t(\cdot,v,w,\beta)$ is $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-quasisounivex at y, ϕ_t is increasing, and $\phi_t(0)=0$;*
(iii) $\sum_{t\in M}\rho_t(x,y) > 0$;

(d) (i) *for each $t \in L_1$, $\Pi_t(\cdot,y,u,v,w,\alpha,\beta)$ is strictly $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-pseudosounivex at y, for each $t \in L_2$, $\Pi_t(\cdot,y,u,v,w,\alpha,\beta)$ is $(\mathcal{F},b, \phi_t,\rho_t,\theta)$-quasisounivex at y, and for each $t\in L$, ϕ_t is increasing and $\phi_t(0)=0$, where $\{L_1,L_2\}$ is a partition of L;*
(ii) *for each $t \in M\setminus L$, $\Lambda_t(\cdot,v,w,\beta)$ is strictly $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-pseudosounivex at y, ϕ_t is increasing, and $\phi_t(0)=0$;*
(iii) $\sum_{t\in M}\rho_t(x,y) \geqq 0$;

(e) (i) *for each $t \in L_1 \neq \emptyset$, $\Pi_t(\cdot,y,u,v,w,\alpha,\beta)$ is strictly $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-pseudosounivex at y, for each $t \in L_2$, $\Pi_t(\cdot,y,u,v,w,\alpha,\beta)$ is $(\mathcal{F},b, \phi_t, \rho_t,\theta)$-quasisounivex at y, and for each $t\in L$, ϕ_t is increasing and $\phi_t(0)=0$, where $\{L_1,L_2\}$ is a partition of L;*

(ii) *for each* $t \in M \setminus L$, $\Lambda_t(\cdot, v, w, \beta)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasisounivex at* y, ϕ_t *is increasing, and* $\phi_t(0) = 0$;

(iii) $\sum_{t\in M} \rho_t(x, y) \geqq 0$;

(f) (i) *for each* $t \in L$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasisounivex at* y, ϕ_t *is increasing, and* $\phi_t(0) = 0$;

(ii) *for each* $t \in (M \setminus L)_1 \neq \emptyset$, $\Lambda_t(\cdot, v, w, \beta)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-pseudosounivex at* y, *for each* $t \in (M \setminus L)_2$, $\Lambda_t(\cdot, v, w, \beta)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasisounivex at* y, *and for each* $t \in L$, ϕ_t *is increasing and* $\phi_t(0) = 0$, *where* $\{(M \setminus L)_1, (M \setminus L)_2\}$ *is a partition of* $M \setminus L$;

(iii) $\sum_{t\in M} \rho_t(x, y) \geqq 0$;

(g) (i) *for each* $t \in L_1$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-pseudosounivex at* y, *for each* $t \in L_2$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasisounivex at* y, *and for each* $t \in L$, ϕ_t *is increasing and* $\phi_t(0) = 0$, *where* $\{L_1, L_2\}$ *is a partition of* L;

(ii) *for each* $t \in (M \setminus L)_1$, $\Lambda_t(\cdot, v, w, \beta)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-pseudosounivex at* y, *for each* $t \in (M \setminus L)_2$, $\Lambda_t(\cdot, v, w, \beta)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasisounivex at* y, *and for each* $t \in M \setminus L$, ϕ_t *is increasing and* $\phi_t(0) = 0$, *where* $\{(M \setminus L)_1, (M \setminus L)_2\}$ *is a partition of* $M \setminus L$;

(iii) $\sum_{t\in M} \rho_t(x, y) \geqq 0$;

(iv) $L_1 \neq \emptyset$, $(M \setminus L)_1 \neq \emptyset$, *or* $\sum_{t\in M} \rho_t(x, y) > 0$.

Then $\varphi(x) \nleqslant \psi^{II}(s)$.

Proof. (a) Suppose to the contrary that $\varphi(x) \leqslant \psi^{II}(s)$. This implies that

$$F_i(x) + \|A_i x\|_{a(i)} \leqq F_i(y) + \langle \alpha^i, A_i y\rangle, \quad i \in \underline{p},$$

with strict inequality holding for at least one index $\nu \in \underline{p}$. Therefore, for each $t \in L$, we have

$$\sum_{i\in I_t} u_i[F_i(x) + \|A_i x\|_{a(i)}] \leqq \sum_{i\in I_t} u_i[F_i(y) + \langle \alpha^i, A_i y\rangle]. \tag{4.9}$$

Since for each $t \in L$,

$$\begin{aligned}
&\Pi_t(x, u, v, w, \alpha, \beta)\\
\leqq\ & \sum_{i\in I_t} u_i[F_i(x) + \|\alpha^i\|^*_{a(i)} \|A_i x\|_{a(i)}] + \sum_{j\in J_t} v_j[G_j(x) + \|\beta^j\|^*_{b(j)} \|B_j x\|_{b(j)}]\\
&\text{(by Lemma 3.1 and the primal feasibility of } x)\\
\leqq\ & \sum_{i\in I_t} u_i[F_i(x) + \|A_i x\|_{a(i)}] + \sum_{j\in J_t} v_j[G_j(x) + \|B_j x\|_{b(j)}] \ \text{(by (4.6) and (4.7))}\\
\leqq\ & \sum_{i\in I_t} u_i[F_i(x) + \|A_i x\|_{a(i)}] \ \text{(by the primal feasibility of } x)\\
\leqq\ & \sum_{i\in I_t} u_i[F_i(y) + \langle \alpha^i, A_i y\rangle] \ \text{(by (4.9))}
\end{aligned}$$

$$\leqq \sum_{i\in I_t} u_i[F_i(y) + \langle \alpha^i, A_i y\rangle] + \sum_{j\in J_t} v_j[G_j(y) + \langle \beta^j, B_j y\rangle] + \sum_{k\in K_t} w_k H_k(y)$$
$$- \frac{1}{2}\Big\langle z, \Big[\sum_{t\in I_t} u_i\nabla^2 F_i(y) + \sum_{j\in J_t} v_j\nabla^2 G_j(y) + \sum_{k\in K_t} w_k\nabla^2 H_k(y)\Big]z\Big\rangle \text{ (by (4.4))}$$
$$= \Pi_t(y,u,v,w,\alpha,\beta) - \frac{1}{2}\big\langle z, \nabla^2\Pi_t(y,u,v,w,\alpha,\beta)z\big\rangle,$$

and so

$$\phi_t\Big(\Pi_t(x,u,v,w,\alpha,\beta) - \Pi_t(y,u,v,w,\alpha,\beta) + \frac{1}{2}\big\langle z, \nabla^2\Pi_t(y,u,v,w,\alpha,\beta)z\big\rangle\Big) \leqq 0,$$

it follows from (i) that

$$\mathcal{F}\Big(x,y;b(x,y)\Big\{\sum_{i\in I_t} u_i[\nabla F_i(y) + A_i^T\alpha^i] + \sum_{j\in J_t} v_j[\nabla G_j(y) + B_j^T\beta^j]$$
$$+ \sum_{k\in K_t} w_k\nabla H_k(y) + \Big[\sum_{i\in I_t} u_i\nabla^2 F_i(y) + \sum_{j\in J_t} v_j\nabla^2 G_j(y) + \sum_{k\in K_t} w_k\nabla^2 H_k(y)\Big]z\Big\}\Big)$$
$$< -\rho_t(x,y)\|\theta(x,y)\|^2.$$

Summing over $t \in L$ and using the sublinearity of $\mathcal{F}(x,y;\cdot)$, we get

$$\text{(4.10)}\quad \mathcal{F}\Big(x,y;b(x,y)\Big\{\sum_{i=1}^{p} u_i[\nabla F_i(y) + A_i^T\alpha^i]$$
$$+ \sum_{t\in L}\Big\{\sum_{j\in J_t} v_j[\nabla G_j(y) + B_j^T\beta^j] + \sum_{k\in K_t} w_k\nabla H_k(y)\Big\}$$
$$+ \Big\{\sum_{i=1}^{p} u_i\nabla^2 F_i(y) + \sum_{t\in L}\Big[\sum_{j\in J_t} v_j\nabla^2 G_j(y) + \sum_{k\in K_t} w_k\nabla^2 H_k(y)\Big]\Big\}z\Big\}\Big)$$
$$< -\sum_{t\in L}\rho_t(x,y)\|\theta(x,y)\|^2.$$

Following the steps used in the proof of part (a) of Theorem 3.1, one can easily see that our assumptions in (ii) lead to

$$\mathcal{F}\Big(x,y;b(x,y)\Big\{\sum_{j\in J_t} v_j[\nabla G_j(y) + B_j^T\beta^j] + \sum_{k\in K_t} w_k\nabla H_k(y)$$
$$+ \Big[\sum_{j\in J_t} v_j\nabla^2 G_j(y) + \sum_{k\in K_t} w_k\nabla^2 H_k(y)\Big]z\Big\}\Big)$$
$$\leqq -\rho_t(x,y)\|\theta(x,y)\|^2 \text{ for each } t \in M\backslash L.$$

Summing these inequalities over $t\in M\backslash L$ and using the sublinearity of $\mathcal{F}(x,y;\cdot)$, we obtain

$$\text{(4.11)}\quad \mathcal{F}\Big(x,y;b(x,y)\sum_{t\in M\backslash L}\Big\{\sum_{j\in J_t} v_j[\nabla G_j(y) + B_j^T\beta^j] + \sum_{k\in K_t} w_k\nabla H_k(y)$$

$$+\Big[\sum_{j\in J_t} v_j\nabla^2 G_j(y)+\sum_{k\in K_t} w_k\nabla^2 H_k(y)\Big]z\Big\}\Big)$$
$$\leqq -\sum_{t\in M\backslash L}\rho_t(x,y)\|\theta(x,y)\|^2.$$

Now combining (4.10) and (4.11) and using the sublinearity of $\mathcal{F}(x,y;\cdot)$ and (iii), we see that

$$\mathcal{F}\Big(x,y;b(x,y)\Big\{\sum_{i=1}^{p}u_i[\nabla F_i(y)+A_i^T\alpha^i]+\sum_{j=1}^{q}v_j[\nabla G_j(y)+B_j^T\beta^j]$$
$$+\sum_{k=1}^{r}w_k\nabla H_k(y)+\Big[\sum_{i=1}^{p}u_i\nabla^2 F_i(y)+\sum_{j=1}^{q}v_j\nabla^2 G_j(y)+\sum_{k=1}^{r}w_k\nabla^2 H_k(y)\Big]z\Big\}\Big)$$
$$< -\sum_{t\in M}\rho_t(x,y)\|\theta(x,y)\|^2,$$

which contradicts (4.3). Hence, $\varphi(x)\nleq\psi^{II}(s)$.

(b)-(g): The proofs are similar to that of part (a). □

Theorem 4.2 (Strong Duality). *Let x^* be a normal efficient solution of (P) and assume that any one of the seven sets of conditions set forth in Theorem 4.1 is satisfied for all feasible solutions of (DII). Then there exist $u^*\in U$, $v^*\in\mathbb{R}^q_+$, $w^*\in\mathbb{R}^r$, $\alpha^{*i}\in\mathbb{R}^{m_i}$, and $\beta^{*j}\in\mathbb{R}^{n_j}$, $j\in\underline{q}$, such that $s^*\equiv(x^*,z^*=0,u^*,v^*,w^*,\alpha^*,\beta^*)$ is an efficient solution of (DII) and $\varphi(x^*)=\psi^{II}(s^*)$.*

Proof. The proof is similar to that of Theorem 3.2. □

The four duality models discussed in this section collectively contain a fairly large number of special cases. They subsume a variety of existing dual problems and include a number of new duality formulations for several classes of single- and multiple-objective nonlinear programming problems.

5. Concluding remarks

In this paper, we have established, in a unified framework, a fairly large number of second-order duality results under a variety of generalized $(\mathcal{F},b,\phi,\rho,\theta)$-sounivexity assumptions for a multiobjective programming problem containing arbitrary norms (and square roots of positive semidefinite quadratic forms). Each one of these duality results can easily be modified and restated for each one of the five special cases of the prototype problem (P) designated as $(P1)$ - $(P5)$ in Section 1, and hence they collectively subsume a vast number of second-order duality results previously established by different methods for various classes of nonlinear programming problems with multiple and conventional objective functions. Furthermore, the style and techniques employed in this paper can be utilized for developing similar results for some other classes of optimization problems involving more general types of second-order convex functions. These include discrete and continuous minmax fractional programming problems, various classes of semiinfinite programming problems, and certain types of continuous-time programming problems. In particular, employing similar techniques, one can investigate the following important problems with continuous max and multiple objective functions:

$$\underset{x\in\mathbb{F}}{\text{Minimize}}\ \max_{y\in Y} \frac{f(x,y)+\|A(y)x\|_a}{g(x,y)-\|B(y)x\|_b},$$

$$\underset{x\in\mathbb{F}}{\text{Minimize}}\ \left(\frac{f_1(x)+\|A_1x\|_{a(1)}}{g_1(x)-\|B_1x\|_{b(1)}},\dots,\frac{f_p(x)+\|A_px\|_{a(p)}}{g_p(x)-\|B_px\|_{b(p)}}\right).$$

We shall explore the possibility of developing various second-order duality models for these classes of optimization problems in subsequent papers.

References

[1] B. Aghezzaf, *Second order mixed type duality in multiobjective programming problems*, J. Math. Anal. Appl. **285** (2003), no. 1, 97–106.

[2] I. Ahmad and Z. Husain, *Nondifferentiable second-order symmetric duality*, Asia-Pac. J. Oper. Res. **22** (2005), no. 1, 19–31.

[3] ______, *Second order (F,α,ρ,d)-convexity and duality in multiobjective programming*, Inform. Sci. **176** (2006), no. 20, 3094–3103.

[4] ______, *Second order symmetric duality in multiobjective programming involving cones*, Optim. Lett. (2012) (forthcoming).

[5] ______, *On multiobjective second order symmetric duality with cone constraints*, European J. Oper. Res. **204** (2010), no. 3, 402–409.

[6] C. R. Bector and B. K. Bector, *(Generalized)-binvex functions and second order duality for a nonlinear programming problem*, Congr. Numer. **52** (1986), 37–52.

[7] ______, *On various duality theorems for second-order duality in nonlinear programming*, Cahiers Centre Études Rech. Opér. **28** (1986), no. 4, 283–292.

[8] C. R. Bector and S. Chandra, *Second-order duality for generalized fractional programming*, Contributions to mathematical economics and optimization theory, 11–28, Methods Oper. Res., 56, Athenäum/Hain/Hanstein, Königstein, 1986.

[9] ______, *Second order symmetric and self dual programs*, Opsearch **23** (1986), no. 2, 89–95.

[10] ______, *First and second order duality for a class of nondifferentiable fractional programming problems*, J. Inform. Optim. Sci. **7** (1986), no. 3, 335–348.

[11] ______, *(Generalized)-bonvexity and higher order duality for fractional programming*, Opsearch **24** (1987), no. 3, 143–154.

[12] C. R. Bector, S. Chandra, S. Gupta, and S. K. Suneja, *Univex sets, functions, and univex nonlinear programming*, in Generalized convexity (Pécs, 1992), 3–18, Lecture Notes in Econom. and Math. Systems, 405, Springer, Berlin, 1994.

[13] C. R. Bector, S. Chandra, and I. Husain, *Second-order duality for a minimax programming problem*, Opsearch **28** (1991), 249–263.

[14] R. R. Egudo and M. A. Hanson, *Second order duality in multiobjective programming*, Opsearch **30** (1993), 223–230.

[15] T. R. Gulati and I. Ahmad, *Second order symmetric duality for nonlinear minimax mixed integer programming problems*, European J. Oper. Res. **101** (1997), no. 1, 122–129.

[16] T. R. Gulati, I. Ahmad, and I. Husain, *Second order symmetric duality with generalized convexity*, Opsearch **38** (2001), no. 2, 210–222.

[17] M. Hachimi and B. Aghezzaf, *Second order duality in multiobjective programming involving generalized type-I functions*, Numer. Funct. Anal. Optim. **25** (2004), no. 7-8. 725–736.
[18] M. A. Hanson, *Second order invexity and duality in mathematical programming*, Opsearch **30** (1993), 313–320.
[19] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.
[20] I. Husain and Z. Jabeen, *Second order duality for fractional programming with support functions*, Opsearch **41** (2004), no. 2, 121–134.
[21] V. Jeyakumar, *p-convexity and second order duality*, Util. Math. **29** (1986), 71–85.
[22] ———, *First and second order fractional programming duality*, Opsearch **22** (1985), no. 1, 24–41.
[23] J. C. Liu, *Second order duality for minimax programming*, Util. Math. **56** (1999), 53–63.
[24] O. L. Mangasarian, *Second- and higher-order duality in nonlinear programming*, J. Math. Anal. Appl. **51** (1975), no. 3, 607–620.
[25] S. K. Mishra, *Second order generalized invexity and duality in mathematical programming*, Optimization **42** (1997), no. 1, 51–69.
[26] ———, *Second order symmetric duality in mathematical programming with F-convexity*, European J. Oper. Res. **127** (2000), no. 3, 507–518.
[27] S. K. Mishra and N. G. Rueda, *Higher-order generalized invexity and duality in mathematical programming*, J. Math. Anal. Appl. **247** (2000), no. 1, 173–182.
[28] B. Mond, *Second order duality for nonlinear programs*, Opsearch **11** (1974), no. 2-3, 90–99.
[29] B. Mond and T. Weir, *Generalized convexity and higher-order duality*, J. Math. Sci. **16-18** (1981-1983), 74–94.
[30] ———, *Generalized concavity and duality*, in Generalized Concavity in Optimization and Economics, S. Schaible and W. T. Ziemba, editors, pp. 263–279, Academic Press, New York, 1981.
[31] B. Mond and J. Zhang, *Duality for multiobjective programming involving second-order V-invex functions*, in Proceedings of the Optimization Miniconference II, B. M. Glover and V. Jeyakumar, editors, pp. 89–100, University of New South Wales, Sydney, Australia, 1995.
[32] ———, *Higher order invexity and duality in mathemaical programming*, in Generalized Convexity, Generalized Monotonicity: Recent Results, J. P. Crouzeix, *et al.*, editors, pp. 357–372, Kluwer Academic Publishers, printed in the Netherlands, 1998.
[33] R. B. Patel, *Second order duality in multiobjective fractional programming*, Indian J. Math. **38** (1996), no. 1, 39–46.
[34] M. N. Vartak and I. Gupta, *Duality theory for fractional programming problems under η-convexity*, Opsearch **24** (1987), 163–174.
[35] X. Yang, *Generalized convex duality for multiobjective fractional programs*, Opsearch **31** (1994), no. 2, 155–163.
[36] X. Q. Yang, *Second-order global optimality conditions for convex composite optimization*, Math. Programming **81** (1998), no. 3, Ser. A, 327–347.
[37] X. M. Yang, X. Q. Yang, and K. L. Teo, *Nondifferentiable second order symmetric duality in mathematical programming with F-convexity*, European J. Oper. Res. **144** (2003), 554–559.
[38] ———, *Huard type second-order converse duality for nonlinear programming*, Appl. Math. Lett. **18** (2005), no. 2, 205–208.
[39] X. M. Yang, X. Q. Yang, K. L. Teo, and S. H. Hou, *Second order duality for nonlinear programming*, Indian J. Pure Appl. Math. **35** (2004), no. 5, 699–708.
[40] G. J. Zalmai, *Optimality conditions and duality models for a class of nonsmooth constrained fractional variational problems*, Optimization **30** (1994), no. 1, 15–51.

[41] ———, *Generalized (η, ρ)-invex functions and global semiparametric sufficient efficiency conditions for multiobjective fractional programming problems containing arbitrary norms*, J. Global Optim. **36** (2006), no. 1, 51–85.

[42] ———, *Generalized second-order $(\mathcal{F}, \beta, \phi, \rho, \theta)$-univex functions and parametric duality models in semiinfinite discrete minmax fractional programming*, Adv. Nonlinear Var. Inequal. **15** (2012), no. 2, 63–91.

[43] J. Zhang and B. Mond, *Second order b-invexity and duality in mathematical programming*, Util. Math. **50** (1996), 19–31.

[44] ———, *Second order duality for multiobjective nonlinear programming involving generalized convexity*, in Proceedings of the Optimization Miniconference III, B. M. Glover, B. D. Craven, and D. Ralph, editors, pp. 79–95, University of Ballarat, 1997.

Department of Mathematics and Computer Science
Northern Michigan University
Marquette, Michigan 49855, USA
E-mail address: gzalmai@nmu.edu