

Solving Mixed Strategy Nash-Cournot Equilibria under Generation and Transmission Constraints in Electricity Market

Kwang-Ho Lee[†]

Abstract – Generation capacities and transmission line constraints in a competitive electricity market make it troublesome to compute Nash Equilibrium (NE) for analyzing participants' strategic generation quantities. The NE can cause a mixed strategy NE rather than a pure strategy NE resulting in a more complicated computation of NE, especially in a multiplayer game. A two-level hierarchical optimization problem is used to model competition among multiple participants. There are difficulties in using a mathematical programming approach to solve a mixed strategy NE. This paper presents heuristics applied to the mathematical programming method for dealing with the constraints on generation capacities and transmission line flows. A new formulation based on the heuristics is provided with a set of linear and nonlinear equations, and an algorithm is suggested for using the heuristics and the newly-formulated equations.

Keywords: Cournot model, Electricity market, Generation capacity, Mixed strategy, Nash equilibrium (NE), Power transfer distribution factor (PTDF), Transmission congestion.

1. Introduction

In order for an economically proficient electricity market to take root, it is paramount that prices in a spot market are settled competitively among market participants. Competitive prices implement appropriate incentives for designing and operating electricity markets. Notwithstanding, the demand side of limited price responsiveness and the strategic behaviors of the generation companies, have deviated the prices somewhat from competitive levels.

Using game theory to model firm's strategic production quantity in order to analyze the phenomenon of the price deviation within such a setting has been attempted copiously. The most extensively used among oligopoly market models has been the Cournot model in Nash Equilibrium (NE) analysis of generation wholesale markets [1-4]. In this context generation firms determine a strategic quantity based on the rivals' expected productions. Albeit the Cournot model does not absolutely conform to the typical market rules for electricity markets, it accommodates an upper bound on end results in markets where offers can be altered from interval to interval. In conjunction, the Cournot model capably explains capacity withholding from the market and it is also argued in [5] that generation capacity withholding, in order to manipulate prices, was a strategy extensively employed in the electricity market of England and Wales in the 1990s.

Numerous efforts to develop a solution method for determining the NE of generation quantity games in electricity markets have led to some obstacles remaining unsolved even today. One such demanding problem is regarding the transmission line limits, since the market clearing mechanism becomes complicated by the transmission congestion [6-8]. Another is in relation to a multiplayer game where more than three players participate [7, 8]. In considering realistic electricity market, it is pertinent to keep in mind that there are somewhat more than two but less than ten major firms.

The transmission line congestion status is determined by the distribution of generation quantities. Accordingly, the decision space can be divided into subsets depending on whether the transmission lines are congested or uncongested [6]. This leads to discontinuities in the reaction curves [9] and precipitates the profit functions to be non-differentiable and non-concave. In this instance, there may not abide a pure strategy NE. Rather there is a mixed strategy equilibrium where players find it optimal to randomly choose between strategies [10].

The mathematical programming approach [11-13] can decipher for the NE of a multiplayer game with differentiable and concave profit functions. Nonetheless when a transmission line is congested, this process has adversities in concluding equilibria due to the profit function being non-differentiable and non-concave. Though an artificial intelligence based approach [14, 15] addresses to find a global optimum using this method, the mixed strategy NE and its probability are arduous to compute. Conversely, the

[†] Corresponding Author: Department of Electrical Engineering, Dankook University, Yongin, Korea (khlee@dku.edu).
Received: August 6, 2012; Accepted: February 14, 2013

payoff matrix approach [7, 8, 16, 17] for two or three players can acquire mixed strategy NEs by discretization of the decision space. Notwithstanding, it is troublesome to utilize this approach to finding mixed strategies in a multiplayer game because of the dimensional limit to three players and the computational burden.

The Nikaido-Isoda function and a relaxation algorithm (NIRA) are combined in [18-20] to calculate the NE for a multiplayer game. The NIRA method is attractive in that the most advanced computational routine required is minimization of a multivariate function (Nikaido-Isoda function). It is also its merit that the method can be applied to a wide class of problems, including non-differential payoffs and coupled constraints games [18]. However, in case that the best response functions of the players do not intersect, the method might have difficulty in calculating the mixed strategy NE.

In this study, a new algorithm based on heuristics is identified for finding a mixed strategy NE in a multiplayer game based on the network configuration. Two intriguing phenomena are found and proposed here to characterize the equilibrium as a result of investigating manifold cases of mixed strategy NEs in all manners of networks and markets. The first phenomenon is the existence of a “key player” in a mixed NE who controls the transmission congestion status (congested or uncongested). The second is that the player whose generator is in the specific bus as determined by the power transfer distribution factor (PTDF) can be the key player. When the mixed strategy solution by the heuristics is bigger than the generation capacity, it does not meet the generation capacity constraint. Therefore two more heuristics are proposed for the NE within the generation capacity.

Numerical examples applied to IEEE 30-bus 6-player systems furnished to illustrate the validity of the heuristics and the algorithm. These are based on the two-level hierarchical optimization converted into a new framework with a set of linear equations and a single nonlinear equation. An algorithm is also presented for discerning a mixed NE by applying a solution of the linear and nonlinear equations.

2. Nash Equilibrium Of Cournot Model

A. Market and cournot model

Cournot equilibria have been utilized widely to study the electricity market. The supplier adopts its profit-maximizing quantity of generation in the belief that the quantities supplied by other producers are fixed and do not react to the firm’s quantity changes within Cournot competition.

Assume there are N strategic firms having N generating units each. Firm $i \in \{1, \dots, N_g\}$ has a unit i with a generating cost function $C_i(q_i) = b_i q_i + 0.5 m_i q_i^2$ and a

marginal cost $c_i(q_i) = b_i + m_i q_i$, where q_i represents the unit i ’s generation power quantity, b_i and m_i are cost coefficients. The inverse demand function at bus $j \in \{1, \dots, N_d\}$ is $p_j = a_j - r_j d_j$, where d_j is the power demand, a_j and r_j are the coefficients.

B. Two-Level optimization

The objective of the market operator (MO) is quite dissimilar from the other participants. It attempts to maximize the demand side benefit as defined by consumer’s surplus and payment, instead of maximizing profits. Formulating the MO’s objective into a quadratic program, we have

$$\text{Max}_{d_j} B(d) = \sum_{j \in D} (a_j d_j - 0.5 r_j d_j^2) \quad (1)$$

$$\text{s.t.} \quad \sum_{i \in G} q_i - \sum_{j \in D} d_j = 0 \quad (2)$$

$$0 \leq T_l \leq T_{l,\max} \quad \forall l \in L \quad (3)$$

where D and G are the sets of all demand buses and generating units respectively, L is the set of all transmission lines, and T_l acts as the power flow on the transmission line l . The constraints consists of a power balance equality (2), and transmission power inequalities (3).

Each rational strategic generation firm maximizes its profits (revenue minus generating costs) by selecting its own generation parameter accepting as given the strategic parameters of other firms [3]. Formulating the firm i ’s objective into a quadratic program with generation capacity limits, we have

$$\text{Max}_{q_i} \pi_i = p_i q_i - (b_i q_i + 0.5 m_i q_i^2) \quad (4)$$

$$\text{s.t.} \quad p_i = a_i - r_i d_i \quad (5)$$

$$q_i \leq q_{Mi} \quad (6)$$

where π_i is the profit from unit i , and p_i is a nodal price determined by the local demand, d_i at the bus unit i , and q_{Mi} is the capacity limit of q_i .

In this paper, locational marginal pricing is adopted as a pricing method of the MO in the electricity market. As generating firms’ optimization needs to be solved with the MO’s problem simultaneously, these optimizations of decentralized decision makers require hierarchical coordination. At the upper level in (4)~(5), producers initiate a decision of selling energy, while, at the lower level in (1)~(3), the MO endeavors to maximize the demand side benefit based upon the suppliers’ quantities.

C. Congestion and Mixed Strategies

The two-level hierarchical optimization can be solved easily by the mathematical programming method, once no

inequality binds underlying decision space with the Cournot model. Let the solved quantity parameters be q^* . At a pure strategy NE, the strategies of all participants satisfy

$$\pi_i(q_i^*, q_{-i}^*) \geq \pi_i(q_i, q_{-i}^*) \quad \forall i \in G \quad (7)$$

where q_i^* is the solved quantity parameter of firm i , q_i is the possible quantity parameter firm i can choose, and q_{-i}^* is the solved quantity parameter set of all participants excluding firm i . By unilaterally altering their choices, none of the generating firms can improve their profits.

Discouragingly, the transmission limit constraints lead to problem complexity depending on whether the transmission constraint is binding or not. In case of congestion occurring on a transmission line, the decision space becomes divided into sub-regions according to whether the line is congested or uncongested. This gives rise to discontinuities in the reaction curves and causes the profit functions to be non-differentiable and non-concave [21]. In this way, ultimately, the analytic method becomes ineffective. It may be that no such pure strategies satisfy the definition of Nash equilibrium (7). Instead, the firms may discover that they must play a combination of pure strategies, choosing amongst them randomly. This is a “mixed strategy,” which is specified by the probability distribution of the choice of pure strategies [7, 10].

3. Heuristics for mixed strategy NE

A. Example of a mixed strategy NE

For understanding of mixed strategy NE, an electricity power market is given with a simple network as shown in Fig. 1. The system consists of 3 generating firms, and 3 buses with a local market at each bus.

The marginal cost functions of firm F_1, F_2, F_3 , and the inverse demand functions are respectively, $MC_1 = 10 + 0.3q_1$, $MC_2 = 20 + 0.4q_2$, $MC_3 = 15 + 0.45q_3$, $p_1 = 70 - 0.7d_1$, $p_2 = 80 - 0.5d_2$, $p_3 = 90 - 0.4d_3$. The transmission lines are assumed to be lossless and have the reactance satisfying $x_{12} = x_{13} = 2x_{23}$.

Initially assume that there are no limits of transmission flow and generation capacity. On the basis of the Cournot model, the clearing prices are determined by the benefit

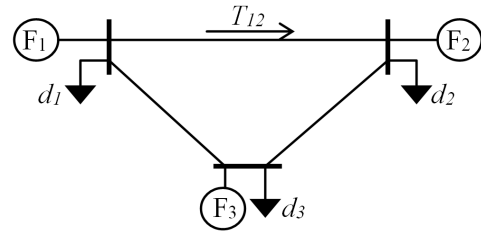


Fig. 1. Diagram of 3-bus system with a congested line

maximization of demands located at each bus. Solving the two-level optimization yields a pure strategy NE; $q_1 = 84.21$, $q_2 = 51.82$, $q_3 = 55.71$. Such choices lead to the clearing prices of $p_1 = p_2 = p_3 = 49.47$, and the power flow of $T_{12} = 23.8$.

Now assume the flow limit on the line joining bus 1 to bus 2 is $T_{max} = 15$. Taking the inequality into consideration gives a mixed strategy NE; $q_1 = 60.23$, $q_2 = [56.1, 46.3]$, $q_3 = 60.95$ with probability [0.49, 0.51]. At the NE, player 2 chooses a mixed strategy consisting of two pure strategies, while player 1 and 3 choose pure strategies. In this paper, a player choosing a mixed strategy is defined as a “key player,” while other player choosing a pure strategy is called as a normal player.

The expected profit of each firm is shown in Fig. 2. The expected profit with respect to its generation quantity is computed with the other players’ quantities fixed. The highest point in the curve corresponds to equilibrium strategy. In the case of π_2 , there are two peak points having equivalent values, since F_2 chooses a mixed strategy. This means that each player has no incentive to deviate from its choice, given the choices of the other players.

B. Distribution factors

The dc power flow is widely used to approximate the power system. The dc power flow equations, including all buses except the slack bus, are ([22, 23]):

$$\mathbf{B}\theta = \mathbf{P} \quad (8)$$

where \mathbf{B} is bus susceptance matrix, θ is the vector of bus voltage angles, and \mathbf{P} is the vector of bus real power injections. The nodal power balance equations are implied in (7). The real power flow on the branch between bus i and j is

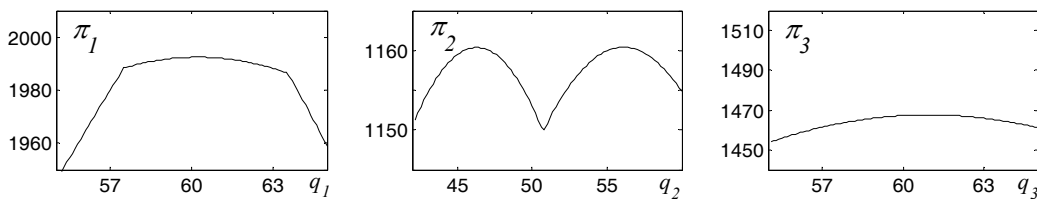


Fig. 2. Expected profit of each firm at NE with transmission capacity

$$T_{ij} = b_{ij}(\theta_i - \theta_j) \quad (9)$$

where b_{ij} is branch susceptance between bus i and j . The vector of power flows on all branches is

$$\mathbf{T} = \mathbf{H}\theta \quad (10)$$

where \mathbf{H} is the product of the branch susceptance diagonal matrix and an appropriate incidence matrix of branches with buses [2]. If the k th transmission line connects bus i and bus j , then

$$h_{ki} = b_{ij}, h_{kj} = -b_{ij}, h_{kl} = 0, \forall l \neq i, l \neq j. \quad (11)$$

Substituting (8) into (10), we have

$$\mathbf{T} = \mathbf{H} \cdot \mathbf{B}^{-1} \mathbf{P} \quad (12)$$

Define $\mathbf{D}_f = \mathbf{H} \cdot \mathbf{B}^{-1}$, whose elements indicate the sensitivity of branch flows to nodal net injection and withdrawal at the reference bus.

C. Heuristics about “Key Player”

A new algorithm for finding mixed strategies results from investigating numerous cases of mixed NEs in a congested multiplayer power market. Some patterns were extracted from relations among the mixed strategies, key player, and the bus of key player. These patterns are organized into two heuristics.

Certain conditions are needed to validate the heuristics: 1) the market analysis model is the Cournot model or the supply function equilibrium model; 2) the demand is not strategic; and 3) congestion occurs on one transmission line. If a different model like Bertrand model is used instead of Cournot, the heuristics may not be valid any longer. However, condition 3) is just for simplicity of understanding, and the algorithm can be used in multi-congested line cases as shown in numerical results.

Heuristic I: There exists one key player, and the others are normal players at the NE.

In other words, only one player chooses randomly amongst two pure strategies with probabilities α and β ($\alpha + \beta = 1$); that is, the player chooses a mixed strategy, and the others choose pure strategies. The key player chooses amongst the two choices with the probabilities, while the normal players choose their pure strategies. One of the two choices gives rise to congestion on a line. It is called a congestion strategy. The other, on the contrary, makes no congestion. It is called an uncongestion strategy. The α and β show the probabilities of an uncongestion and a congestion strategy, respectively. In the 3-bus case, among the key player’s choice, $q_2 = 46.3$ is a congestion strategy, while $q_2 = 56.1$ is an uncongestion strategy.

Heuristic II: Location of the key player is the bus that

has the most negative value of PTDF on the congested line.

The PTDF indicates the sensitivity of line flows to nodal power injection. Therefore the bus having the most negative value on the congested line can be interpreted as the most effective location to control the congestion by withholding supply. Also, it seems to be the best location to exercise market power by using the congestion. In the 3-bus case, the PTDF values of all buses on the line from bus 1 to bus 2 with a reference of bus 3 are 0, -0.6, and -0.4, respectively. Firm 2 at bus 2 has the most negative among the strategic participants, thus it plays a key player role in the NE.

D. Heuristics about generation capacity limits

A key player locates in receiving area over a congested line. If a key player lowers its generation quantity, then the players locating in the sending area hope to increase their generation for supplying the demand in receiving area over the congested line. Thus the less quantity a key player chooses, the more the line is congested. So among the mixed strategies of a key player, a congestion strategy is smaller than uncongestion strategy.

The generation capacity limits can affect NE of all the players. When uncongestion strategy of a key player is bigger than its generation capacity limit, the strategy is impractical and useless.

Heuristic III: When a key player has a mixed strategy NE, congestion strategy is smaller than uncongestion strategy, and uncongestion strategy is equal or less than its generation capacity limit.

In a solution algorithm, we proposed two steps; considering transmission limit only, and considering both of transmission and generation capacity limits. The solution after the first step may not satisfy the generation capacity constraints. If uncongestion strategy solved in the first step is bigger than the generation capacity limit of a key player, then uncongestion strategy is set to the generation capacity in the second step.

Heuristic IV: If a generation capacity limit of a key player is quite less than uncongestion strategy from the first step, then the role of key player may be transferred to other player in the NE.

A key player with a mixed strategy NE uses two conflicting cards (uncongestion strategy, congestion strategy). But when uncongestion strategy is strongly restricted by the generation capacity, the controlling power of a key player using the two cards is weakened. If the generation capacity limit becomes less than a certain point, a next candidate for a key player takes the role of key player. The next candidate means a player at the bus that has the secondly negative value of PTDF on the congested line. As for the certain point, it is named in this paper as the “critical” generation capacity of a key player. It means the lowest limit of generation quantity for a key player to wield its mixed strategies. The critical generation capacity

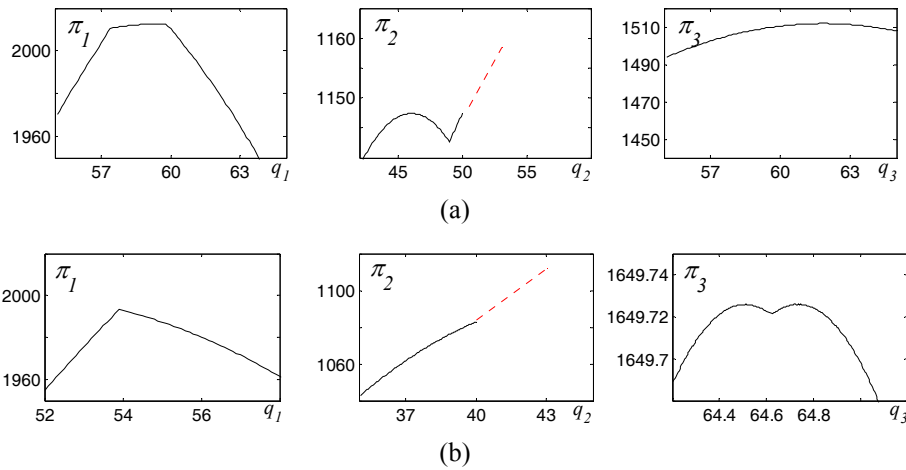


Fig. 3. Expected profits at NE with generation capacities, $q_{M2}=50$ (a), 40 (b)

Table 1. NE at Different Generation Capacity Limits in 3 bus system

	$q_{M2}=50$			$q_{M2}=40$		
	(q_u, q_c)	(α, β)		(q_u, q_c)	(α, β)	
F1	59.25, 0	1, 0		53.9, 0	1, 0	
F2	50, 46.0	0.44, 0.56		40, 0	1, 0	
F3	61.85, 0	1, 0		64.7, 64.5	0.19, 0.81	

is a little less than the congestion strategy solved in the first step. Those are shown in a numerical simulation, but it is not verified or derived mathematically.

Generation capacity limits are imposed on the former example of 3 bus system. Table 1 shows the solutions of generation quantity, and the probabilities at NE corresponding to generation capacity limits of $q_{M2}=50$ and 40 . When the limit is 50 which is smaller than the uncongestion strategy in Fig. 2, the player F2 still has the role of key player. The expected profits are shown in Fig. 3(a). Although the dotted line for π_2 has higher profits than profit at NE, it is not allowed due to the generation capacity limit 50 . When the limit is 40 , however, F2's strategy becomes restricted more, and F3 who has the next PTFD value takes the role of key player. Fig. 3(b) shows the expected profits and verifies the NE condition. F3's profits increases due to the key player role, but the others' decreases than (a). The critical generation capacity is computed as 45.697 in this example. This is a little less than 46.30 which is the congestion strategy of F2 under a generation capacity limit free.

4. Solution Method

A. Expected profit balance of key player

An insight to a mixed strategy NE is the following: *All pure strategies that are played as part of the mixed strategy NE have the same expected profit* [24]. If one strategy

yields lower profit than another, then we should play only the strategy that profits more to the exclusion of the strategy that profits less. This is just domination of one strategy by another, when probability is involved.

The key player has two choices in a mixed strategy: an uncongestion strategy and a congestion strategy. Therefore, the expected profit from an uncongestion strategy should be equal to that from a congestion strategy. So the key player's profits satisfy

$$\pi_u(q_u, q_{-i}^*, \alpha) = \pi_c(q_c, q_{-i}^*, \beta) \quad (13)$$

where q_u and q_c are an uncongestion strategy and a congestion strategy of the key player, respectively, while q_{-i}^* is the pure strategies of normal players. The coefficients α and $\beta (=1-\alpha)$ are the probabilities of choosing q_u and q_c , respectively. The π_u and π_c are the profits of the key player corresponding to the q_u and q_c , respectively. This condition of expected profit balance of the key player is the nonlinear equation with a variable α .

B. Modified optimization formulation

The expected profit of the key player depends on the probabilities of the two different states: uncongestion and congestion. The normal players' expected profits also depend on the key player's choice. Therefore, the optimization of (4) at the upper level needs to be changed into the forms with expected values.

The optimality conditions for the upper level problem of maximizing the expected profits of the key players are:

$$\partial \pi_x^u / \partial q_x^u = (m_x + r_x s_x^u) q_x^u + r_x d_x^u + b_x - a_x = 0 \quad (14)$$

$$\partial \pi_x^c / \partial q_x^c = (m_x + r_x s_x^c) q_x^c + r_x d_x^c + b_x - a_x = 0 \quad (15)$$

where the subscript 'x' denotes the key player, d_x^u and d_x^c are the demand powers at bus x, dispatched by the MO

corresponding to q_x^u and q_x^c , respectively. The sensitivities of demand quantity are included which are defined as $s_i^u = \partial d_i^u / \partial q_i$, $s_i^c = \partial d_i^c / \partial q_i$. These are shown in Appendix A with the optimal conditions of the MO.

The optimal condition to the expected profits of the normal players i is

$$\begin{aligned} \partial \pi_i / \partial q_i &= \partial \{ \alpha \cdot \pi_i^u + \beta \cdot \pi_i^c \} / \partial q_i \\ &= \partial \{ (\alpha \cdot p_i^u + \beta \cdot p_i^c) q_i - C_i(q_i) \} / \partial q_i \\ &= -(m_i + \alpha \cdot s_i^u + \beta \cdot s_i^c) q_i \\ &\quad - r_i (\alpha \cdot d_i^u + \beta \cdot d_i^c) - b_i + a_i = 0 \quad \forall i \in G_{-x} \end{aligned} \quad (16)$$

where the G_{-x} is the sets of all generation firms of normal players. The p_i^u and p_i^c are the market prices at bus i when the network conditions are in an uncongestion and congestion, respectively.

C. Conversion to single variable problem

Rewriting the optimal conditions of generation firms in (14)~(16) as a matrix form, we have

$$\mathbf{T}_u \cdot \mathbf{d}^u + \mathbf{T}_c \cdot \mathbf{d}^c + \mathbf{T}_s \cdot \mathbf{q}_a = \mathbf{b}_s \quad (17)$$

where \mathbf{q}_a is the generation variables as specified in (A.1) in the Appendix, \mathbf{d}^u and \mathbf{d}^c are the demand powers at all the buses in an uncongestion and a congestion situations, respectively. The coefficient matrices, \mathbf{T}_u , \mathbf{T}_c , and \mathbf{T}_s contain the variable α (or β) as defined in Appendix A, and \mathbf{b}_s is described in (A.11).

We consider the number of equations. There is one nonlinear equation from (13) corresponding to the upper level optimization, $N_d + 1$ linear equations for uncongestion situation from (A.6) from the lower level optimization, $N_d + 2$ linear equations for congestion situation from (A.7) from the lower level optimization, and $N_g + 1$ linear equations for generation firms from (17) corresponding the upper level optimization. Therefore, there are one nonlinear and $N_g + 2N_d + 4$ linear equations.

Rearranging the linear equations, we have

$$\begin{pmatrix} \mathbf{M}_u & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_c & \mathbf{0} \\ \mathbf{T}_u & \mathbf{0} & \mathbf{T}_c \end{pmatrix} \begin{pmatrix} \mathbf{d}^u \\ \mathbf{d}^c \\ \mathbf{q}_a \end{pmatrix} + \begin{pmatrix} -\mathbf{e}_u^t \\ -\mathbf{e}_c^t \\ \mathbf{h}_s^t \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{a} \\ \mathbf{b}_s \end{pmatrix} \quad (18)$$

where \mathbf{M}_u , \mathbf{a} , \mathbf{e}_u , \mathbf{e}_c , λ and μ are described in (A.6) and (A.7), and \mathbf{M}_c is the square matrix in (A.7).

Therefore, with a congestion line and with its key player guessed, the equilibrium of the problem is obtained by solving the following simplified set of equations;

$$\text{Nonlinear: } f(\alpha, \mathbf{x}) = 0 \quad (19)$$

$$\text{Linear: } \mathbf{M}_o(\alpha) \cdot \mathbf{x} = \mathbf{b}_o \quad (20)$$

where f re-expresses (13), \mathbf{M}_o , \mathbf{x} , and \mathbf{b}_o are the coefficient matrix, the variable vector, and the right-hand-side constant vector in (18) respectively. Therefore, it can be written as only one nonlinear equation with single variable as

$$f(\alpha, \mathbf{M}_o^{-1} \cdot \mathbf{b}_o) = 0. \quad (21)$$

This equation can be solved easily by a line search method such as a bisection algorithm, since the function value is monotonic in the variable α . This solution process considers only transmission line constraints but generation capacity, because the solution of (21) does not guarantee to satisfy the generation capacity constraints.

D. Two steps for considering generation capacity limits

The mixed strategy NE and its probability are hard to compute using conventional mathematical method. In this paper, two steps are proposed, the first is the solution process of (21) based on the heuristics I and II.

The second step is to check the solution of (21) for meeting the generation capacity constraints and to modify (21) and to solve it based on the heuristics III and IV. If the generation capacity constraints are satisfied after the first step, the solution is the NE and the second step is not required. If a generation capacity limit of a player is smaller than its strategy in the solution of (21), then its strategy in NE is the capacity limit and the others in NE are obtained by solving the modified (21). If the modified (21) does not have any solution, it is required to change the role of key player to the next player.

5. Numerical Results

A. Sample system

The test system used in this paper is a modified IEEE 30-bus system [16] as shown in Fig. 4. Six firms are assumed to participate in the electricity market and own the six generators, with each player having ownership of one generator. Generation marginal cost data are shown in Table 2. At the buses, 1, 13, 22, and 27 where the generation firms are connected, we add the demands. The inverse demand function for all demands are assumed to be identical for simplicity as $p_j = 6.0 - 0.2d_j \quad \forall j \in D$.

First, the line limits are assumed to be big enough to cause no congestion in the equilibrium. The results is the pure strategy NE as follows: $\mathbf{q} = [47.35, 58.66, 21.94, 24.61, 21.94, 41.43]$. The price is 4.27 and identical at all buses, which is higher than the perfect competition price of 4.09. The demand power is 8.637 at each bus, and the total power is 215.93 MW, less than 238.33 at perfect

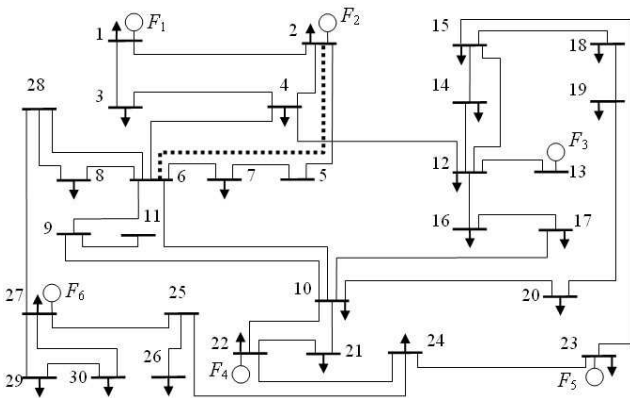


Fig. 4. Diagram of IEEE 30-bus system

Table 2. Generator Marginal Cost Functions

Firm	Bus	b_i	m_i	q_{max}
1	1	2	0.04	80
2	2	1.75	0.035	80
3	13	3	0.05	40
4	22	1	0.125	30
5	23	3	0.05	50
6	27	3.25	0.01668	55

competition. The power flows on line l_A between bus 2 and bus 6, and line l_B between bus 12 and bus 15 are 24.94 MW, and 10.6 MW respectively.

B. Mixed strategy equilibria

To consider congestion, let the limit of line l_A be 20 MW while keeping other limits unchanged. Then, there is no pure strategy satisfying the NE condition.

By looking at the topology of the network, it is hard to guess the key player who is using a mixed strategy with respect to the line l_A dotted in Fig. 4. But the PTDF gives information about this. The PTDFs corresponding to the line l_A are [0.0, 0.057, -0.261, -0.294, -0.279, -0.307]. Because the 6th value is the most negative, the firm F_6 located at bus 27 is guessed as the key player.

By setting up a single variable Eq. (21) of a probability α that F_6 chooses as an uncongestion strategy, and solving the equation, the mixed strategy NE is obtained as follows; $[q_1 \sim q_5] = [42.03, 47.25, 23.39, 25.19, 23.36]$, and F_6 's choice, $[q_6^u, q_6^c] = [44.68, 40.60]$ with probabilities $[\alpha, \beta] = [0.864, 0.136]$. In the uncongestion situation, the price is 4.35 with demand power 8.236 at each bus, and the power flow on the line l_A is 19.8 MW. In the congestion situation, the prices at the buses are shown in Fig. 5. All the nodal prices are a little higher than those of the uncongestion case. The price of the bus 2, sending end of the line l_A , is the lowest, and that of the bus 8, next to the receiving end of the line l_A , is the highest.

To verify the solution is a NE, the expected profit of each firm is illustrated as shown in Appendix B. The expected profit corresponding to the NE displays the

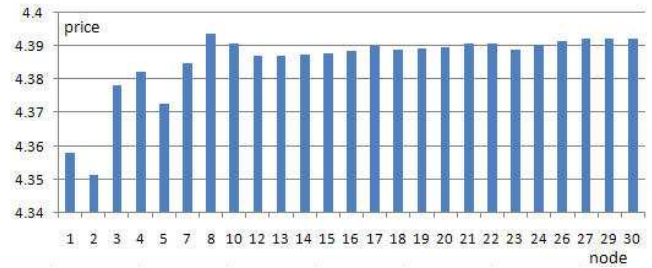


Fig. 5. Nodal prices in congestion case

highest profit for all the expected profit functions. This meets the definition of NE. Hence, the NE is verified to correspond to the heuristic that only one player chooses a mixed strategy consisting of two pure strategies, and the other players choose pure strategies.

C. Solutions as line limit changes

In a solution process of the first step, if the obtained value α is bigger than 1.0, it means the solution is a pure strategy NE with all the lines uncongested. Conversely, if α is less than 0, it corresponds to a pure strategy NE with a line congested, instead of a mixed strategy. If α has a value between 0 and 1, then the solution is a mixed strategy NE with a line congested. The changes of α and the changes of the key player's mixed strategy are investigated as the power flow limit of the line l_A changes, and are shown in Figs. 6 and 7.

The range of the line limit T_{max} is divided as I, II, and III in Figs. 6 and 7. In the range II ($T_{max} = 7.0 \sim 24.9$), the probability α of F_6 to choose q_6^u in a mixed strategy

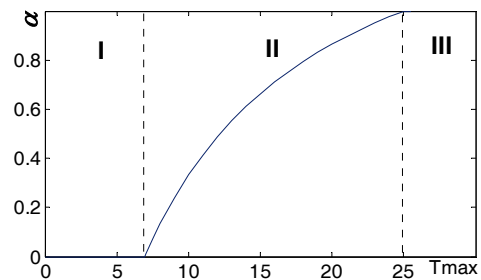


Fig. 6. Probability changes as line limit changes

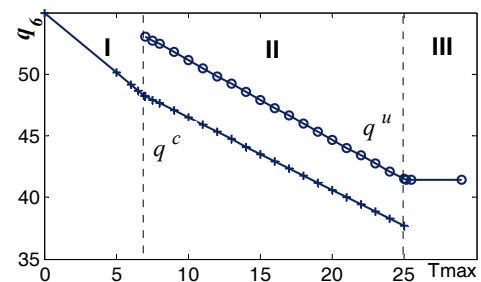


Fig. 7. Quantity parameter changes limit changes as line limit changes

varies from 0 to 1. However, a congestion case pure strategy is obtained in range I, while an uncongestion case pure strategy is obtained in range III as shown in Fig. 7. In range I, the pure strategy q_6^c varies, while in range III, the pure strategy q_6^u keeps constant value 41.43 as in the previous uncongested situation.

As another case, let the limit of the line l_B between bus 12 and bus 15 be 8.0 MW neglecting the limit of the line l_A . The PTDFs to the line l_B are [0.0, -0.002, 0.171, -0.128, -0.363, -0.099]. Hence, due to the 5th value, the firm F_5 is selected as the key player to the line l_B . The computation of (21) gives an equilibrium as follows; $[q_1 \sim q_4, q_6] = [45.08, 55.47, 15.09, 25.34, 45.14]$, and the F_5 's choice $[q_5^u, q_5^c] = [22.89, 18.12]$ with $\alpha < 0$. So, the probability of uncongestion is zero, and the F_5 chooses a pure strategy of q_5^c which causes a congestion on line l_B .

D. Multiple congestions

So far, the algorithm has been presented based on the condition in Section III.C that a transmission line congestion, if any, occurs on just one line. This was needed for simplicity to describe the form of the equations such as (18). Multiple congestion cases are also solved by the proposed algorithm with some modifications on the procedure and the equations.

For example, assume the congestions on lines l_A and l_B with limits 20 MW and 8 MW, respectively. The F_5 and the F_6 are decided as key players by the PTDF values as in the former examples. Since there are two key players, there are four cases relating to congestion or noncongestion of the two, such as UU, UC, CU, CC. As an example, UC means that one of two key players chooses the uncongestion strategy; the other chooses the congestion strategy. Therefore, the generation variable vector \mathbf{q} is defined as $[q_1 \sim q_4, q_5^u, q_5^c, q_6^u, q_6^c]$. The variables of the power demands \mathbf{d} are also categorized into four cases as well as λ and μ , for example, $d^{uu}, d^{uc}, d^{cu}, d^{cc}$.

With these categorized variables, the Eq. (18) is extended to the four cases. Solving the two nonlinear equations as (13) of two variables α_5 and α_6 , with the two-dimensional search, the NE is as follows; $[q_1 \sim q_4] = [41.81, 47.49, 17.45, 25.63]$, the F_5 's choice $[q_5^u, q_5^c] = [23.75, 18.25]$ with probability of uncongestion $\alpha_5 = 0.210$, and the F_6 's choice $[q_6^u, q_6^c] = [46.96, 42.32]$ with probability of uncongestion $\alpha_6 = 0.886$. Therefore, the probability of the UU that line l_A is not congested, and line l_B is not congested at the NE is $\alpha_5 \cdot \alpha_6 = 0.186$. By the same method, the probabilities of UC, CU, and UU are $\alpha_5 \cdot \beta_6 = 0.024$, $\beta_5 \cdot \alpha_6 = 0.700$, and $\beta_5 \cdot \beta_6 = 0.090$, respectively. This solution was also verified as a NE shown in Fig. 9 in appendix B.

E. Generation capacity constraints

In order to consider the generation capacity constraints, we modify gradually F_6 's generation capacity limit from 55

to 38 in Table 2, and go back to the situation of section B where the transmission line constraint of only l_A but l_B is considered. The limit of line l_A is assumed as 20MW as before.

After the first step in a solution process, the key player(F_6)'s choice is $[q_6^u, q_6^c] = [44.68, 40.60]$. Until the capacity limit decreases to the uncongestion strategy 44.68, the NE in section B does not change. During it decreases 44.68~ 40.265, F_6 keeps the role of key player, but the controlling power of key player drops gradually, and its profit also drops. The number 40.265 is the critical generation limit in this case, and it is very close to the congestion strategy 40.60 in section B. When the capacity limit of F_6 is 38, less than the critical generation limit, the role of key player is transferred to F_4 whose value of PTDF is -0.294 in section B, and the second priority after F_6 . The computation of (21) with an assumption of a key player F_4 gives an NE as follows; $[q_1 \sim q_3, q_5, q_6] = [41.94, 46.56, 24.11, 24.06, 38]$, and F_4 's choice $[q_4^u, q_4^c] = [25.56, 25.18]$ with probability of uncongestion $\alpha_4 = 0.844$. To verify the solution is a NE, the expected profit of each firm is illustrated as shown in Fig. 10 in Appendix B.

6. Conclusion

An algorithm for solving a mixed strategy transmission-and-generation-constrained Nash Equilibrium is presented in this paper. The difficulties of finding a mixed strategy in a multiplayer game are lessened greatly by introducing four heuristics. In case of one transmission line being congested, it is proposed that only one player may use a mixed strategy, while the others use a pure strategy. The key player who uses a mixed strategy is determined by the power transfer distribution factors. Under the heuristics, the two-level optimization is converted into a new framework with a nonlinear and linear equations. When a generation capacity limit restricts strongly the strategy of a key player, the role of a key player may be transferred to the other player. A new key player is determined also by the PTDF.

The algorithm for finding a key player, and using a solution of the nonlinear equation is tested on the IEEE 30-bus system. The equilibrium solved by the method is verified to satisfy the NE condition. The proposed algorithm is shown to be effective in reducing the search space for finding a key player, and efficient in computing a mixed strategy in a power market with transmission line congestions. The possibility of extension of the proposed method to multiple line congestions is shown in the numerical results for the two key player case.

Acknowledgements

The present research was conducted by the research fund

of Dankook University in 2011.

$$\begin{pmatrix} \Lambda_r \mathbf{e}_d \\ \mathbf{e}_d^t & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{d}^u \\ \lambda^u \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -\mathbf{e}_u^t \end{pmatrix} \cdot \mathbf{q}^a = \begin{pmatrix} \mathbf{a} \\ 0 \end{pmatrix} \quad (\text{A.6})$$

Appendix A

Sensitivities of generation quantities

We rearrange the generation firms as $1, \dots, N_g-1$ for the normal players, and N_g, N_g+1 for the key player's uncongestion variable and congestion variable, respectively. The generation variable \mathbf{q}_a is defined as an augmented vector with a dimension $(N_g+1) \times 1$;

$$\mathbf{q}_a = [\mathbf{q}_x^t, q^u, q^c]^t \quad (\text{A.1})$$

where \mathbf{q}_x is a column vector of generation variables of the normal players.

The incidence matrices \mathbf{N}_u and \mathbf{N}_c with a dimension $(N_g+1) \times N_d$ are defined as, for simple matrix forms of the equations,

$$\mathbf{N}_u(i, j) = \begin{cases} 1, & i < N_g, \text{ if the } i\text{th normal player is at the } j\text{th bus,} \\ i = N_g, & \text{ if the key player is located at the } j\text{th bus.} \\ 0, & \text{ Otherwise.} \end{cases}$$

$$\mathbf{N}_c(i, j) = \begin{cases} 1, & i < N_g, \text{ if the } i\text{th normal player is at the } j\text{th bus,} \\ i = N_g+1, & \text{ if the key player is located at the } j\text{th bus.} \\ 0, & \text{ Otherwise.} \end{cases}$$

At the lower level of the hierarchical optimization, the Lagrangian of the MO's objective function is

$$L = B(d) + \lambda \cdot (\sum_i q_i - \sum_j d_j) + \mu \cdot (T_{l_{\max}} - T_l) \quad (\text{A.2})$$

where $B(d)$ is the demand side benefit as specified in (1), λ and μ are the multipliers on equality and inequality constraints, respectively, and $T_{l_{\max}}$ is the power flow limit on a congested line l . The transmission power on the line is expressed as

$$T_l = [\mathbf{h}_g^t, \mathbf{h}_d^t] \cdot [\mathbf{q}_a^t, \mathbf{d}^t]^t \quad (\text{A.3})$$

where \mathbf{d} is a column vector of the demand power, \mathbf{h}_g and \mathbf{h}_d are the column vectors of the sensitivities of branch flows to \mathbf{q} and \mathbf{d} from (11).

The optimality conditions of the MO for the uncongested case and congested case are, respectively

$$\partial L / \partial d_j^u = a_j - r_j d_j^u - \lambda^u = 0, \quad \forall j \in D \quad (\text{A.4})$$

$$\partial L / \partial d_j^c = a_j - r_j d_j^c - \lambda^c = 0, \quad \forall j \in D \quad (\text{A.5})$$

Rearranging the linear equations (A.4) corresponding to the uncongestion variables with the supply-demand balance equation as (2), we have

where $\Lambda_r = \text{diag}(r_1, \dots, r_{N_d})$, \mathbf{e}_d is a column vector consisting of N_d number of 1, \mathbf{d}^u is a column vector consisting of power demand in the uncongested case, \mathbf{a} is a column vector containing the intercepts of the demand functions, and $\mathbf{e}_u = \mathbf{N}_u^t \cdot \mathbf{e}_d$ with a dimension $(N_g+1) \times 1$.

Similarly, rearranging the equations (A.5) of congestion variables with the balance equation and the congestion binding equation, we have

$$\begin{pmatrix} \mathbf{M}_u & \mathbf{h}_d \\ \mathbf{h}_d^t & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{d}^c \\ \mu \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -\mathbf{e}_c^t \end{pmatrix} \cdot \mathbf{q}^a = \begin{pmatrix} \mathbf{a} \\ 0 \\ T_{l_{\max}} \end{pmatrix} \quad (\text{A.7})$$

where \mathbf{M}_u is the square matrix in (A.6), and $\mathbf{e}_c = \mathbf{N}_c^t \cdot \mathbf{e}_d$ with a dimension $(N_g+1) \times 1$.

The sensitivities of power demands with respect to generation quantities, s_i^u and s_i^c , are obtained from the linear equations (A.6) and (A.7), respectively. On the other hand, the optimality conditions for generating firms at the upper level are expressed as (18). The coefficient matrices in (18) are

$$\mathbf{T}_u = \mathbf{N}_u \cdot \text{diag}(\delta_i r_i), \quad \forall i \in D \quad (\text{A.8})$$

$$\mathbf{T}_c = \mathbf{N}_c \cdot \text{diag}(\varepsilon_i r_i), \quad \forall i \in D \quad (\text{A.9})$$

$$\mathbf{T}_s = \text{diag}\{m_i + z_i(\alpha \cdot s_i^u + \beta \cdot s_i^c), m_x + r_x \cdot s_x^u, m_x + r_x \cdot s_x^c\}, \quad \forall i \in G_{-x} \quad (\text{A.10})$$

$$\mathbf{b}_s = [w_i - b_i, a_x - b_x, a_x - b_x]^t, \quad \forall i \in G_{-x} \quad (\text{A.11})$$

where $\delta_i = 1$ and $\varepsilon_i = 1$ when $i = x$, otherwise, $\delta_i = \alpha$ and $\varepsilon_i = \beta$. The z_i , and w_i are the i th element of the vectors, $z = \mathbf{N}_u \cdot [r_1, \dots, r_{N_d}]^t$, $w = \mathbf{N}_u \cdot [a_1, \dots, a_{N_d}]^t$, respectively.

Appendix B

Profit curves for NE verification

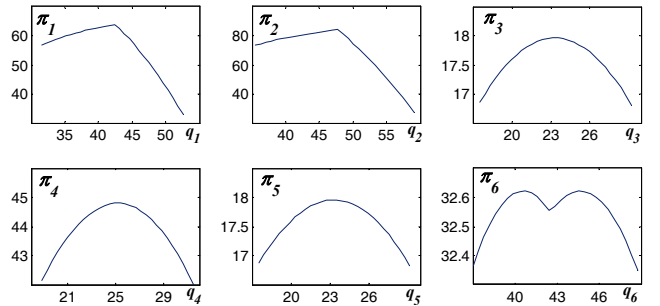


Fig. 8. Expected profits at NE with line l_A congested in IEEE 30-bus system

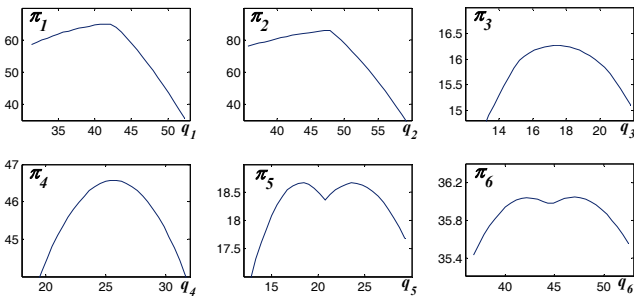


Fig. 9. Expected profits at NE with line l_A and line l_B congested in IEEE 30-bus system

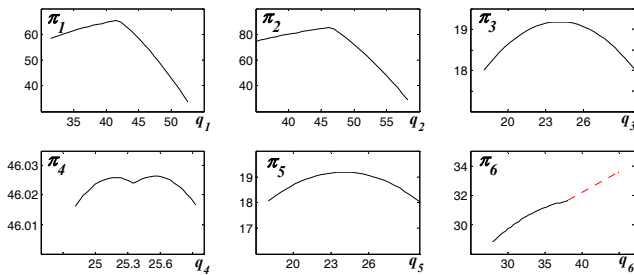


Fig. 10. Expected profits at NE with line l_A congested and generation capacity limit of F_6 as 38 in IEEE 30-bus system

References

[1] P. D. Klemperer and M. A. Meyer, "Supply function equilibria in oligopoly under uncertainty," *Econometrica*, Vol. 57, pp. 1243-1277, 1989.

[2] H. Nui and R. Baldick, "Supply function equilibrium bidding strategies with fixed forward contracts," *IEEE Trans. Power Syst.*, Vol. 20, No. 4, pp. 1859-1867, Nov. 2005.

[3] W. Xian, L. Yuzeng, and Z. Shaohua, "Oligopolistic equilibrium analysis for electricity market: a nonlinear complementarity approach," *IEEE Trans. Power Syst.*, Vol. 19, No. 3, pp. 1348-1355, Aug. 2004.

[4] V. P. Gountis, and A. G. Bakirtzis, "Efficient determination of Cournot equilibria in electricity markets," *IEEE Trans. Power Syst.*, Vol. 19, No. 4, pp. 1837-1844, Nov. 2004.

[5] F.A. Wolak and R. H. Patrick, "The Impact of market rules and market structure on the price determination process in the England and Wales electricity market," Univ. California Energy Inst., Berkeley, PWP-047, Apr. 1997.

[6] P. F. Correica, T. J. Overbye, and I. A. Hiskens, "Searching for noncooperative equilibria in centralized electricity markets," *IEEE Trans. Power Syst.*, Vol. 18, No. 4, pp. 1417-1424, Nov. 2003.

[7] K.H. Lee and R. Baldick, "Tuning of discretization in bimatrix game approach to power system market analysis," *IEEE Trans. Power Syst.*, Vol. 18, No. 2,

pp. 830-836, May 2003.

[8] K. H. Lee and R. Baldick, "Solving three-player games by the matrix approach with application to an electric power market," *IEEE Trans. Power Syst.*, Vol. 18, No. 4, pp. 1573-1580, Nov. 2003.

[9] J. D. Weber and T.J. Overbye, "An individual welfare maximization algorithm for electricity markets," *IEEE Trans. Power Syst.*, Vol. 17, No. 3, pp. 590-596, Aug. 2002.

[10] D. Fudenberg and J. Tirole, *Game Theory*. Cambridge, MA: MIT Press, 1991.

[11] B.F. Hobbs, "Linear complementarity model models of Nash-Cournot competition in bilateral and POOLCO power market," *IEEE Trans. Power Syst.*, Vol. 16, No. 2, pp. 194-202, May 2001.

[12] A. L. Motto and F. D. Galiana, "Coordination in markets with nonconvexities as mathematical program with equilibrium constraints-part I: a solution procedure," *IEEE Trans. Power Syst.*, Vol. 19, No. 1, Feb. 2004.

[13] B.F. Hobbs, "Strategic gaming analysis for electric power systems: an MPEC approach," *IEEE Trans. Power Syst.*, Vol. 15, No. 2, pp. 638-645, May 2000.

[14] T.C. Price, "Using co-evolutionary programming to simulate strategic behavior in markets," *J. Evol. Econ.* Vol. 7, pp.219-254, 1997.

[15] C. Richter and G. Sheble, "Genetic algorithm evolution of utility bidding strategies for the competitive marketplace," *IEEE Trans. Power Syst.*, Vol.13, No.1, pp.256-261, Feb. 1998.

[16] R.W. Ferrero, S.M. Shahidehpour, and V.C. Ramesh, "Transaction analysis in deregulated power systems using game theory," *IEEE Trans. Power Syst.*, Vol.12, No. 3, pp. 1340-1347, Aug. 1997.

[17] C.E. Lemke and J.T. Howson, "Equilibrium points of bimatrix game," *SIAM J. Appl. Math.*, Vol. 12, pp. 413-423, 1964.

[18] J. Contreras, M. Klusch, and J. B. Krawczyk, "Numerical solutions to Nash-Cournot equilibria in coupled constraint electricity markets," *IEEE Trans. Power Syst.*, Vol. 19, No. 1, pp. 195-206, Feb. 2004.

[19] A. V. Heusinger and C. Kanzow, "Optimization reformulations of the generalized Nash equilibrium problem using Nikaido-Isoda-type functions," *Computational Optimization and Applications*, Vol. 43, No. 2, pp. 353-377, 2009.

[20] J. B. Krawczyk, Jacek and Zuccolo, NIRA-3: An improved MATLAB package for finding Nash Equilibria in infinite games, in Working Paper, Dec. 2006.

[21] Severin Borenstein, James Bushnell, and Steven Stoff, "The Competitive effects of transmission capacity in a deregulated electricity industry," *RAND Journal of Economics*, Vol. 31, No. 2, pp. 294-325, Summer 2000.

[22] A. J. Wood and B. F. Wollenberg, *Power Generation*,

Operation, and Control, New York: Wiley-Interscience, 1996.

- [23] S. Song, J. Jeong, Y. T. Yoon and S. Moon, "Model of Information Exchange for Decentralized Congestion Management," *Journal of Electrical Engineering & Technology*, Vol .7, No. 2, pp. 141-150, 2012
- [24] Roy Gardner, *Games for Business and Economics*, John Wiley& Sons, Inc. 2003.



Kwan-Ho Lee He received his B.S., M.S., and Ph.D. degrees from Seoul National University in 1988, 1990, and 1995, respectively, in Electrical Engineering. He conducted research on reliability enhancement of power systems in the Korea Electrical Power Research Institute. He is presently a Professor in the Department of Electrical and Electronics Engineering at Dankook University, Yongin City, Korea.