# THE EFFECT OF INFLATION RISK AND SUBSISTENCE CONSTRAINTS ON PORTFOLIO CHOICE

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ABSTRACT. The optimal portfolio selection problem under inflation risk and subsistence constraints is considered. There are index bonds to invest in financial market and it helps to hedge the inflation risk. By applying the martingale method, the optimal consumption rate and the optimal portfolios are obtained explicitly. Furthermore, the quantitative effect of inflation risk and subsistence constraints on the optimal polices are also described.

## 1. INTRODUCTION

The inflation rate has an crucial impact on the individual's financial activities. If the inflation rate is high, the value of individual's wealth is decreasing and it sometimes gives restrictions to consume for living. Therefore the economic individual should incorporate the inflation risk while his financial activities like consumption and investment. The inflation risk is more sensitive to the long term perspectives so the portfolio choice problems under inflation risk are concerned with the pension fund managers or retirees. But there exist many difficulties when we treat the inflation risk. This is because the inflation risk is not hedgeable in real world. While the market uncertainty is manageable with risky assets in market the price process representing the inflation rate has no instrument to hedge the risk in financial market. As noticed in Gong and Remolona [3] and Kothari and Shanken [4], to manage the inflation risk there have been issued the inflation-indexed bond from the British government in 1981 and now its volume in 2008 is over $1.5 trillion among the international debt market. Thus it is possible for the economic agent to extend his investment scope into the index bond so he can manage the inflation risk in complete market.

We address the optimal portfolio problem of the investor who faces the inflation risk and subsistence constraints. As mentioned above, our individual problem is more likely close to retirees so their living cost is also an important factor to consider. The subsistence constraints implies that the economic agents should always consume with a higher level than predetermined amount. This is one of the typical habit formation which is the usual patterns of individual's consumption behavior. The subsistence constraints and inflation risk will affect the optimal

consumption and portfolios in a different way. Our object is to figure out their quantitative effects.

Our problem is the extended version of Merton [7, 8] with the inflation risk and subsistence constraints. This problem is also studies by Gong and Li [2] and they also give an explanations of the role of the index bond. We differentiate the method to resolve the optimization problem from their studies. Whereas they use a dynamic programming method and derive the Hamilton-Jacobi-Bellman equation, we apply the martingale method and also obtain not only value function but also the optimal controls explicitly. Moreover, we supplements the quantitative results by adding the effect of inflation risk and subsistence constraints on investments and comparison results with the classical Merton's solution. These supplementary studies give more adequate explanations about the role of index bonds when the investor has a subsistence constraints. We verify that the inflation risk is an important factor to select his optimal polices and the subsistence constraints makes the investor to reduce his consumption and investment on stock.

This paper is also related to other literatures about inflation risk and subsistence constraints. Brennan and Xia [1], Munk et al. [9], and Gong and Li [2] tackle the inflation risk by using dynamic programming method. Especially Brennan and Xia [1] and Munk et al. [9] consider the problem under stochastic environment where the interest rate is also stochastic so their methods need more than one state variables, which makes hard to resolve. The subsistence constraints also widely studied by Lakner and Nygren [6], Sethi, Taksar, and Presman [11], Gong and Li [2] and Shin et al. [10]. Most common feature of subsistence constraints is that the constraints lower the optimal consumption and portfolio level compared to the original problem and we have the same implications.

This paper is organized as follows. Section 2 explains the financial market setup and Section 3 states the optimization problem with martingale method. The closed-form solutions and numerical results are given in Section 4 and Section 5. Section 6 concludes.

## 2. FINANCIAL MARKET

We consider the continuous time financial market in which there are three kinds of assets to be invested. The financial assets are a risk free bond, an index bond, and a stock and the investor has to choose his investment ratios for each asset. It is assumed that the risk free bond has a constant nominal interest rate $R$ and stock price follows the geometric Brownian motion with constant drift $\mu$ and constant volatility $\sigma$. The return of an index bond is consist of real interest rate $r$ and inflation rate which reduces the real financial asset value. The inflation rate is expressed by the price process and it is also supposed to evolve

$$\frac{dP_t}{P_t} = \mu_p dt + \sigma_p dW_t^1,$$

where $\mu_p$ and $\sigma_p$ are constant coefficients and $W_t$ is the standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Therefore the risk free bond $B_t$, index bond $I_t$, and risky asset $S_t$ are

respectively governed by

$$\frac{dB_t}{B_t} = Rdt,$$

$$\frac{dI_t^0}{I_t^0} = rdt + \frac{dP_t}{P_t},$$

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma_s dW_t^2,$$

where $W_t^2$ is also the standard Brownian motion which is independent of $W_t^1$.[1] Since there exist two different risk sources and two assets containing each risk, the market is complete. In our financial setup, the investor can hedge the inflation risk by investing the index bond.

We denote the investor's consumption rate by $c_t$ and it is assumed to be progressively measurable with respect to $\mathcal{F}_t$, positive such that for all $t \geq 0$,

$$\int_0^\infty c_s ds < \infty, \text{ a.s.}$$

The portfolios for each assets are also denoted by $\pi_t^0, \pi_t^1$ and $\pi_t^2$ and they are supposed to be $\mathcal{F}_t$-measurable, adapted such that for all $t \geq 0$,

$$\int_0^\infty \pi_s^{i\,2} ds < \infty, \text{ almost surely(a.s.)}, \qquad i = 0, 1, 2.$$

Then investor's inflation adjusted real wealth process $X_t$ is derived by

$$dX_t = \pi_t^0 X_t \frac{dB_t}{B_t} + \pi_t^1 X_t \frac{dI_t}{I_t} + \pi_t^2 X_t \frac{dS_t}{S_t} - c_t dt - X_t \frac{dP_t}{P_t}$$

$$= [R + \pi_t^1(r - R) + \pi_t^2(\mu_s - R) - \mu_p(\pi_t^0 + \pi_t^2)]X_t dt$$

$$- c_t dt - (\pi_t^0 + \pi_t^2)X_t \sigma_p dW_t^1 + \pi_t^2 X_t \sigma_s dW_t^2, \qquad (2.1)$$

where $c_t$ is the real consumption and the initial wealth is a positive constant $X_0 = x$. Let us define a admissible consumption-portfolio plan pair $(c, \pi)$ if $X_t > 0$ for all $t \geq 0$.

To make the dynamic wealth process as static constraint, we introduce the equivalent martingale measure $\mathbb{Q}$ generated from the following exponential martingale process

$$\xi_t = e^{-\frac{1}{2}\int_0^t (\theta_1^2 + \theta_2^2)ds - \int_0^t \theta_1 dW_s^1 - \int_0^t \theta_2 dW_s^2},$$

where

$$\theta_1 = \sigma_p^{-1}(r + \mu_p - R),$$

$$\theta_2 = \sigma_s^{-1}(\mu_s - R).$$

Then the equivalent martingale measure $\mathbb{Q}$ is defined by

$$\mathbb{P}^{\mathbb{Q}}(A) \triangleq \mathbb{E}[\xi_T \mathbf{1}_A], \qquad \text{for } A \in \mathcal{F}_T.$$

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[1]The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is endowed with an augmented filtration $\{\mathcal{F}\}_{t=0}^\infty$ generated by the independent Brownian motions $W_t^1$ and $W_t^2$. It can be assumed to be correlated between $B_t$ and $W_t$. However, the correlated Brownian motions can be restructured into two independent Brownian motions.

By Girsanov's theorem, under the measure $\mathbb{Q}$ the standard Brownian motions are rewritten as,

$$d\widetilde{W}_t^1 = dW_t^1 + \theta_1 dt,$$
$$d\widetilde{W}_t^2 = dW_t^2 + \theta_2 dt.$$

Therefore, under the new measure $\mathbb{Q}$ the inflation risk adjusted wealth process becomes

$$dX_t = [rX_t - c_t]dt - (\pi_t^0 + \pi_t^2)X_t\sigma_p d\widetilde{W}_t^1 + \pi_t^2 X_t\sigma_s d\widetilde{W}_t^2.$$

We can verify that under the equivalent martingale measure $\mathbb{Q}$, the risk adjusted wealth process is the wealth process with real interest rate only. This implies that the inflation risk is hedgeable in financial market so we can choose a unique optimal portfolios under inflation risk. If we define the pricing kernel as $H_t = e^{-rt}\xi_t$, the wealth dynamics (2.1) is represented by

$$\mathbb{E}\left[\int_0^\infty H_t c_t dt\right] \le x. \tag{2.2}$$

This relation is the static wealth constraints for the dynamic wealth process and it plays a crucial role to apply the martingale method. Moreover the static wealth constraint is the same as that of an investor who does not face the inflation risk with no investment on an index bond.

We consider an additional constraints on consumption. The investor with no income stream represents the defined benefit pension fund managers or retirees and for the case of retirees they face the standard living cost. With the inflation risk, this subsistence constraints affect their optimal consumption rate and optimal portfolios either. Formally, we suggest that the investor has a subsistence constraints which restrict the consumption rate such that

$$c_t \ge L, \quad \text{for all} \quad t \ge 0, \tag{2.3}$$

where $L$ is the constant least living cost. Now we are ready for stating our objective function.

## 3. OPTIMIZATION PROBLEM

Our objective is maximizing the expected utility maximization by choosing consumption and portfolios. In this paper, the investor's preference is supposed to be defined by CRRA(constant relative risk aversion) utility function so that his/her satisfaction depends on his/her wealth level.[2] With the constant risk aversion $\gamma$, the utility function is expressed by

$$u(c_t) = \frac{1}{1-\gamma}c_t^{1-\gamma}.$$

It is easily checked that $u(\cdot)$ is a strictly increasing function and strictly concave. Then the agent's expected utility maximization problem with discount factor $\beta$ is represented as follows.

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[2]It is not hard to extend the problem with general utility which satisfies the regular conditions. Since we want to show the quantitative results for optimal polices, the concrete preference of the investor is more appropriate.

**Problem 1.** *An investor who can invest his wealth on an index bond and stock wants to maximize his expected utility function:*

$$V(x) = \max_{c,\pi} \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t) dt\right] \tag{3.1}$$

*subject to the wealth constraint (2.2) and the subsistence constraints (2.3).*

The agent's optimal choice variables are consumption rate and investments for index bond and risky assets. Then the savings are the remaining wealth after investing and consumption. To resolve this optimization problem we apply the martingale method, which is a main difference with Gong and Li [2]. They resolve the same optimization problem using dynamic programming principle which induces the HJB(Hamilton-Jacobi-Bellman) equation with suitable boundary conditions.

To apply the martingale method, we introduce the Lagrange multiplier $\lambda$ so that the dual value function is defined by

$$J(\lambda) = \max_{c,\pi} \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t) dt\right] - \lambda \mathbb{E}\left[\int_0^\infty H_t c_t dt\right]. \tag{3.2}$$

Since the CRRA utility function is concave and non-decreasing and has derivative function which is continuous, positive, and strictly decreasing, there exists the continuous inverse function $I : (0, \infty) \to (\bar{x}, \infty)$ of $u'(\cdot)$. Then the convex dual of a utility function $u(\cdot)$ is the function

$$\widetilde{u}(y) = \sup_{x \in \mathbb{R}} \{u(x) - xy\}, \qquad y \in \mathbb{R},$$

and it is easily check that

$$\tilde{u}(y) = [u(I(y)) - yI(y)]\mathbf{1}_{\{0 < y \le \tilde{y}\}} + [u(L) - Ly]\mathbf{1}_{\{y \ge \tilde{y}\}},$$

where $\tilde{y} = L^{-\gamma}$ and $I(y) = y^{-\frac{1}{\gamma}}$ for the CRRA utility function. The boundary $\tilde{y}$ is the threshold where the investor bounds the subsistence constraints so under that circumstance the investor's consumption rate should be constant as his minimum level $L$. With this covex dual function $\tilde{u}(\cdot)$, the dual value function can be written as

$$J(\lambda) = \mathbb{E}\left[\int_0^\infty e^{-\beta t} \tilde{u}(y_t) dt\right]$$

where $y_t = \lambda e^{\beta t} H_t$ and has a following differential form

$$\frac{dy_t}{y_t} = (\beta - r)dt - \theta_1 dW_t^1 - \theta_2 dW_t^2. \tag{3.3}$$

The optimal consumption rate is also determined while constructing the dual value function and it has the form of

$$c_t^* = \begin{cases} I(y), & 0 < y \le \tilde{y}, \\ R, & y \ge \tilde{y}. \end{cases}$$

If we confirm the dual value function, the value function of the original problem is given by the relation which can be obtained by finding the minimizing multiplier $\lambda$.

**Lemma 3.1.** *When the dual value function is given by the equation in (3.2), the value function of the primal problem is determined by*

$$V(x) = \inf_{\lambda > 0} \{J(\lambda) + \lambda x\}.$$

*Proof.* The value function $V(x)$ is obtained from $J(\lambda)$ by the Legendre transform inverse formula. See Section 3.8 of Karatzas and Shreve [5]. □

The optimal wealth process is determined in Lemma 3.1 by the first order conditoin and its dynamics totally depends on the process in (3.3). Therefore the optimal portfolios are obtained from comparing the wealth dynamics in (2.1) with the optimal wealth process.

## 4. THE SOLUTIONS

From the duality and Lemma 3.1, the investor's value function is obtained from the dual value function $J(\lambda)$. Our first step is a derivation of dual value function. Before that let us denote the two real solutions to the following quadratic equation by $\alpha_+ > 0, \alpha_- < 0$.

$$\frac{1}{2}(\theta_1^2 + \theta_2^2)\alpha^2 + \left(\beta - r - \frac{1}{2}(\theta_1^2 + \theta_2^2)\right)\alpha - \beta = 0.$$

**Proposition 4.1.** *The dual value function defined in (3.2) is obtained from*

$$J(\lambda) = \begin{cases} A\lambda^{\alpha_+} + \frac{\gamma}{K(1-\gamma)}\lambda^{-\frac{1-\gamma}{\gamma}}, & 0 < y \le \tilde{y}, \\ B\lambda^{\alpha_-} - \frac{L}{r}\lambda + \frac{L^{1-\gamma}}{\beta(1-\gamma)}, & y > \tilde{y}, \end{cases}$$

*where the constants $A, B$ and $K$ are determined by*

$$A = \frac{\frac{1}{K}\left(\frac{\gamma\alpha_-}{1-\gamma} + 1\right) + \frac{\alpha_- - 1}{r} - \frac{\alpha_-}{\beta(1-\gamma)}}{\alpha_+ - \alpha_-}L^{1-\gamma+\gamma\alpha_+},$$

$$B = \frac{\frac{1}{K}\left(\frac{\gamma\alpha_+}{1-\gamma} + 1\right) + \frac{\alpha_+ - 1}{r} - \frac{\alpha_+}{\beta(1-\gamma)}}{\alpha_+ - \alpha_-}L^{1-\gamma+\gamma\alpha_-}.$$

$$K = r + \frac{\beta - r}{\gamma} - \frac{1-\gamma}{2\gamma^2}(\theta_1^2 + \theta_2^2). \tag{4.1}$$

*Proof.* For the CRRA utility function, the dual value function in (3.2) is characterized by

$$J(\lambda) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left\{\frac{\gamma}{1-\gamma}\left(\lambda e^{\beta t}H_t\right)^{-\frac{1-\gamma}{\gamma}}\mathbf{1}_{\{0<y\le\tilde{y}\}} + \left(\frac{L^{1-\gamma}}{1-\gamma} - L\left(\lambda e^{\beta t}H_t\right)\right)\mathbf{1}_{\{y\ge\tilde{y}\}}\right\}dt\right].$$

The second term represents the value function when the investor bounds the subsistence constraints so he consumes only $L$.

With $y_t$ in (3.3), if we define the indirect utility function at time $t$ by

$$\phi(t, y) = \mathbb{E}_t\left[\int_t^\infty e^{-\beta s}\left\{\frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}}\mathbf{1}_{\{0<y\le\tilde{y}\}} + \left(\frac{L^{1-\gamma}}{1-\gamma} - Ly\right)\mathbf{1}_{\{y\ge\tilde{y}\}}\right\}ds\right],$$

then by Feynmman-Kac's formula, we have the PDE(partial differential equation) for $\phi(t, y)$

$$\begin{cases} \mathcal{L}\phi(t, y) + e^{-\beta t}\frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}} = 0, & 0 < y \leq \tilde{y} \\ \mathcal{L}\phi(t, y) + e^{-\beta t}\left(\frac{L^{1-\gamma}}{1-\gamma} - Ly\right) = 0, & y \geq \tilde{y} \end{cases}$$

where the differential operator is defined by

$$\mathcal{L}\phi(t, y) = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial y}y(\beta - r) + \frac{1}{2}\frac{\partial^2\phi}{\partial y^2}y^2(\theta_1^2 + \theta_2^2).$$

If we suggest the separable solution as $\phi(t, y) = e^{-\beta t}v(y)$, the PDE is reduced to the ODE(ordinary differential equation) for $v(y)$:

$$\begin{cases} \frac{1}{2}y^2v''(y)(\theta_1^2 + \theta_2^2) + yv'(y) - \beta v(y) + \frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}} = 0, & 0 < y \leq \tilde{y} \\ \frac{1}{2}y^2v''(y)(\theta_1^2 + \theta_2^2) + yv'(y) - \beta v(y) + \frac{L^{1-\gamma}}{1-\gamma} - Ly = 0, & y \geq \tilde{y} \end{cases} \quad (4.2)$$

From the variation of parameter, we can get the solution to the ODE by adding the general solution and particular solution. For the region $0 < y \leq \tilde{y}$, the general solution part for positive power remains and the opposite part remains for $y \geq \tilde{y}$ i.e.,

$$v_g(v) = \begin{cases} Ay^{\alpha_+}, & 0 < y \leq \tilde{y} \\ By^{\alpha_-}, & y \geq \tilde{y} \end{cases}$$

The particular solutions are also easily obtained by guessing

$$v_p(v) = \begin{cases} A'y^{-\frac{1-\gamma}{\gamma}}, & 0 < y \leq \tilde{y} \\ B'y + C, & y \geq \tilde{y} \end{cases}$$

If we substitute the function and its derivatives into the ODE (4.2), with the constant $K$ in (4.1) the coefficient $A'$, $B'$ and $C$ are given by $A = \frac{\gamma}{K(1-\gamma)}, B' = -\frac{L}{r}$, and $C = -\frac{L^{1-\gamma}}{\beta(1-\gamma)}$. Thus the dual value function is a sum of the general and particular solutions($v(y) = v_g(y) + v_p(y)$) and the coefficients $A$ and $B$ are determined by the smooth-pasting condition at $\tilde{y} = L^{-\gamma}$.

Accordingly, the dual value function is derived by

$$J(\lambda) = \phi(0, \lambda) = v(y).$$

$\square$

Since the dual value function is obtained explicitly, by Lemma 3.1 the value function defined in (3.1) is able to be expressed explicitly either.

**Theorem 4.1.** *The value function in (3.1) of the economic agent who faces the inflation risk and has subsistence constraints is given by*

$$V(x) = \begin{cases} B\left[\frac{\frac{L}{r} - x}{B\alpha_-}\right]^{\frac{\alpha_-}{\alpha_- - 1}} + \left(x - \frac{L}{r}\right)\left[\frac{\frac{L}{r} - x}{B\alpha_-}\right]^{\frac{1}{\alpha_- - 1}} + \frac{L^{1-\gamma}}{\beta(1-\gamma)}, & L/r < x \leq \tilde{x} \\ A(\lambda^*)^{\alpha_+} + \frac{\gamma}{K(1-\gamma)}(\lambda^*)^{-\frac{1-\gamma}{\gamma}} + \lambda^*x, & x \geq \tilde{x} \end{cases},$$

*where $\lambda^*$ and the boundary value $\tilde{x}$ are determined by*

$$A\alpha_+(\lambda^*)^{\alpha_+-1} - \frac{1}{K}(\lambda^*)^{-\frac{1-\gamma}{\gamma}} + x = 0,$$

$$\tilde{x} \triangleq -A\alpha_+ L^{-\gamma\alpha_++\gamma} + \frac{1}{K}L^{1-\gamma} = -B\alpha_- L^{-\gamma\alpha_-+\gamma} + \frac{L}{r}.$$

*Proof.* The value function of the original problem is easily confirmed from Lemma 3.1 and Proposition 4.1. Since $-J'(\lambda)$ is decreasing function, there exists one-to-one correspondence between $y$ and $x$. This implies that the interval $(0, \tilde{y}]$ corresponds to $[\tilde{x}, \infty)$ and $(\tilde{y}, \infty)$ corresponds to $[L/r, \tilde{x})$. Here $L/r$ is the minimum wealth level due to the subsistence constraints.

Thus for $x \in [L/r, \tilde{x})$, the optimal $\lambda^*$ is induced by

$$\lambda^* = \left[\frac{\frac{L}{r} - x}{B\alpha_-}\right]^{\frac{1}{\alpha_--1}},$$

and the value function is obtained from substituting optimal $\lambda^*$ in Lemma 3.1. On the contrary for $x \in [\tilde{x}, \infty)$ the $\lambda^*$ is the solution to the algebraic equation derived also in Lemma 3.1. Finally the boundary $\tilde{x}$ is determined at $\tilde{y}$.                                                   $\square$

In contrast with the optimal consumption rate, it is hard to find the optimal portfolios directly. This is the big drawback of martingale method compared to dynamic programming principle. The optimal wealth process, however, contains the optimal portfolios and its differential form gives the optimal portfolios explicitly. The closed-form solutions for optimal policies are given in the next theorem.

**Theorem 4.2.** *The investor who wants to maximize his expected utility in (3.1) has the optimal wealth process as*

$$X_t^* = \begin{cases} -B\alpha_-\left(y_t^{\lambda^*}\right)^{\alpha_--1} + \frac{L}{r}, & L/r < x \le \bar{x}, \\ -A\alpha_+\left(y_t^{\lambda^*}\right)^{\alpha_+-1} + \frac{1}{K}\left(y_t^{\lambda^*}\right)^{-\frac{1}{\gamma}}, & x > \bar{x}. \end{cases} \tag{4.3}$$

*Furthermore, the investor's optimal consumption rate and portfolios are also given by*

$$c_t^* = \begin{cases} R, & L/r < x \le \bar{x}, \\ \left(y_t^{\lambda^*}\right)^{-\frac{1}{\gamma}}, & x > \bar{x}, \end{cases}$$

$$\pi_t^{0*} = \begin{cases} -\left(\frac{\theta_1}{\sigma_p} + \frac{\theta_2}{\sigma_s}\right)(\alpha_- - 1)\left(\frac{L}{r} - x\right), & L/r < x \le \bar{x}, \\ -\left(\frac{\theta_1}{\sigma_p} + \frac{\theta_2}{\sigma_s}\right)\left[A\alpha_+(\alpha_+ - 1)\left(y_t^{\lambda^*}\right)^{\alpha_+-1} + \frac{1}{\gamma K}\left(y_t^{\lambda^*}\right)^{-\frac{1}{\gamma}}\right], & x > \bar{x}, \end{cases}$$

$$\pi_t^{1*} = \begin{cases} \frac{\theta_1}{\sigma_p}(\alpha_- - 1)\left(\frac{L}{r} - x\right) + x, & L/r < x \le \bar{x}, \\ \frac{\theta_1}{\sigma_p}\left[A\alpha_+(\alpha_+ - 1)\left(y_t^{\lambda^*}\right)^{\alpha_+-1} + \frac{1}{\gamma K}\left(y_t^{\lambda^*}\right)^{-\frac{1}{\gamma}}\right] + x, & x > \bar{x}, \end{cases}$$

$$\pi_t^{2*} = \begin{cases} \frac{\theta_2}{\sigma_s}(\alpha_- - 1)\left(\frac{L}{r} - x\right), & L/r < x \leq \bar{x}, \\ \frac{\theta_2}{\sigma_s}\left[A\alpha_+(\alpha_+ - 1)\left(y_t^{\lambda^*}\right)^{\alpha_+ - 1} + \frac{1}{\gamma K}\left(y_t^{\lambda^*}\right)^{-\frac{1}{\gamma}}\right], & x > \bar{x}, \end{cases}.$$

*Proof.* From the first order condition in Lemma 3.1, the optimal wealth process is induced as a function of $\lambda^*$. Therefore with the dynamics of $y_t^{\lambda^*}$ in (3.3), we can obtain the optimal wealth process $X_t^*$.

For $x \geq \tilde{x}$, the optimal wealth dynamics is governed by

$$dX_t^* = \left[-A\alpha_+(\alpha_+ - 1)\left(y_t^{\lambda^*}\right)^{\alpha_+ - 1} - \frac{1}{\gamma K}\left(y_t^{\lambda^*}\right)^{-\frac{1}{\gamma}}\right]\{(\beta - r)dt - \theta_1 dW_t^1 - \theta_2 dW_t^2\}$$

$$+ \frac{1}{2}\left[-A\alpha_+(\alpha_+ - 1)(\alpha_+ - 2)\left(y_t^{\lambda^*}\right)^{\alpha_+ - 1} + \frac{1 + \gamma}{\gamma^2 K}\left(y_t^{\lambda^*}\right)^{-\frac{1+\gamma}{\gamma}}\right](\theta_1^2 + \theta_2^2)dt$$

$$= [r + \pi_t^{2*}(\mu_s - R) + \pi_t^{1*}(r - R) - (\pi_t^{0*} + \pi_t^{2*})\mu_p]X_t^* dt - c_t^* dt$$

$$+ \pi_t^{2*}X_t^*\sigma_s dB_t - (\pi_t^{0*} + \pi_t^{2*})X_t^*\sigma_p dW_t$$

where the optimal policies are defined by

$$c_t^* = \left(y_t^{\lambda^*}\right)^{-\frac{1}{\gamma}},$$

$$-(\pi_t^{0*} + \pi_t^{2*})X_t^* = \frac{\theta_2}{\sigma_p}\left[-A\alpha_+(\alpha_+ - 1)\left(y_t^{\lambda^*}\right)^{\alpha_+ - 1} - \frac{1}{\gamma K}\left(y_t^{\lambda^*}\right)^{-\frac{1}{\gamma}}\right],$$

$$\pi_t^{2*}X_t^* = \frac{\theta_1}{\sigma_s}\left[-A\alpha_+(\alpha_+ - 1)\left(y_t^{\lambda^*}\right)^{\alpha_+ - 1} - \frac{1}{\gamma K}\left(y_t^{\lambda^*}\right)^{-\frac{1}{\gamma}}\right].$$

Similarly, for $L/r < x \leq \tilde{x}$, we have

$$dX_t^* = \left[-B\alpha_-(\alpha_- - 1)\left(y_t^{\lambda^*}\right)^{\alpha_- - 1}\right]\{(\beta - r)dt - \theta_1 dW_t^1 - \theta_2 dW_t^2\}$$

$$+ \frac{1}{2}\left[-B\alpha_-(\alpha_- - 1)(\alpha_- - 2)\left(y_t^{\lambda^*}\right)^{\alpha_- - 1}\right](\theta_1^2 + \theta_2^2)dt$$

$$= [r + \pi_t^{2*}(\mu_s - R) + \pi_t^{1*}(r - R) - (\pi_t^{0*} + \pi_t^{2*})\mu_p]X_t^* dt - c_t^* dt$$

$$+ \pi_t^{2*}X_t^*\sigma_s dB_t - (\pi_t^{0*} + \pi_t^{2*})X_t^*\sigma_p dW_t$$

where the optimal policies are defined by

$$c_t^* = R,$$

$$-(\pi_t^{0*} + \pi_t^{2*})X_t^* = \frac{\theta_2}{\sigma_p}\left[-B\alpha_-(\alpha_- - 1)\left(y_t^{\lambda^*}\right)^{\alpha_- - 1}\right],$$

$$\pi_t^{2*}X_t^* = \frac{\theta_1}{\sigma_s}\left[-B\alpha_-(\alpha_- - 1)\left(y_t^{\lambda^*}\right)^{\alpha_- - 1}\right].$$

By substituting the optimal wealth in (4.3) into each optimal policies, we get the results. □

We have the intuitive results for the optimal portfolios. In reality, the nominal interest rate is higher than the sum of real interest rate and inflation rate so the generator $\theta_1$ of the equivalent martingale measure $\mathbb{Q}$ has a negative value in usual. So the saving amount is composed of two components which are borrowing for stock investment and index bond related amount. We have more graphical results in the next section.

## 5. NUMERICAL RESULTS

In this section, we verify the quantitative effect on optimal policies. The inflation risk and subsistence constraints have impact on the optimal policies in different ways. To analyze the subsistence constraints in optimal consumption and portfolio, the results for the classical Merton's problem are revisited. For more concrete example, the parameters in our model are adopted from Gong and Li [2] and this would give the comparison results with theirs. Let us set the parameters as follows.

$$\gamma = 2, \quad \beta = 0.07, \quad L = 0.5, \quad R = 0.07, \quad r = 0.04,$$

$$\mu = 0.09, \quad \sigma = 0.2, \quad \mu_p = 0.035, \quad \sigma_p = 0.05.$$

The inflation rate parameter $\mu_p$ could have various values but we define inflation rate as a high level to show its impact more concretely.

Fig. 1 shows the optimal consumption rate. The thin line is the results for Merton's problem and thick line is our optimal consumption rate. Near the minimum wealth level, the investors's consumption rate is constant and it approach to the Merton's solution as wealth increases. This is the similar results in Shin et al. [10] in which the subsistence constraints are considered only.
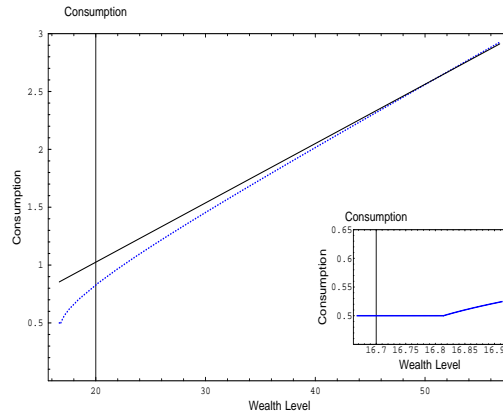


FIGURE 1. This figure is the optimal consumption rate when the investor faces the inflation risk and subsistence constraints with the parameters $\gamma = 2, \beta = 0.07, L = 0.5, R = 0.07, r = 0.03, \mu_s = 0.09, \sigma_s = 0.2, \mu_p = 0.035, \sigma_p = 0.05$. The thick line is our optimal consumption rate and the thin line is the results of the classical Merton's problem
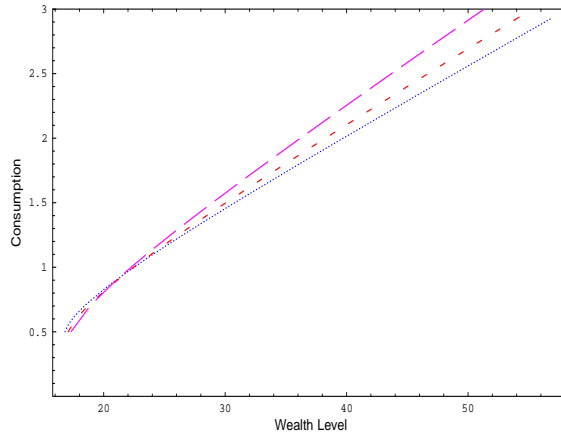
FIGURE 2. This figure is a comparison result of the optimal consumption rate with different inflation rate. The parameters are given by $\gamma = 2, \beta = 0.07, L = 0.5, R = 0.07, r = 0.03, \mu_s = 0.09, \sigma_s = 0.2, \mu_p = 0.035, \sigma_p = 0.05$. The dotted line and dashed line are the optimal consumptions of the model with $\mu_p = 0.03$ and $\mu_p = 0.025$ respectively, and the other line is our benchmark model with $\mu_p = 0.035$.

The effect of inflation risk is described in Fig. 2. As the inflation rate increases, the optimal consumption rate decreases as we expected. The plain line is the our benchmark model with inflation rate $\mu_p = 0.35$ and the dashed line is the line with $\mu_p = 0.25$. We can confirm that the inflation risk is very important determinant to consume.

We have interesting results for the portfolio ratios. In Fig. 3 the overall results are similar with that of Gong and Li [2] but in Fig. 4, we have additional results for the high inflation rate case. The thick line represents the stock investment and dashed line is the investment on index bond and the dotted line is the savings after stock and index bond investment. Moreover the thin line represents the Merton's optimal portfolio. In Fig. 3, the investor invests his considerable portion of wealth to index bond near the minimum wealth level but the amount shrinks as his wealth increases. This is because the role of index bond is the reducing volatility of wealth so near the minimum wealth level he reduce stock investment and hedge inflation risk by using index bond. If he has enough wealth not to care about the inflation risk, he acts as if there are no inflation risk and index bond in financial market. In the other words, his financial market setup is similar to that of Merton's problem. Therefore, it is also confirmed that the stock investment approach to the Merton's solution as wealth increases.

Fig. 4 shows additional portfolios when the inflation risk is high so that the index bond premium defined by $(r + \mu_p - R)$ is positive. The main difference with Fig. 3 is the sign of index bond premium. This is the very intuitive results, however. Since there is no limits to borrowing, the investor borrow from bank as much as possible and invest index bond and stock because the return of index bond and stock is always higher than savings.
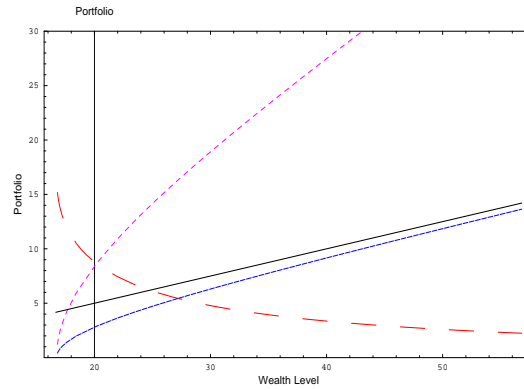
FIGURE 3. This figure is the optimal portfolios when the investor faces the inflation risk and subsistence constraints with the parameters $\gamma = 2, \beta = 0.07, L = 0.5, R = 0.07, r = 0.03, \mu_s = 0.09, \sigma_s = 0.2, \mu_p = 0.035, \sigma_p = 0.05$. The thick line is the optimal stock investment and dashed line is the index bond investments. The dotted line represents the savings after investing stock and index bond. The thin line is the optimal portfolio of the classical Merton's problem
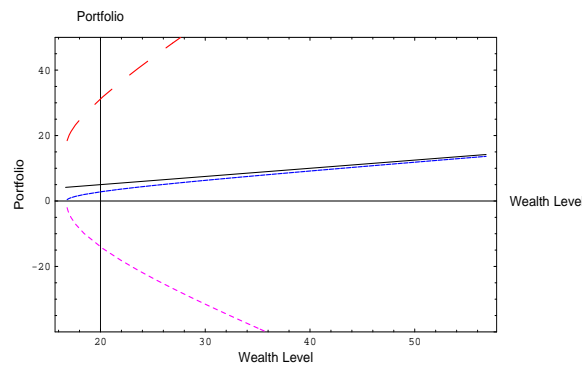


FIGURE 4. This figure is the optimal portfolios when the investor faces the inflation risk and subsistence constraints with the parameters $\gamma = 2, \beta = 0.07, L = 0.5, R = 0.07, r = 0.03, \mu_s = 0.09, \sigma_s = 0.2, \mu_p = 0.045, \sigma_p = 0.05$. The thick line is the optimal stock investment and dashed line is the index bond investments. The dotted line represents the savings after investing stock and index bond. The thin line is the optimal portfolio of the classical Merton's problem

We have comparison results for the optimal portfolios in Fig. 5. The left figure is the optimal stock investments and the right figure is the index bond holdings. In a similar manner of optimal consumption comparison, the plain line is our benchmark case and the dashed line is the smallest inflation rate case. In our model, the lower inflation rate the higher return on savings so the investor use 'short-selling' strategy when the inflation rate is low enough. The relatively high return on savings also reduces the stock investments.
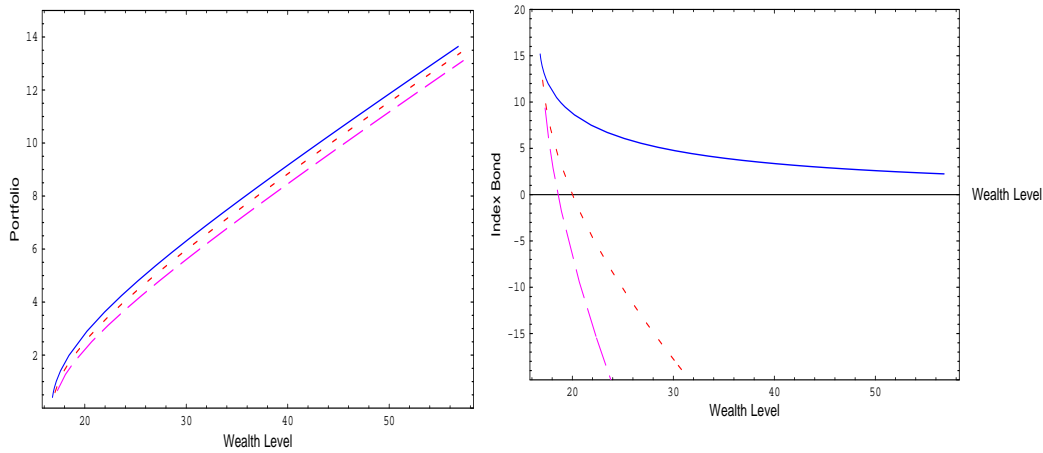


FIGURE 5. These figures are comparison results of the optimal stock investments and index bond investments with different inflation rate. The parameters are given by $\gamma = 2, \beta = 0.07, L = 0.5, R = 0.07, r = 0.04, \mu = 0.09, \sigma = 0.2, \mu_p = 0.035, \sigma_p = 0.05$. The dotted line and dashed line are the optimal consumptions of the model with $\mu_p = 0.03$ and $\mu_p = 0.025$ respectively, and the other line is our benchmark model with $\mu_p = 0.035$.

## 6. CONCLUSION

In this paper we investigate the optimal portfolio selection problem of the investor who faces the inflation risk and subsistence constraints. Our paper offers a different method to resolve the problem from Gong and Li [2]. Since the inflation risk is the new source of uncertainty in the financial market, the standard dynamic programming principle approach has a difficulty in treat the new uncertainty. As Gong and Li [2], our financial market has an index bond to hedge inflation risk so we can find the unique equivalent martingale measure to apply the martingale method. Nevertheless, our approach has other advantages compared to DPP approach such as deriving optimal consumption and no need of verification theorem for optimal policies. Furthermore other extension models like voluntary retirement are also applicable easily.

We also supplement the numerical results when the investor faces the inflation risk and subsistence constraints. At enough wealth level, the investor acts as if his economy is same as

that of Merton's problem. Moreover, it is verified that the inflation risk is important determinant when he selects his consumption and portfolio near the minimum wealth level. Our results could be extended to the case of correlated risks with little difficulty but the main characteristics would have similar results for our model.

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