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ON MULTIPLIERS OF BCC-ALGEBRAS

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Abstract. In this paper, we introduced the notion of multiplier of a BCC-algebra, and gave some properties of BCC-algebras. Also, we characterized kernels and normal ideals of multipliers on BCC-algebras.

1. Introduction

A BCK-algebra is an important class of logical algebras introduced by K. Iséki ([5]) and was extensively investigated by several researchers. The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [10]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori ([7]) introduced a notion of BCC-algebras, and W. A. Dudek ([1, 2]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. C. Prabpayak and U. LerrawatIn ([8]) introduced the derivation of BCC-algebra. In [9] a partial multiplier on a commutative semigroup $(A, \cdot)$ has been introduced as a function F from a nonvoid subset D_F of A into A such that $F(x) \cdot y = x \cdot F(y)$ for all $x, y \in D_F$. In this paper, we introduce the notion of multiplier of a BCC-algebra, and give some properties of BCC-algebras. Also, we characterize Kernel of multipliers on BCC-algebras.

2. Preliminary

An algebra $X = (X, *, 0)$ of type (2,0) is called a *BCC-algebra* if it satisfies the following axioms: For all $x, y, z \in X$,

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(1) $((x * y) * (z * y)) * (x * z) = 0$,
(2) $x * x = 0$,
(3) $0 * x = 0$,
(4) $x * 0 = x$,
(5) $x * y = 0$ and $y * x = 0$ imply $x = y$.
Note that $(x * y) * x = 0$ from (1).

A subset S is subalgebra of X if $x * y \in S$ for all $x, y \in S$. Note that ([2]) *a BCC-algebra is a BCK-algebra iff it satisfies the identity*
(6) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$,
which holds in all BCK-algebras. Methods of construction of BCC-algebras from the given BCK-algebras are described in [2] and [3].

The class of all BCC-algebra is a quasivariety ([7]), but many subclasses of this quasivariety form variety ([2]). Also the quasivariety of all BCK-algebras has many well described subclasses which are varieties. On any BCC-algebra X (similarly, as in the case of BCK-algebras) one can define the natural order $\leq$ putting
(7) $x \leq y$ *if and only if* $x * y = 0$ for all $x, y \in X$,

It is not difficult to verify that this order is partial and 0 is its smallest element. Moreover, for all $x, y, z \in X$
(8) $(x * y) * (z * y) \leq x * z$,
(9) $x \leq y$ *implies* $x * z \leq y * z$ *and* $z * y \leq z * x$.

For elements x and y of a BCC-algebra X, we denote $x \wedge y = y * (y * x)$.

A BCC-algebra is said to be *commutative* if it satisfies for all $x, y \in X$,

$$x * (x * y) = y * (y * x), \quad \text{i.e.,} \quad x \wedge y = y \wedge x.$$

Let X be a BCC-algebra. A subset I of a BCC-algebra X is called an *ideal* if it satisfies
(1) $0 \in I$,
(2) If $y \in I$ and $x * y \in I$, then $x \in I$ for all $x, y \in X$.

3. Multipliers of BCC-algebras

In what follows, let X denote a BCC-algebra unless otherwise specified.

Definition 3.1. Let X be a BCC-algebra. A self-map f of X is called a *multiplier* if

$$f(x * y) = f(x) * y$$

for all $x, y \in X$.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ a set in which "$*$" are defined by

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	2	0

It is easy to check that $(X, *)$ is a BCC-algebra. Define a map $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 1, 2 \\ 1 & \text{if } x = 3 \end{cases}$$

Then it is easy to check that f is a multiplier of a BCC-algebra X.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ a set in which "$*$" are defined by

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	1
3	3	3	3	0

It is easy to check that $(X, *)$ is a BCC-algebra. Define a map $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 1, 2 \\ 3 & \text{if } x = 3 \end{cases}$$

Then it is easy to check that f is a multiplier of a BCC-algebra X.

Example 3.4. The identity mapping ϵ, the unit mapping $\iota : a \longmapsto 1$ are multipliers of X.

Proposition 3.5. Let f be a multiplier of X. Then we have $f(x*f(x)) = 0$ for all $x \in X$.

Proof. Let $x \in X$. Then $f(x * f(x)) = f(x) * f(x) = 0$. This completes the proof. $\square$

Definition 3.6. A self-map f of X is said to be *regular* if $f(0) = 0$.

Proposition 3.7. Let f be a regular multiplier of X. Then the following identities hold.

(i) $f(x) \leq x$ for all $x \in X$.

(ii) $f(x * y) \leq f(x) * f(y)$ for all $x, y \in X$.

Proof. (i) For all $x \in X$, we have $0 = f(0) = f(x * x) = f(x) * x$, that is, $f(x) \leq x$.

(ii) Since $f(y) \leq y$ for all $y \in X$, we have $f(x * y) = f(x) * y \leq f(x) * f(y)$ by (9). □

Definition 3.8. Let X be a BCC-algebra and f be a self-map of X. If $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in X$, f is said to be *isotone*.

Proposition 3.9. Let f be a regular multiplier of X. If f is an endomorphism on X, then f is isotone.

Proof. Let $x \leq y$. Then $x * y = 0$ and $0 = f(0) = f(x * y) = f(x) * f(y)$. Hence $f(x) \leq f(y)$. This completes the proof. □

Proposition 3.10. Let f is a non-expansive map on a BCC-algebra X, i.e., $f(x) \leq x$ for all $x \in X$. Then $f(x) * y \leq x * f(y)$ for all $x, y \in X$.

Proof. Suppose that f is a non-expansive map on X and $x, y \in X$. Then $f(x) \leq x$ and $f(y) \leq y$. Hence $f(x) * y \leq x * y$ and $x * y \leq x * f(y)$ by (9). It follows that $f(x) * y \leq x * f(y)$. □

Proposition 3.11. Let f be a multiplier of a BCC-algebra. Define $f^2(x) = f(f(x))$ for all $x \in X$. If $f^2 = f$, then f is regular.

Proof. Since $0 = x * x$ for all $x \in X$, we have $f(0) = f(f(0) * f(0)) = f^2(0) * f(0) = f(0) * f(0) = 0$. □

Let X be a BCC-algebra and f_1, f_2 two self-maps. We define $f_1 \circ f_2 : X \to X$ by

$$(f_1 \circ f_2)(x) = f_1(f_2(x))$$

for all $x \in X$.

Proposition 3.12. Let X be a BCC-algebra and f_1, f_2 two multipliers. Then $f_1 \circ f_2$ is also a multiplier of X.

Proof. Let X be a BCC-algebra and f_1, f_2 two multipliers. Then we have

$$\begin{aligned}(f_1 \circ f_2)(a * b) &= f_1(f_2(a * b)) \\ &= (f_1(f_2(a) * b)) \\ &= f_1(f_2(a)) * b \\ &= (f_1 \circ f_2)(a) * b\end{aligned}$$

for all $a, b \in X$. This completes the proof. □

Let X be a BCC-algebra and f_1, f_2 two self-maps. We define $(f_1 \wedge f_2)(x)$ by

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$$

for all $x \in X$.

Proposition 3.13. Let X be a BCC-algebra and f_1, f_2 two multipliers. Then $f_1 \wedge f_2$ is also a multiplier of X.

Proof. Let X be a BCC-algebra and f_1, f_2 two multipliers. Then we have

$$\begin{aligned}(f_1 \wedge f_2)(a * b) &= f_1(a * b) \wedge f_2(a * b) \\ &= f_1(a) * b \wedge f_2(a) * b \\ &= f_2(a) * b * ((f_2(a) * b) * (f_1(a) * b)) \\ &= (f_2(a) * b) * (f_2(a) * f_1(a)) \\ &= ((f_2(a) * (f_2(a) * f_1(a))) * b \\ &= (f_1(a) \wedge f_2(a)) * b \\ &= (f_1 \wedge f_2)(a) * b.\end{aligned}$$

for all $a, b \in X$. This completes the proof. □

Let X_1 and X_2 be two BCC-algebras. Then $X_1 \times X_2$ is also a BCC-algebra with respect to the point-wise operation given by

$$(a, b) * (c, d) = (a * c, b * d)$$

for all $a, c \in X_1$ and $b, d \in X_2$.

Proposition 3.14. Let X_1 and X_2 be two BCC-algebras. Define a map $f : X_1 \times X_2 \to X_1 \times X_2$ by $f(x, y) = (x, 0)$ for all $(x, y) \in X_1 \times X_2$. Then f is a multiplier of $X_1 \times X_2$ with respect to the point-wise operation.

Proof. Let $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. The we have

$$\begin{aligned}(f((x_1, y_1) * (x_2, y_2))) &= f(x_1 * x_2, y_1 * y_2)\\ &= (x_1 * x_2, 0)\\ &= (x_1 * x_2, 0 * y_2)\\ &= (x_1, 0) * (x_2, y_2)\\ &= f(x_1, y_1) * (x_2, y_2).\end{aligned}$$

Therefore f is a multiplier of the direct product $X_1 \times X_2$. □

Let f be a multiplier of X. Define a set $Fix_f(X)$ by

$$Fix_f(X) := \{x \in X \mid f(x) = x\}$$

for all $x \in X$.

Proposition 3.15. Let f be a multiplier of X. If $x \in Fix_f(X)$, then we have $(f \circ f)(x) = x$.

Proof. Let $x \in Fix_f(X)$. Then we have

$$(f \circ f)(x) = f(f(x)) = f(x) = x.$$

This completes the proof. □

Proposition 3.16. Let f be a multiplier of a BCC-algebra X. Then $Fix_f(X)$ is a subalgebra of X.

Proof. Let f be a multiplier of X and $x, y \in Fix_f(X)$. Then we have $f(x) = x$ and $f(y) = y$, and so $f(x * y) = f(x) * y = x * y$. This implies $x * y \in Fix_f(X)$. This completes the proof. □

Proposition 3.17. Let X be a BCC-algebra and f be a multiplier of X. If $x \in X$ and $y \in Fix_f(X)$, then $x \wedge y \in Fix_f(X)$.

Proof. Let f be a multiplier and $y \in Fix_f(X)$. Then we have $f(y) = y$, and so

$$f(x \wedge y) = f(y * (y * x)) = f(y) * (y * x)$$
$$= y * (y * x)) = x \wedge y.$$

This completes the proof. $\square$

Let us recall from [9] that the composition of two multipliers f and g of a BCC-algebra X is a multiplier of X where $(f \circ g)(x) = f(g(x))$ for all $x \in X$.

Theorem 3.18. Let f and g be two multipliers of X such that $f \circ g = g \circ f$. Then the following conditions are equivalent.

(i) $f = g$.

(ii) $f(X) = g(X)$.

(iii) $Fix_f(X) = Fix_g(X)$.

Proof. (i)$\Rightarrow$ (ii): It is obvious.

(ii) $\Rightarrow$ (iii): Assume that $f(X) = g(X)$. Let $x \in Fix_f(X)$. Then $x = f(x) \in f(X) = g(X)$. Hence $x = g(y)$ for some $y \in X$. Now $g(x) = g(g(y)) = g^2(y) = g(y) = x$. Thus $x \in Fix_g(X)$. Therefore, $Fix_f \subseteq Fix_g(X)$. Similarly, we can obtain $Fix_g(X) \subseteq Fix_f(X)$. Thus $Fix_f(X) = Fix_g(X)$.

(iii) $\Rightarrow$ (i): Assume that $Fix_f(X) = Fix_g(X)$. Let $x \in X$. Since $f(x) \in Fix_f(X) = Fix_g(X)$, we have $g(f(x)) = f(x)$. Also, we obtain $g(x) \in Fix_g(X) = Fix_f(X)$. Hence we get $f(g(x)) = g(x)$. Thus we have

$$f(x) = g(f(x)) = (g \circ f)(x) = (f \circ g)(x) = f(g(x)) = g(x).$$

Therefore, f and g are equal in the sense of mappings. $\square$

Let f be a multiplier of X. Define a $Kerf$ by

$$Kerf = \{x \in X \mid f(x) = 0\}$$

for all $x \in X$.

Proposition 3.19. Let f be a multiplier of X. Then $Kerf$ is a subalgebra of X.

Proof. Let f be a multiplier of X. Let $x, y \in Kerf$. Then $f(x) = 0$ and $f(y) = 0$. Hence we have $f(x * y) = f(x) * y = 0 * y = 0$, and so $x * y \in Kerf$. Thus $Kerf$ is a subalgebra of X. $\square$

Proposition 3.20. Let X be a commutative BCC-algebra. If $y \in Kerf$ and $x \leq y$, then we have $x \in Kerf$.

Proof. Let f be a multiplier of X. If $y \in Kerf$ and $x \leq y$. Then $f(y) = 0$ and $x * y = 0$.

$$\begin{aligned} f(x) &= f(x * 0) = f(x * (x * y)) \\ &= f(y * (y * x)) = f(y) * (y * x) = 0 * (y * x) \\ &= 0, \end{aligned}$$

and so $x \in Kerf$. This completes the proof. □

Theorem 3.21. Let f be a multiplier of X and an endomorphism. Then $Kerd$ is an ideal of X.

Proof. Clearly, $0 \in Kerf$. Let $y \in Kerf$ and $x * y \in Kerf$. Then we have $f(y) = f(x * y) = 0$, and so

$$0 = f(x * y) = f(x) * f(y) = f(x) * 0 = f(x).$$

This implies $x \in Kerf$. This completes the proof. □

Definition 3.22. Let X be a BCC-algebra. A non-empty set I of X is called a *normal ideal* if it satisfies the following conditions:
(i) $0 \in I$,
(ii) $x \in I$ and $y \in X$ imply $x * y \in I$.

Example 3.23. Let $X = \{0, 1, 2, 3\}$ a set in which "$*$" are defined by

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

It is easy to check that $(X, *)$ is a BCC-algebra. Let $I = \{0, 2\}$. Then it is easy to check that I is a normal ideal of a BCC-algebra X.

Theorem 3.24. Let f be a multiplier of a BCC-algebra X. For any normal ideal I of X, both $f(I)$ and $f^{-1}(I)$ are normal ideals of X.

Proof. Clearly, $0 = f(0)$. Let $x \in X$ and $a \in f(I)$. Then $a = f(s)$ for some $s \in I$. Now $a * x = f(s) * x = f(s * x) \in f(I)$ because $s * x \in I$. Therefore $f(I)$ is a normal ideal of X. Since I is a normal ideal of X, we obtain $f(0) = 0 \in I$. Hence $0 = f^{-1}(I)$. Let $x \in X$ and $a \in f^{-1}(I)$. Then $f(a) \in I$. Since I is a normal ideal, we get $f(a * x) = f(a) * x \in I$. Hence $a * x \in f^{-1}(I)$. Therefore $f^{-1}(I)$ is a normal ideal of X. $\square$

Proposition 3.25. For any multiplier f of a BCC-algebra X, $Kerf$ is a normal ideal of X.

Proof. Clearly, $0 \in Kerf$. Let $a \in Kerf$ and $x \in X$. Then $f(a * x) = f(a) * x = 0 * x = 0$. Hence $a * x \in Kerf$, which implies that $Kerf$ is a normal ideal of X. $\square$

Lemma 3.26. Let f be a multiplier of a BCC-algebra X. Then $Im(f) = Fix_f(X)$.

Proof. Let $x \in Fix_f(X)$. Then $x = f(x) \in Im(f)$. Hence $Fix_f(X) \subseteq Im(f)$. Now let $a \in Im(f)$. Then we get $a = f(b)$ for some $b \in X$. Thus $f(a) = f(f(b)) = f(b) = a$, which implies $Im(f) \subseteq Fix_f(X)$. Therefore, $Im(f) = Fix_f(X)$. This completes the proof. $\square$

Theorem 3.27. Let f be a multiplier of a BCC-algebra X. Then we have

(i) $Fix_f(X)$ is a normal ideal of X.

(ii) $Im(f)$ is a normal ideal of X.

Proof. (i) Since $f(0) = 0$, we have $0 \in Fix_f(X)$. Let $x \in X$ and $a \in Fix_f(X)$. Then $f(a) = a$ Now $f(a * x) = f(a) * x = a * x$. Hence $a * x \in Fix_f(X)$. Therefore, $Fix_f(X)$ is a normal ideal of X.

(ii) Obviously, $0 = f(0)$. Let $x \in X$ and $a \in Im(f)$. Then $a = f(b)$ for some $b \in X$. Now $a * x = f(b) * x = f(b * x) \in f(X)$. Therefore, $Im(f)$ is a normal ideal of X. $\square$

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