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# Some Characterizations of the Choquet Integral with Respect to a Monotone Interval-Valued Set Function

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**Abstract**

Intervals can be used in the representation of uncertainty. In this regard, we consider monotone interval-valued set functions and the Choquet integral. This paper investigates characterizations of monotone interval-valued set functions and provides applications of the Choquet integral with respect to monotone interval-valued set functions, on the space of measurable functions with the Hausdorff metric.

**Keywords:** Interval-valued set function, Choquet integral, Hausdorff metric, Subadditivity of the Choquet integral

## 1. Introduction

Axiomatic characterizations of the Choquet integral have been introduced by Choquet [1], Murofushi et al [2, 3], Wang [4] and Campos-Bolanos [5] as an interesting extension of the Lebesgue integral. Other researchers have studied various convergence problems on monotone set functions and, on sequences of measurable functions, as well as applications. For example, the convergence in the (C) mean [6], on decision-making problems [7, 8], on the Choquet weak convergence [9], on the monotone expectation [10], and on the aggregation approach [11].

In the past decade, it has been suggested to use intervals in order to represent uncertainty, for example, for economic uncertainty [12], for fuzzy random variables [13], in interval-probability [14], for martingales of multi-valued functions [15], in the integrals of set-valued functions [16], in the Choquet integrals of interval-valued (or closed set-valued) functions [17–22], and for interval-valued capacity functions [23]. Couso-Montes-Gil [24] studied applications under the sufficient and necessary conditions on monotone set functions, i.e., the subadditivity of the Choquet integral with respect to monotone set functions.

Intervals are useful in the representation of uncertainty. We shall consider monotone interval-valued set functions and the Choquet integral with respect to a monotone interval-valued set function of measurable functions. Based on the results of Couso-Motes-Gil [24], we shall provide characterizations of monotone interval-valued set functions as well as applications of the Choquet integral regarding a monotone interval-valued set function in the space of measurable functions with the Hausdorff metric.

In Section 2, we list definitions and basic properties for the monotone set functions, the Choquet integrals and for the various convergence notions in the space of measurable functions.

In Section 3, we define a monotone interval-valued set function and the Choquet integral with respect to a monotone interval-valued set function of measurable functions, and we discuss

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their properties. We also investigate various convergences in the Hausdorff metric on the space of intervals as well as the characterizations of the Choquet integral with respect to a monotone interval-valued set function of measurable functions.

In Section 4, we give a brief summary of our results and conclusions.

## 2. Preliminaries and Definitions

In this section, we consider monotone set functions, also called fuzzy measures, and the Choquet integral defined by Choquet [1]. The Choquet integral [1] generalizes the Lebesgue integral to the case of monotone set functions. Let $X$ be a non-empty set, and let $\mathcal{A}$ denote a $\sigma$-algebra of subsets of $X$. Let $\mathbb{R} = (-\infty, \infty)$, $\overline{\mathbb{R}} = [-\infty, \infty]$, $\mathbb{R}^+ = [0, \infty)$ and $\overline{\mathbb{R}}^+ = [0, \infty]$. First we define, monotone set functions, the Choquet integral, the different types of convergences, and the uniform integrability of measurable functions as follows:

**Definition 2..1.** [2,3,5,24] (1) A mapping $\mu : \mathcal{A} \longrightarrow \overline{\mathbb{R}}^+$ is said to be a set function if $\mu(\emptyset) = 0$.

(2) A set function $\mu$ is said to be monotone if

$$\mu(A) \leq \mu(B) \text{ whenever } A, B \in \mathcal{A} \text{ and } A \subset B. \tag{1}$$

(3) A set function $\mu$ is said to be continuous from below (or lower semi-continuous) if for any sequence $\{A_n\} \subset \mathcal{A}$ and $A \in \mathcal{A}$ such that

$$A_n \uparrow A, \text{ then } \lim_{n\to\infty} \mu(A_n) = \mu(A). \tag{2}$$

(4) A set function $\mu$ is said to be continuous from above (or upper semi-continuous) if for any sequence $\{A_n\} \subset \mathcal{A}$ and $A \in \mathcal{A}$ such that

$$\mu(A_1) < \infty \text{ and } A_n \downarrow A, \text{ then } \lim_{n\to\infty} \mu(A_n) = \mu(A). \tag{3}$$

(5) A set function $\mu$ is said to be continuous if it is continuous from above and continuous from below.

(6) A set function $\mu$ is said to be subadditive if $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$, then

$$\mu(A \cup B) \leq \mu(A) + \mu(B). \tag{4}$$

(7) A set function $\mu$ is said to be submodular if $A, B \in \mathcal{A}$, then

$$\mu(A \cap B) + \mu(A \cup B) \leq \mu(A) + \mu(B). \tag{5}$$

(8) A set function $\mu$ is said to be null-additive if

$$\mu(E \cup F) = \mu(E) \text{ for any } E \in \mathcal{A} \text{ whenever } \mu(F) = 0. \tag{6}$$

**Definition 2..2.** [2,3,5,24] Let $\mu$ be a monotone set function on $\mathcal{A}$. (1) If $f : X \longrightarrow \mathbb{R}^+$ is a non-negative measurable function, then the Choquet integral of $f$ with respect to $\mu$ is defined by

$$(C)\int f\, d\mu = \int_0^\infty \mu_f(\alpha)\, d\alpha \tag{7}$$

where

$$\mu_f(\alpha) = \mu(\{x \in X | f(x) > \alpha\}) \tag{8}$$

for all $\alpha \in \mathbb{R}^+$ and the integral on the right-hand side is the Lebesgue integral of $\mu_f$.

(2) If $f : X \longrightarrow \mathbb{R}$ is a real-valued measurable function, then the Choquet integral of $f$ with respect to $\mu$ is defined by

$$(C)\int f\, d\mu = (C)\int f^+\, d\mu - (C)\int f^-\, d\mu^* \tag{9}$$

where $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$, $A^c$ is the complementary set of $A$, and $\mu^*$ is the conjugate of $\mu$, that is,

$$\mu^*(A) = \mu(X) - \mu(A^c) \text{ for all } A \in \mathcal{A}. \tag{10}$$

(3) A measurable function $f$ is said to be $\mu$-integrable if the Choquet integral of $f$ on $X$ exists.

We note that

$$\begin{aligned} \mu_f(\alpha) &= \mu(\{x \in X | f(x) > \alpha\}) \\ &= \mu\left(\{x \in X | f^+(x) > \alpha\}\right) \\ &= \mu_{f^+}(\alpha), \end{aligned} \tag{11}$$

for all $\alpha \in \mathbb{R}^+$ and

$$\begin{aligned} \mu^*_f(\alpha) &= \mu(X) - \mu(\{x \in X | f(x) > \alpha\}^c) \\ &= \mu(X) - \mu(\{x \in X | f(x) \leq \alpha\}) \\ &= \mu(X) - \mu\left(\{x \in X | f^-(x) \leq \alpha\}\right) \\ &= \mu^*_{f^-}(\alpha), \end{aligned} \tag{12}$$

for all $\alpha \in \mathbb{R}^- = (-\infty, 0)$. Thus, we have

$$\begin{aligned} & (C)\int f\,d\mu \\ = \;& (C)\int f^+\,d\mu - (C)\int f^-\,d\mu^* \\ = \;& \int_0^\infty \mu_f(\alpha)\,d\alpha - \int_{-\infty}^0 \mu_f^*(\alpha)\,d\alpha \\ = \;& \int_0^\infty \mu_f(\alpha)\,d\alpha - \int_{-\infty}^0 \mu_f^*(\alpha) \\ = \;& \int_0^\infty \mu_f(\alpha)\,d\alpha + \int_{-\infty}^0 \left(-\mu_f^*(\alpha)\right)\,d\alpha \\ = \;& \int_0^\infty \mu_f(\alpha)\,d\alpha + \int_{-\infty}^0 (\mu_f(\alpha) - \mu(X))d\alpha. \quad (13) \end{aligned}$$

We introduce almost everywhere convergence, convergence in $\mu$-mean, and uniform $\mu$-integrability as follows:

**Definition 2..3.** Let $\mu$ be a monotone set function on a measurable space $(X, \mathcal{A})$, $\{f_n\}$ a sequence of measurable functions from $X$ to $\mathbb{R}$, and $f$ a measurable function from $X$ to $\mathbb{R}$.

(1) A sequence $\{f_n\}$ almost everywhere converges to $f$ if there exists a measurable and null subset $N \in \mathcal{A}$, $\mu(N) = 0$ such that

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad \text{for all } x \in N^c. \quad (14)$$

(2) A sequence $\{f_n\}$ converges in $\mu$-mean to $f$ if

$$\lim_{n\to\infty} (C)\int \mid f_n - f \mid \,d\mu = 0, \quad (15)$$

where $|\cdot|$ is the absolute value on $\mathbb{R}$.

**Definition 2..4.** [24] Let $\mu$ be a monotone set function on $\mathcal{A}$ and $I \subset \mathbb{N}$ an index set. A class of real-valued measurable functions $\{f_n\}_{n\in I}$ is said to be uniform $\mu$-integrable if

$$\text{(i)}\ \sup_{n\in I} (C)\int |f_n| d\mu < \infty, \quad (16)$$

$$\begin{aligned} \text{(ii)} \quad & \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ such that} \sup_{n\in I} (C)\int_A |f_n| d\mu < \varepsilon \\ & \text{if } A \in \mathcal{A} \text{ and } \mu(A) < \delta(\varepsilon). \quad (17) \end{aligned}$$

Now, we recall from [24] the subadditivity of the Choquet integral, the equivalence between the convergence in mean and the uniform integrability of a sequence of measurable functions.

**Theorem 2..5.** (Subadditivity for the Choquet integral) Let $(X, \mathcal{A})$ be a measurable space. If a monotone set function $\mu : \mathcal{A} \longrightarrow \mathbb{R}^+$ is submodular and $f, g : X \longrightarrow \mathbb{R}$ are real-valued measurable functions, then we have

$$(C)\int (|f| + |g|)\,d\mu \le (C)\int |f|\,d\mu + (C)\int |g|\,d\mu. \quad (18)$$

**Theorem 2..6.** Let $(X, \mathcal{A})$ be a measurable space. If a monotone set function $\mu : \mathcal{A} \longrightarrow \mathbb{R}^+$ is subadditive and $f, g : X \longrightarrow \mathbb{R}$ are measurable functions with disjoint support, that is, $\{x \in X \mid f(x) > 0\} \cap \{x \in X \mid g(x) > 0\} = \emptyset$, then we have

$$(C)\int (f + g)\,d\mu \le (C)\int f\,d\mu + (C)\int g d\mu. \quad (19)$$

## 3. Main Results

In this section, we consider intervals, interval-valued functions, and the Aumann integral of measurable interval-valued functions. Let $I(\mathbb{R})$ be the class of all bounded and closed intervals (intervals, for short) in $\mathbb{R}$ as follows:

$$I(\mathbb{R}) = \{[a_l, a_r] \mid a_l, a_r \in \mathbb{R} \text{ and } a_l \le a_r\}. \quad (20)$$

For any $a \in \mathbb{R}$, we define $a = [a, a]$. Obviously, $a \in I(\mathbb{R})$ [18–21].

Recall that if $(\mathbb{R}, \mathfrak{M}, m)$ is the Lebesgue measure space and $C(\mathbb{R})$ is the set of all closed subsets of $\mathbb{R}$, then the Aumann integral of a closed set-valued function $F : \mathbb{R} \longrightarrow C(\mathbb{R})$ is defined by

$$(A)\int F dm = \{\int g\,dm | g \in S(F)\}, \quad (21)$$

where $S(F)$ is the set of all integrable selections of $F$, that is,

$$\begin{aligned} S(F) \quad = \quad & \{g : \mathbb{R} \longrightarrow \mathbb{R} | \int \mid g \mid dm < \infty \\ & \text{and } g(\alpha) \in F(\alpha)\ m - a.e.\}, \quad (22) \end{aligned}$$

where $m - a.e.$ means almost everywhere in the Lebesgue measure $m$, and $|g|$ is the absolute value of $g$ [15,16]. In [13,23], we can see that $(A)\int F dm$ is a nonempty bounded and closed interval in $\mathbb{R}$ whenever $F$ is an interval-valued function as in the following theorem.

**Theorem 3..1.** If an interval-valued function $F = [g_l, g_r] : \mathbb{R} \longrightarrow I(\mathbb{R})$ is measurable and integrably bounded, then $g_l, g_r \in$

$S(F)$ and

$$(A)\int F dm = \left[\int g_l dm, \int g_r dm\right], \tag{23}$$

where the two integrals on the right-hand side are the Lebesgue integral with respect to $m$.

Note that we write $\int g\, dm = \int_{-\infty}^{\infty} g(\alpha)\, d\alpha$ for all bounded continuous function $g$. Let $C(\mathbb{R})$ be the class of all closed subsets of $\mathbb{R}$. We recall that the Hausdorff metric $d_H : C(\mathbb{R}) \times C(\mathbb{R}) \longrightarrow \mathbb{R}^+$ is defined by

$$d_H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B}|x-y|, \sup_{y\in B}\inf_{x\in A}|x-y|\right\}, \tag{24}$$

for all $A, B \in C(\mathbb{R})$. It is well-known that for all $\bar{a} = [a_l, a_r]$, $\bar{b} = [b_l, b_r] \in I(\mathbb{R})$,

$$d_H(\bar{a},\bar{b}) = \max\{|a_l - b_l|, |a_r - b_r|\}. \tag{25}$$

Next, we shall define monotone interval-valued set functions and discuss their characterization.

**Definition 3..2.** (1) A mapping $\bar{\mu} : \mathcal{A} \longrightarrow I(\bar{\mathbb{R}}^+)$ is said to be an interval-valued set function if $\bar{\mu}(\emptyset) = 0$.

(2) An interval-valued set function $\bar{\mu}$ is said to be monotone if

$$\bar{\mu}(A) \le \bar{\mu}(B) \text{ whenever } A, B \in \mathcal{A} \text{ and } A \subset B. \tag{26}$$

(3) An interval-valued set function $\bar{\mu}$ is said to be continuous from below if for any sequence $\{A_n\} \subset \mathcal{A}$ and $A \in \mathcal{A}$ such that $A_n \uparrow A$, then

$$d_H - \lim_{n\to\infty} \bar{\mu}(A_n) = \bar{\mu}(A), \tag{27}$$

that is,

$$\lim_{n\to\infty} d_H(\bar{\mu}(A_n), \bar{\mu}(A)) = 0.$$

(4) An interval-valued set function $\bar{\mu}$ is said to be continuous from above if for any sequence $\{A_n\} \subset \mathcal{A}$ and $A \in \mathcal{A}$ such that $\bar{\mu}(A_1)$ is a bounded interval and $A_n \downarrow A$, then

$$d_H - \lim_{n\to\infty} \bar{\mu}(A_n) = \bar{\mu}(A). \tag{28}$$

(5) An interval-valued set function $\bar{\mu}$ is said to be continuous if it is both continuous from above and continuous from below.

(6) An interval-valued set function $\bar{\mu}$ is said to be subadditive if $A, B \in \mathcal{A}$, then

$$\bar{\mu}(A \cup B) \le \bar{\mu}(A) + \bar{\mu}(B). \tag{29}$$

(7) An interval-valued set function $\bar{\mu}$ is said to be submodular if $A, B \in \mathcal{A}$, then

$$\bar{\mu}(A \cup B) + \bar{\mu}(A \cap B) \le \bar{\mu}(A) + \bar{\mu}(B). \tag{30}$$

(8) An interval-valued set function $\bar{\mu}$ is said to be null-additive if

$$\bar{\mu}(E \cup F) = \bar{\mu}(E) \text{ for any } E \text{ whenever } \bar{\mu}(F) = 0. \tag{31}$$

From Definition 3.2 and Eq. (25), we can directly derive the following theorem [23, 25].

**Theorem 3..3.** (1) A mapping $\bar{\mu} = [\mu_l, \mu_r]$ is an interval-valued set function if only only if $\mu_l$ and $\mu_r$ are set functions, and $\mu_l \le \mu_r$.

(2) An interval-valued set function $\bar{\mu} = [\mu_l, \mu_r]$ is monotone if only only if the set functions $\mu_l$ and $\mu_r$ are monotone.

(3) An interval-valued set function $\bar{\mu} = [\mu_l, \mu_r]$ is continuous from below if only only if the set functions $\mu_l$ and $\mu_r$ are continuous from below, and $\mu_l \le \mu_r$.

(4) An interval-valued set function $\bar{\mu} = [\mu_l, \mu_r]$ is continuous from above if only only if the set functions $\mu_l$ and $\mu_r$ are continuous from above, and $\mu_l \le \mu_r$.

(5) An interval-valued set function $\bar{\mu}$ is subadditive if and only if the set functions $\mu_l$ and $\mu_r$ are subadditive, and $\mu_l \le \mu_r$.

(6) An interval-valued set function $\bar{\mu}$ is submodular if and only if the set functions $\mu_l$ and $\mu_r$ are submodular, and $\mu_l \le \mu_r$.

(7) An interval-valued set function $\bar{\mu}$ is null-additive if and only if the set functions $\mu_l$ and $\mu_r$ are null-additive, and $\mu_l \le \mu_r$.

By using Definition 2.2 and Theorem 3.3, we define the Choquet integral of a non-negative measurable function with respect to a continuous from below and monotone interval-valued set function as follows:

**Definition 3..4.** (1) The Choquet integral of a non-negative measurable function $f : X \longrightarrow \mathbb{R}^+$, with respect to a monotone interval-valued set function $\bar{\mu}$, is defined by

$$(C)\int f\, d\bar{\mu} = (A)\int \bar{\mu}_f\, dm \tag{32}$$

where $m$ is the Lebesgue measure on $\mathbb{R}$ and the integral on the right-hand side is the Aumann integral with respect to $m$ of $\bar{\mu}_f$.

(2) The Choquet integral of a real-valued measurable function $f : X \longrightarrow \mathbb{R}$, with respect to a monotone interval-valued set function $\bar{\mu}$, is defined by

$$(C)\int f\,d\bar{\mu} = (C)\int f^+\,d\bar{\mu} - (C)\int f^-\,d\bar{\mu}^*, \tag{33}$$

where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, and $\bar{\mu}^*$ is the conjugate of $\bar{\mu}$, that is,

$$\bar{\mu}^*(A) = \bar{\mu}(X) - \bar{\mu}(A^c) \ \text{ for all } \ A \in \mathcal{A}. \tag{34}$$

(3) A measurable function $f$ is said to be $\bar{\mu}$-integrable if $(C)\int f\,d\bar{\mu} \in I(\mathbb{R})\backslash\{\emptyset\}$.

We note that Eq. (36) implies

$$(A)\int \bar{\mu}_f\,dm = \int_0^\infty \bar{\mu}_f(\alpha)\,d\alpha,$$

where $\bar{\mu}_f(\alpha) = \bar{\mu}\left(\{x \in X | f(x) > \alpha\}\right)$ for all $\alpha \in \mathbb{R}^+$. By the definition of $\bar{\mu}^*$, we easily get the following theorem.

**Theorem 3..5.** (1) A monotone interval-valued set function $\bar{\mu}$ is continuous from below (resp. from above) if and only if $\bar{\mu}^*$ is continuous from above (resp. from below).

(2) If $\bar{\mu} = [\mu_l, \mu_r]$ is a monotone interval-valued set function and $\mu_l(X) = \mu_r(X)$, then $\bar{\mu}^* = [\mu_r^*, \mu_l^*]$, where $\mu_r^*(A) = \mu_r(X) - \mu_r(A^c)$ and $\mu_l^*(A) = \mu_l(X) - \mu_l(A^c)$ for all $A \in \mathcal{A}$.

In [21], we can find the theorem below. This gives a useful and interesting tool for the application of the Choquet integral of a non-negative measurable function $f$, with respect to a monotone interval-valued set function $\bar{\mu}$.

**Theorem 3..6.** ([23, Lemma 2.5 (i) and (v)]) Let $f$ be a non-negative measurable function and $\bar{\mu} = [\mu_l, \mu_r]$ a monotone interval-valued function. If $\bar{\mu}$ is continuous from above and we take $F(\alpha) = \bar{\mu}(\{x \in X | f(x) > \alpha\})$ for all $\alpha \in \mathbb{R}^+$, then we have

(1) $F$ is continuous from above, and

(2)

$$(A)\int_{\mathbb{R}} F\,dm = \left[\int_0^\infty \mu_{lf}(\alpha)\,d\alpha, \int_0^\infty \mu_{rf}(\alpha)\,d\alpha\right],$$

where $m$ is the Lebesgue measure and $\mu_{lf}(\alpha) = \mu_l(\{x \in X | f(x) > \alpha\})$ and $\mu_{rf}(\alpha) = \mu_r(\{x \in X | f(x) > \alpha\})$.

Note that Theorem 3.6(2) implies the following equation (36) under the same condition of $f$ and $\bar{\mu}$;

$$\begin{aligned}(C)\int f\,d\bar{\mu} &= (A)\int_{\mathbb{R}^+} F(\alpha)\,dm(\alpha) \\ &= \left[\int_0^\infty \mu_{lf}(\alpha)\,d\alpha, \int_0^\infty \mu_{rf}(\alpha)\,d\alpha\right] \\ &= \left[(C)\int f\,d\mu_l, (C)\int f\,d\mu_r\right].\end{aligned} \tag{36}$$

By using Theorem 3.5 and Eq. (36), we can obtain the following theorem, which is a useful formula for the Choquet integral of a measurable function $f : X \to \mathbb{R}$, with respect to a continuous monotone interval-valued set function.

**Theorem 3..7.** Let $f$ be a measurable function and $\overline{\mu} = [\mu_l, \mu_r]$ a monotone interval-valued set function. If $\overline{\mu}$ is continuous and $\mu_l(X) = \mu_r(X)$, then we have

$$(C)\int f\,d\overline{\mu} = \left[(C)\int f\,d\mu_l, (C)\int f\,d\mu_r\right]. \tag{37}$$

**Proof**. Let $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Since $\overline{\mu}$ is continuous from above, by (40), we have

$$(C)\int f^+\,d\overline{\mu} = \left[(C)\int f^+\,d\mu_l, (C)\int f^+\,d\mu_r\right]. \tag{38}$$

Since $\overline{\mu}$ is continuous from below and $\mu_l(X) = \mu_r(X)$, by Theorem 3.5(2), $\overline{\mu}^* = [\mu_r^*, \mu_l^*]$ is continuous from above. Thus, by (36), we have

$$(C)\int f^-\,d\overline{\mu}^* = \left[(C)\int f^-\,d\mu_r^*, (C)\int f^-\,d\mu_l^*\right]. \tag{39}$$

By Definition 3.4(2), Eq. (38), and Eq. (39), we have the result.

Next, we present the following theorems which give characterizations of the Choquet integral with respect to a monotone interval-valued set function.

**Theorem 3..8.** Let $\overline{\mu} = [\mu_l, \mu_r]$ be a monotone interval-valued set function and let $A \in \mathcal{A}$. If $\overline{\mu}$ is continuous from above, then we have

$$(C)\int_A a\,d\overline{\mu} = \begin{cases} a\overline{\mu}(A) & \text{if } a \geq 0 \\ a\overline{\mu}^*(A) & \text{if } a < 0. \end{cases} \tag{40}$$

**Proof**. If $a \geq 0$ (35), then by Eq. (36), we have

$$\begin{aligned}(C)\int_A a\,d\overline{\mu} &= \int_0^\infty \overline{\mu}_a(\alpha)\,d\alpha \\ &= \left[(C)\int_A a\,d\mu_l, (C)\int_A a\,d\mu_r\right] \\ &= [a\mu_l(A), a\mu_r(A)]\end{aligned}$$

$$
\begin{aligned}
&= a\,[\mu_l(A), \mu_r(A)] \\
&= a\overline{\mu}(A).
\end{aligned}
$$

If $a < 0$, then by Eq. (36), we have

$$
\begin{aligned}
(C)\int_A a\, d\overline{\mu} &= -(C)\int_A (-a)\, d\overline{\mu}^* \\
&= -\int_0^\infty \overline{\mu}^*_{-a}(\alpha)\, d\alpha \\
&= -\left[(C)\int_A (-a)\, d\mu_r^*, (C)\int_A (-a)\, d\mu_l^*\right] \\
&= -\left[-a\mu_r^*(A), -a\mu_l^*(A)\right] \\
&= [a\mu_l^*(A), a\mu_r^*(A)] \\
&= a\,[\mu_r^*(A), \mu_l(A)] \\
&= a\overline{\mu}^*(A).
\end{aligned}
$$

**Theorem 3..9.** Let a monotone interval-valued set function $\overline{\mu} = [\mu_l, \mu_r]$ be continuous from above and let $f$ a non-negative $\overline{\mu}$-integrable function. If $\overline{\mu}$ is continuous from above and $A, B \in \mathcal{A}$ with $A \subset B$, then we have

$$
(C)\int_A f\, d\overline{\mu} \le (C)\int_B f\, d\overline{\mu}. \tag{41}
$$

**Proof**. Since $\overline{\mu} = [\mu_l, \mu_r]$ is a monotone interval-valued set function, by Theorem 3.3 (1) and (2), $\mu_l$ and $\mu_r$ are monotone interval-valued set functions. Thus,

$$
\begin{aligned}
&(C)\int_A f\, d\mu_l \le (C)\int_B f\, d\mu_l \;\text{ and} \\
&(C)\int_A f\, d\mu_r \le (C)\int_B f\, d\mu_r.
\end{aligned} \tag{42}
$$

By Eq. (36) and Eq. (42), we have the result.

We remark that if we take a $\overline{\mu}$-integrable function $f$ which is $f^+ = 0$ and $f^- > 0$, then $(C)\int f\, d\overline{\mu}$ is not monotone, that is, for each pair $A, B \in \mathcal{A}$ with $A \subset B$,

$$
\begin{aligned}
(C)\int_A f\, d\overline{\mu} &= -(C)\int_A f^-\, d\overline{\mu}^* \\
&\ge -(C)\int_B f^-\, d\overline{\mu}^* = (C)\int_B f\, d\overline{\mu}.
\end{aligned} \tag{43}
$$

**Theorem 3..10.** Let $\overline{\mu} = [\mu_l, \mu_r]$ be a monotone interval-valued set function which is continuous from above, and let $A \in \mathcal{A}$. If $f$ and $g$ are non-negative $\overline{\mu}$-integrable functions with $f \le g$, then we have

$$
(C)\int_A f\, d\overline{\mu} \le (C)\int_A g\, d\overline{\mu}. \tag{44}
$$

**Proof**. The proof is similar to the proof of Theorem 3.10.

**Theorem 3..11.** Let $\overline{\mu}_1 = [\mu_{1l}, \mu_{1r}]$ and $\overline{\mu}_2 = [\mu_{2l}, \mu_{2r}]$ be monotone interval-valued set functions, $f$ a non-negative $\overline{\mu}_1$-integrable and $\overline{\mu}_2$-integrable function, and $A \in \mathcal{A}$.

(1) If $\overline{\mu}_1 \le \overline{\mu}_2$, then we have

$$
(C)\int_A f\, d\overline{\mu}_1 \le (C)\int_A f\, d\overline{\mu}_2. \tag{45}
$$

(2) If $\overline{\mu}_1 \subset \overline{\mu}_2$, then we have

$$
(C)\int_A f\, d\overline{\mu}_1 \subset (C)\int_A f\, d\overline{\mu}_2. \tag{46}
$$

**Proof**. (1) Note that $\overline{\mu}_1 \le \overline{\mu}_2$ if and only if $\mu_{1l} \le \mu_{2l}$ and $\mu_{1r} \le \mu_{2r}$. Thus, we have

$$
\begin{aligned}
&(C)\int_A f\, d\mu_{1l} \le (C)\int_A f\, d\mu_{2l} \;\text{ and} \\
&(C)\int_A f\, d\mu_{1r} \le (C)\int_A f\, d\mu_{2r}.
\end{aligned} \tag{47}
$$

By (36) and (47), we have the result.

(2) Note that $\overline{\mu}_1 \subset \overline{\mu}_2$ if and only if $\mu_{2l} \le \mu_{1l}$ and $\mu_{1r} \le \mu_{2r}$. Thus, we have

$$
\begin{aligned}
&(C)\int_A f\, d\mu_{2l} \le (C)\int_A f\, d\mu_{1l} \;\text{ and} \\
&(C)\int_A f\, d\mu_{1r} \le (C)\int_A f\, d\mu_{2r}.
\end{aligned} \tag{48}
$$

By Eq. (36) and eq. (48), we have the result.

Finally, we investigate the subadditivity of the Choquet integral under some conditions for the monotone interval-valued set functions.

**Theorem 3..12.** Let $(X, \mathcal{A})$ be a measurable space. If a continuous monotone interval-valued set function $\overline{\mu} = [\mu_l, \mu_r] : \mathcal{A} \to I(\bar{\mathbb{R}}^+)$, with $\mu_l(X) = \mu_r(X)$, is submodular and $f, g : X \to \bar{\mathbb{R}}$ are measurable functions, then we have

$$
(C)\int (|f| + |g|) d\overline{\mu} \le (C)\int |f| d\overline{\mu} + (C)\int |g| d\overline{\mu}. \tag{49}
$$

**Proof**. Since $\overline{\mu}$ is a submodular monotone interval-valued set function, by Theorem 3.3(6), $\mu_l$ and $\mu_r$ are submodular monotone set functions.

By Theorem 2.5, we have

$$
(C)\int (|f| + |g|) d\mu_l \le (C)\int |f| d\mu_l + (C)\int |g| d\mu_l, \tag{50}
$$

and

$$
(C)\int (|f| + |g|) d\mu_r \le (C)\int |f| d\mu_r + (C)\int |g| d\mu_r. \tag{51}
$$

By Eq. (36), eq. (50), and eq. (51), we have the result.

**Theorem 3..13.** Let $(X, \mathcal{A})$ be a measurable space. If a continuous monotone interval-valued set function $\overline{\mu} = [\mu_l, \mu_r] : \mathcal{A} \to I(\overline{\mathbb{R}}^+)$, with $\mu_l(X) = \mu_r(X)$, is subadditive, and $f, g : X \to \overline{\mathbb{R}}^+$ are measurable functions with disjoint support, then

$$(C)\int (f+g)\, d\overline{\mu} \leq (C)\int f\, d\overline{\mu} + (C)\int g d\overline{\mu}. \quad (52)$$

**Proof**. Since $\overline{\mu}$ is a subadditive monotone interval-valued set function, by Theorem 3.3(5), $\mu_l$ and $\mu_r$ are subadditive monotone set functions. By Theorem 2.6, we have

$$(C)\int (f+g) d\mu_l \leq (C)\int f d\mu_l + (C)\int g d\mu_l, \quad (53)$$

and

$$(C)\int (f+g) d\mu_r \leq (C)\int f d\mu_r + (C)\int g d\mu_r. \quad (54)$$

By Eq. (36), Eq. (53), and Eq. (54), we have the result.

## 4. Conclusions

In this paper, we introduced the concept of a monotone interval-valued set function and, the Aumann integral of a measurable function, with respect to the Lebesgue measure. By using the two concepts, we define the Choquet integral with a monotone interval-valued set function of measurable functions.

From Theorem 3.2, Definition 3.3(3), and the condition that $\mu_l(X) = \mu_r(X)$ of a continuous monotone set function, we can deal with the new concept of the Choquet integral of a monotone interval-valued set function $\overline{\mu} = [\mu_l, \mu_r]$ of measurable functions $f : X \longrightarrow \mathbb{R}$. Theorems 3.3, 3.5, 3.6, 3.7, 3.8, 3.9, and 3.10 are important characterizations of the Choquet integral with respect to a monotone interval-valued set function on the space of non-negative $\overline{\mu}$-integrable functions. Theorem 3.12 and Theorem 3.13 are both, useful and interesting tools, in the application of the Choquet integral with respect to a continuous monotone interval- valued set function.

In the future, by using the results in this paper, we shall investigate various problems and models, for representing monotone uncertain set functions, and for the application of the bi-Choquet integral with respect to a monotone interval-valued set function.

## Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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