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Isoparametric Curve of Quadratic F-Bézier Curve

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Abstract

In this thesis, we consider isoparametric curves of quadratic F-Bézier curves. F-Bézier curves unify C-Bézier curves whose basis is $\{\sin t, \cos t, t, 1\}$ and H-Bézier curves whose basis is $\{\sinh t, \cosh t, t, 1\}$. Thus F-Bézier curves are more useful in Geometric Modeling or CAGD(Computer Aided Geometric Design). We derive the relation between the quadratic F-Bézier curves and the quadratic rational Bézier curves. We also obtain the geometric properties of isoparametric curve of the quadratic F-Bézier curves at both end points and prove the continuity of the isoparametric curve.

Key words : F-Bézier Curve, Quadratic Rational Bézier Curve, Isoparametric Curve, Q-Bézier Curve, H-Bézier Curve

1. Introduction

C-Bézier curve[1,2] is an extension of Beizer curve using $\cos(t)$ and $\sin(t)$, which is used to express circular arc, ellipse, cycloid and helix. Zhang developed the C-B-spline from the C-Bézier curves[3] and C-Bézier curves and surfaces[4]. It is more advantage of F-Bézier curve than rational Bézier curve to able to express helix or cycloid[5]. Chen and Wang[6] presented C-Bézier curves of higher degree.

H-Bézier curve[7] is the curve combined by Bézier curve and hyperbolic functions $\cosh(t)$, $\sinh(t)$. Zhang and Krause[8] unified these curves into F-Bézier curve. Zhang et al.[9] also unified the coefficient of F-Bézier curves using complex number.

Every quadratic F-Bézier curve has the same graph of a quadratic rational Bézier curve, and vice versa. Any isoparametric curve of quadratic Bézier curves is a straight line, but that of F-Bézier curves is not. So we study the isoparametric curve of quadratic F-Bézier curves in this paper. We show that the isoparametric curve is tangent to the polygon at the end point of the curve, or a straight line. We also prove that the isoparametric curves are C^1 continuous but not twice differentiable.

This paper is constructed as follows. In Section 2, the well-known facts about the quadratic rational Bézier curves and the quadratic F-Bézier curves are given. In Section 3, we present some geometric properties of F-Bézier curves, and we summary them in Section 4.

2. Prior Works

In this section we remind the definition and properties of quadratic rational Bézier curve and quadratic F-Beizer curve.

The quadratic rational Bézier curve is well introduced by Farin[10].

Definition 2.1 Quadratic rational Bézier curve is

$$\boldsymbol{r}(t) = \frac{\sum_{i=0}^{2} w_i B_i^2(t) \boldsymbol{b}_i}{\sum_{i=0}^{2} w_i B_i^2(t)}$$

where $\boldsymbol{b}_i$, $i = 0,1,2$, is control point, $w_i > 0$ is weight, and $B_i^2(t) = \binom{2}{i} t^i (1-t)^{2-i}$ is the quadratic Bernstein polynomial. If $w_0 = w_2 = 1$, the quadratic Bézier curve is called by standard form.

Any quadratic rational Bézier curve $\boldsymbol{r}(t)$ with control points $\boldsymbol{b}_i$ and weights $w_i (i = 0,1,2)$ can be reparametrized

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by standard form with new weights $1, \dfrac{w_1}{\sqrt{w_0 w_2}}, 1$ without change of graph[6]. Actually, using the reparametrization

$$t(s) = \frac{s}{\rho(1-s)+s}, \rho = \sqrt{\frac{w_2}{w_0}}$$

new reparametrized quadratic Bézier curve is obtained by

$$\begin{aligned}\hat{r}(s) &= r(t(s)) \\ &= \frac{\rho^2 w_0 B_0^2(s) b_0 + \rho w_1 B_1^2(s) b_1 + w_2 B_2^2(s) b_2}{\rho^2 w_0 B_0^2(s) + \rho w_1 B_1^2(s) + w_2 B_2^2(s)}\end{aligned}$$

which is a standard quadratic rational Bézier curve. $r(t(s))$ and $\hat{r}(s)$ have the same graph. Thus any quadratic rational Bézier curve can be rewritten by the standard form

$$r(t) = \frac{B_0^2(t) b_0 + w B_1^2(t) b_1 + B_2^2(t) b_2}{B_0^2(t) + w B_1^2(t) + B_2^2(t)}$$

where $w = \dfrac{w_1}{\sqrt{w_0 w_2}}$. It is well known that $w < 1$ iff r is ellipse, $w = 1$ iff r is parabola, and $w > 1$ iff r is hyperbola, as shown in Fig. 1[10].

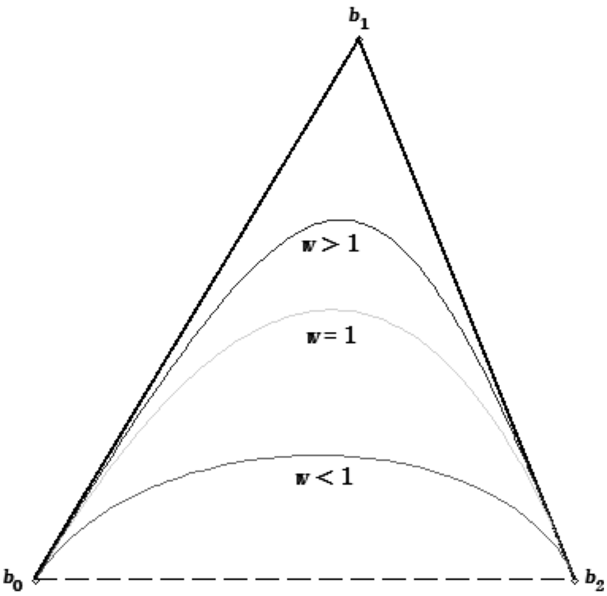

Fig. 1. Quadratic rational Bézier curves with weight $w<1$ (ellipse), $w=1$(parabola), or $w>1$(hyperbola). $[b_0, b_1, b_2]$ is control polygon(black).

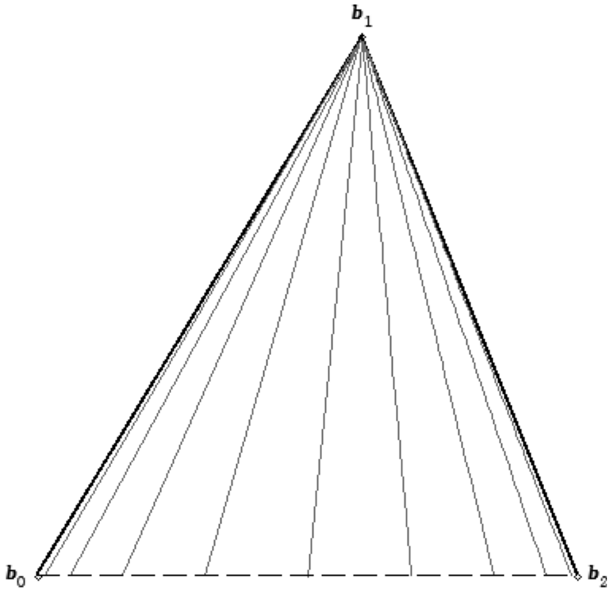

Fig. 2. isoparametric curves of quadratic rational Bézier curves: $r^*(w,t_0)$ for $t_0 = \dfrac{i}{10}, (i=,1,2,\cdots,9)$.

It is well known that the function $r^*:(0,1)\times(0,\infty) \to \triangle p_0 p_1 p_2$ defined by $r^*(w,t) = r(t)$ is homeomorphic on $(0,1)\times(0,\infty)$. For each $t_0 \in (0,1)$, the isoparametric curve is

$$r^*(w,t_0) = b_1 + \frac{B_0^2(t_0)(b_0 - b_1) + B_2^2(t_0)(b_2 - b_1)}{B_0^2(t_0) + w B_1^2(t_0) + B_2^2(t_0)}$$

Hence $r^*(w,t_0)$ is a straight line between the two points $\dfrac{B_0^2(t) b_0 + B_2^2(t) b_2}{B_0^2(t) + B_2^2(t)}$ and b_1 as shown in Fig. 2.

In what follows, we will need to consider the implicit form of a conic section. Any point $(x,y) \in \mathbb{R}^2$ can be written uniquely in terms of barycentric coordinates (τ_0, τ_1, τ_2), where $\tau_0 + \tau_1 + \tau_2 = 1$, with respect to the triangle $\triangle p_0 p_1 p_2 : (x,y) = \tau_0 p_0 + \tau_1 p_1 + \tau_2 p_2$. Consequently any function $f : \triangle b_0 b_1 b_2 \to \mathbb{R}$ can be expressed as a function of τ_0, τ_1, τ_2. The following lemma is well known[10,11].

Lemma 2.2 Let r be a quadratic rational Bézier curve with control points b_0, b_1, b_2 and weight w and let f be defined as

$$f(x,y) = \tau_1^2 - 4w^2 \tau_0 \tau_2.$$

Then r satisfes the equation $f(r(t))=0$ for all $t\in[0,1]$.

Zhang and Krause[9] unified the C-Bézier curve and H-Bézier curve, which is called F-Bézier curve.

Definition 2.3 The F-Bézier curve of degree n with a parameter $\alpha\in(-\pi,\infty)$ is

$$q(t)=\sum_{i=0}^{n} Z_i^n(t) q_i$$

where q_i, $i=0,1,\cdots,n$, is control point, and Z_i^n is the F-basis functions defined by

$$Z_1^1(t)=\begin{cases}\dfrac{\sin(\alpha t)}{\sin(\alpha)} & (-\pi<\alpha<0)\\ t & (\alpha=0)\\ \dfrac{\sinh(\alpha t)}{\sinh(\alpha)} & (\alpha>0)\end{cases}$$

$$Z_0^1(t)=Z_1^1(1-t)$$

and recursively

$$W_i^n(t)=\alpha\int_0^t Z_i^n(t)dt$$
$$(0\leqq i\leqq n,\ n\geqq 1)$$
$$Z_n^n(t)=\frac{W_{n-1}^{n-1}(t)}{W_{n-1}^{n-1}(1)}$$
$$Z_0^n(t)=1-\frac{W_0^{n-1}(t)}{W_0^{n-1}(1)}$$
$$Z_i^n(t)=\frac{W_{i-1}^{n-1}(t)}{W_{i-1}^{n-1}(1)}-\frac{W_i^{n-1}(t)}{W_i^{n-1}(1)}$$
$$(1\leqq i\leqq n-1,\ n\geqq 2).$$

In this paper we concentrate the quadratic F-Bézier curve, which can be rewritten as follows.

Definition 2.4 The quadratic F-Bézier curve is

$$q(t)=\sum_{i=0}^{2} Z_i^2(t) q_i$$

where q_i, $i=0,1,2$, is control point, and Z_i^2 is quadratic F-basis functions defined by

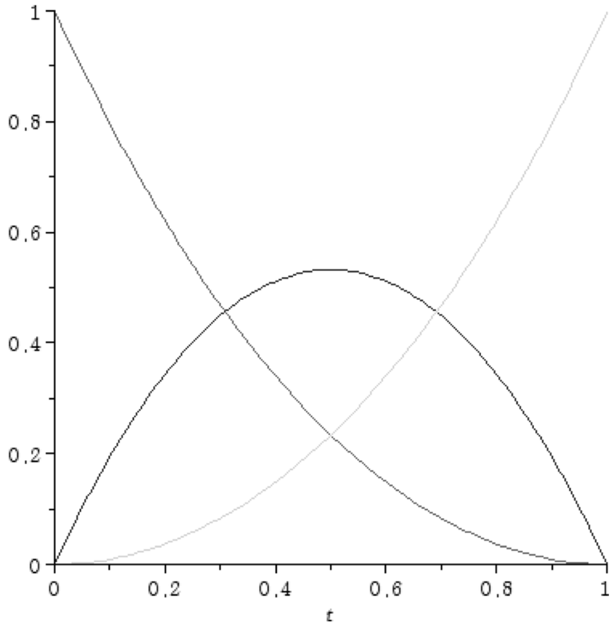

Fig. 3. Basis functions of quadratic F-Bézier curve for $\alpha=\pi/3$: $Z_0^2(t)$(red), $Z_1^2(t)$(blue), $Z_2^2(t)$(green).

$$Z_2^2(t)=\begin{cases}\dfrac{1-\cos(\alpha t)}{1-\cos\alpha} & (-\pi<\alpha<0)\\ t^2 & (\alpha=0)\\ \dfrac{1-\cosh(\alpha t)}{1-\cosh\alpha} & (\alpha>0)\end{cases}$$

and

$$Z_0^2(t)=Z_2^2(1-t)$$
$$Z_1^2(t)=1-Z_0^2(t)-Z_2^2(t).$$

As shown in Fig. 3, the basis functions $Z_i^2(t)$, $i=0,1,2$ satisfy the partition of unity, so the quadratic F-Bézier curves satisfy convex-hull property.

3. Relation between F-Bézier Curve and Quadratic Rational Bézier Curve

In this section we present some geometric properties of the isoparametric curves of the quadratic F-Bézier curves. The results in this section is an improvement of the results in the thesis[12] of the first author in this paper. We consider the nonlinear control polygons of F-Bézier curve and quadratic ratonal Bézier curves

$$q_0, q_1, q_2 \text{ and } b_0, b_1, b_2.$$

Proposition 3.1 F-Bézier curve has the same graph of quadratic rational Bézier curve if and only if $q_i = b_i$ $(i = 0,1,2)$, and

$$w = \begin{cases} \cos(\alpha/2) & (-\pi < \alpha \le 0) \\ \cosh(\alpha/2) & (\alpha > 0). \end{cases}$$

proof. By Definition 2.4, $q(t)$ has the barycentric coordinates $(Z_0^2(t), Z_1^2(t), Z_2^2(t))$, and by Lemma 2.2, $q(t)$ is a conic section iff $Z_1^2(t)/2\sqrt{Z_0^2(t)Z_2^2(t)}$ is a constant which is equal to the weight of the conic.

For $-\pi < \alpha < 0$,

$$\frac{Z_1^2(t)}{2\sqrt{Z_0^2(t)Z_2^2(t)}} = \frac{1-\dfrac{1-\cos(\alpha t)}{1-\cos\alpha}-\dfrac{1-\cos(\alpha(1-t))}{1-\cos\alpha}}{2\sqrt{\dfrac{1-\cos(\alpha t)}{1-\cos\alpha}\cdot\dfrac{1-\cos(\alpha(1-t))}{1-\cos\alpha}}}$$

$$= \frac{\cos(\alpha t)-\cos\alpha+\cos(\alpha(1-t))-1}{2\sqrt{(1-\cos(\alpha t))\cdot(1-\cos(\alpha(1-t)))}}$$

$$=-\frac{1}{2}\frac{\sin\left(\dfrac{\alpha t+\alpha}{2}\right)\sin\left(\dfrac{\alpha t-\alpha}{2}\right)+\sin^2\left(\dfrac{\alpha(1-t)}{2}\right)}{\sin\dfrac{\alpha t}{2}\cdot\sin\dfrac{\alpha(1-t)}{2}}$$

$$=\frac{1}{2}\frac{\sin\left(\dfrac{\alpha t+\alpha}{2}\right)-\sin\left(\dfrac{\alpha(1-t)}{2}\right)}{\sin\left(\dfrac{\alpha t}{2}\right)}$$

$$=\frac{1}{2}\frac{2\cos\left(\dfrac{\alpha}{2}\right)\sin\left(\dfrac{\alpha t}{2}\right)}{\sin\left(\dfrac{\alpha t}{2}\right)}=\cos\left(\frac{\alpha}{2}\right).$$

For $\alpha = 0$, trivially $Z_1^2(t)/2\sqrt{Z_0^2(t)Z_2^2(t)} = 1$.
For $\alpha > 0$, by the identity $\cosh(x) = \cos(ix)$,

$$\frac{Z_{1,2}(t)}{2\sqrt{Z_{0,2}(t)Z_{2.2}(t)}} = \frac{\cosh(\alpha t)-\cosh\alpha+\cosh(\alpha(1-t))-1}{2\sqrt{(1-\cosh(\alpha t))\cdot(1-\cosh(\alpha(1-t)))}} = \cosh\left(\frac{\alpha}{2}\right).$$

Corollary 3.2 If $q_i = b_i$ $(i = 0,1,2)$, and

$$w = \begin{cases} \cos(\alpha/2) & (-\pi < \alpha \le 0) \\ \cosh(\alpha/2) & (\alpha > 0), \end{cases}$$

then for all $t \in [0,1]$

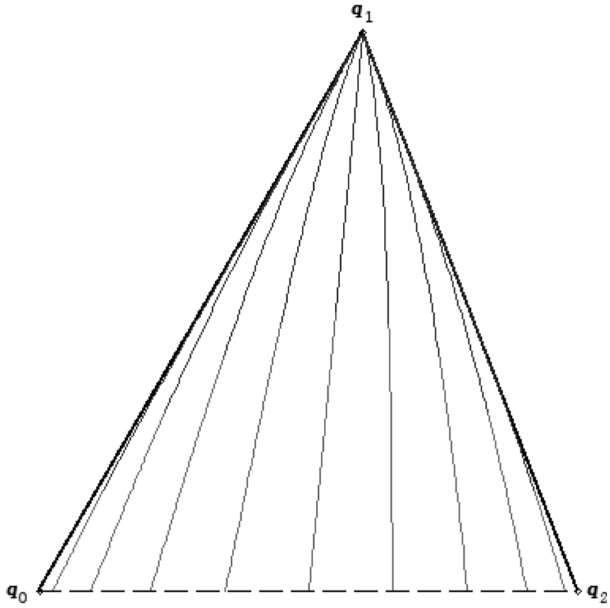

Fig. 4. Isoparametric curves of quadratic F-Bézier curves: $q^*(\alpha,t_0)$ for $t_0 = \frac{i}{10}$, $(i=,1,2,\cdots,9)$, for $-\pi<\alpha<0$ (red) and $0<\alpha<\infty$(blue).

$$r(t) = \begin{cases} q(-\operatorname{arccos}(1-\mu_2(t)(1-\cos\alpha))/\alpha) \\ q(\operatorname{arccosh}(1-\mu_2(t)(1-\cosh\alpha))/\alpha), \end{cases}$$

respectively, where

$$\mu_2(t) = \frac{B_2^2(t)}{B_0^2(t)+wB_1^2(t)+B_2^2(t)}.$$

Actually there are two parameters $\alpha \in (-\pi,\infty)$ and $t \in [0,1]$ in the F-Bézier curve $q(t)$. Let the isoparametric curve $q(t_0)$ with $\alpha \in (-\pi,\infty)$ denoted by $q^*(\alpha,t_0)$, as shown Fig. 4. If $t_0 = 0$ or 1, the the isoparametric curve $q^*(\alpha,t_0)$ is a one point q_0 or q_2.

Proposition 3.3 For each $t_0 \in (0,1)$, the isoparametric curve $q^*(\alpha,t_0)$ starts at $q^*(-\pi,t_0) = \frac{1}{2}((1+\cos(\pi t_0))q_0 + (1-\cos(\pi t_0))q_2)$ and ends at q_1.

proof. By Definition 2.4,

$$\lim_{\alpha\to-\pi} Z_2^2(t) = \frac{1-\cos(\pi t)}{2}$$

$$\lim_{\alpha\to-\pi} Z_0^2(t) = \frac{1+\cos(\pi t)}{2}$$

$$\lim_{\alpha\to-\pi} Z_1^2(t) = 0$$

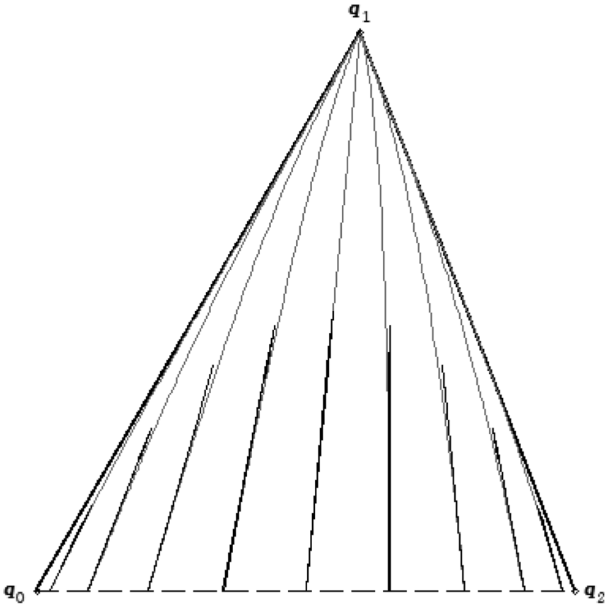

Fig. 5. Tangent vectors(blue) of isoparametric curves(red) of F-Bézier curves $q^*(\alpha,t_0)$ at $\alpha=-\pi$.

so the start point is

$$q^*(-\pi,t_0)=\frac{(1+\cos(\pi t_0))\,q_0+(1-\cos(\pi t_0))\,q_2}{2}$$

and by

$$\lim_{\alpha\to\infty} Z_2^2(t)=\lim_{\alpha\to\infty}\frac{1-\cosh(\alpha t)}{1-\cosh\alpha}=0$$
$$\lim_{\alpha\to-\pi} Z_0^2(t)=\lim_{\alpha\to\infty}\frac{1-\cosh(\alpha(1-t))}{1-\cosh\alpha}=0$$
$$\lim_{\alpha\to-\pi} Z_1^2(t)=1,$$

the end point is $\lim_{\alpha\to\infty} q^*(\alpha,t_0)=q_1$.

Let D_α be the partial derivative with respect to α, and let $Z_i^*(\alpha,t)=Z_i^2(t)$ for $i=0,1,2$.

Proposition 3.4 For each $t_0\in(0,1)$, the isoparametric curve $q^*(\alpha,t_0)$ is of tangent direction $\frac{1}{2}\sin(\pi t_0)$ $(q_1-(1-t_0)q_0-t_0q_2)$ at the start point $q^*(-\pi,t_0)$.

proof. Since for $\alpha\in(-\pi,0)$

$$D_\alpha Z_2^*(\alpha,t)=\frac{\sin(\alpha t)\,t}{1-\cos\alpha}-\frac{(1-\cos(\alpha t))\sin\alpha}{(1-\cos\alpha)^2}$$

we have

$$\lim_{\alpha\to-\pi} D_\alpha Z_2^*(\alpha,t)=-\frac{\sin(\pi t)\,t}{2}$$

and so

$$\lim_{\alpha\to-\pi} D_\alpha Z_0^*(\alpha,t)=-\frac{\sin(\pi t)\,(1-t)}{2}$$
$$\lim_{\alpha\to-\pi} D_\alpha Z_1^*(\alpha,t)=\frac{\sin(\pi t)}{2}.$$

By Definition 2.4,

$$D_\alpha q^*(-\pi,t_0)=\frac{\sin(\pi t_0)}{2}(q_1-(1-t_0)q_0-t_0q_2).$$

Theorem 3.5 For each $t_0\in(0,1)$, the isoparametric curve $q^*(\alpha,t_0)$ is tangent to $\overline{q_0q_1}$, $\overline{m\,q_1}$, or $\overline{q_2q_1}$ at q_1 iff $0<t_0<1/2$, $t_0=1/2$ or $1/2<t_0<1$, respectively, where $m=(q_0+q_2)/2$. Moreover, $q^*(\alpha,1/2)$, $\alpha\in[-\pi,\infty)$ is a line segment $\overline{m\,q_1}$.

proof. For $\alpha>0$, from Definition 2.4,

$$D_\alpha Z_2^*(\alpha,t)=-\frac{\sinh(\alpha t)\,t}{1-\cosh\alpha}+\frac{(1-\cosh(\alpha t))\sinh\alpha}{(1-\cosh\alpha)^2}$$

and so

$$D_\alpha Z_0^*(\alpha,t)=-\frac{\sinh(\alpha(1-t))\,(1-t)}{1-\cosh\alpha}+\frac{(1-\cosh(\alpha(1-t)))\sinh\alpha}{(1-\cosh\alpha)^2}$$
$$D_\alpha Z_1^*(\alpha,t)=-D_\alpha Z_0^*(\alpha,t)-D_\alpha Z_2^*(\alpha,t).$$

Thus $\lim_{\alpha\to\infty}\frac{D_\alpha Z_0^*(\alpha,t)}{D_\alpha Z_2^*(\alpha,t)}=\infty,\ 1,\ \text{or}\ 0$ iff $t<\frac{1}{2}$, $t_0=\frac{1}{2}$, or $t>\frac{1}{2}$, respectively. If $0<t_0<\frac{1}{2}$, then

$$\lim_{t\to\infty}\frac{D_\alpha Z_1^*(\alpha,t)}{D_\alpha Z_0^*(\alpha,t)}=-1,$$

so that

$$\lim_{t\to\infty}\frac{D_\alpha q^*(\alpha,t)}{D_\alpha Z_0^*(\alpha,t)}$$
$$=\lim_{t\to\infty}\frac{D_\alpha Z_0^*(\alpha,t)q_0+D_\alpha Z_1^*(\alpha,t)q_1+D_\alpha Z_2^*(\alpha,t)q_2}{D_\alpha Z_0^*(\alpha,t)}$$
$$=q_0-q_1.$$

If $1/2<t_0<1$, then

$$\lim_{t\to\infty}\frac{D_\alpha Z_1^*(\alpha,t)}{D_\alpha Z_2^*(\alpha,t)}=-1,$$

so that

$$\lim_{t\to\infty}\frac{D_\alpha q^*(\alpha,t)}{D_\alpha Z_2^*(\alpha,t)}=q_2-q_1.$$

If $t_0 = 1/2$, then $Z_0^*(\alpha,1/2) = Z_2^*(\alpha,1/2)$ and
$\boldsymbol{q}^*(\alpha,1/2) = 2Z_0^*(\alpha,1/2)\boldsymbol{m} + (1-2Z_0^*(\alpha,1/2))\boldsymbol{q}_1$.

Since $Z_0^*(\alpha,1/2) \in (0,1/2]$ for $\alpha \in [-\pi,\infty)$, the isoparametric curve $\boldsymbol{q}^*(\alpha,1/2)$ is a line segment $\overline{\boldsymbol{m}\boldsymbol{q}_1}$.

Thus the isoparametric curve $\boldsymbol{q}^*(\alpha,t_0)$ is tangent to $\overline{\boldsymbol{q}_0\boldsymbol{q}_1}$, $\overline{\boldsymbol{m}\boldsymbol{q}_1}$, or $\overline{\boldsymbol{q}_1\boldsymbol{q}_2}$ at $\boldsymbol{q}_1$ iff $0 < t_0 < 1/2$, $t_0 = 1/2$ or $1/2 < t_0 < 1$, respectively.

Proposition 3.6 For each $t_0 \in (0,1)$, the isoparametric curve $\boldsymbol{q}^*(\alpha,t_0)$, $\alpha \in (-\pi,\infty)$ is a C^1 continuous curve.

proof. It is sufficient to show that $\boldsymbol{q}^*(\alpha,t_0)$ is continuous and differentiable at $\alpha = 0$. By the series expansion

$$\frac{1-\cos(\alpha t)}{1-\cos\alpha} = \frac{\frac{t^2}{2!}\alpha^2 - \frac{t^4}{4!}\alpha^4 + \cdots}{\frac{\alpha^2}{2!} - \frac{\alpha^4}{4!} + \cdots} = \frac{t^2 - \frac{2t^4}{4!}\alpha^2 + \cdots}{1 - \frac{2\alpha^2}{4!} + \cdots}$$
$$= t^2 + \frac{t^2(1-t^2)}{12}\alpha^2 + O(\alpha^4)$$
$$\frac{1-\cosh(\alpha t)}{1-\cosh\alpha} = t^2 - \frac{t^2(1-t^2)}{12}\alpha^2 + O(\alpha^4)$$

we have

$$Z_2^*(\alpha,t) = \begin{cases} t^2 + \frac{t^2(1-t^2)}{12}\alpha^2 + \cdots & (-\pi < \alpha < 0) \\ t^2 & (\alpha = 0) \\ t^2 - \frac{t^2(1-t^2)}{12}\alpha^2 + \cdots & (\alpha > 0). \end{cases}$$

Thus $Z_2^*(\alpha,t)$ is continuous and differentiable at $\alpha = 0$. Since $D_\alpha Z_2^*(0,t) = \lim_{\alpha\to 0}\frac{Z_2^*(\alpha,t)}{\alpha} = 0$, $Z_2^*(\alpha,t_0)$, $\alpha \in (-\pi,\infty)$ is C^1 continuous, and so are $Z_1^*(\alpha,t)$ and $Z_0^*(\alpha,t)$ by Definition 2.4. Hence $\boldsymbol{q}^*(\alpha,t_0)$, $\alpha \in (-\pi,\infty)$ is C^1 continuous.

Proposition 3.7 For each $t_0 \in (0,1)$, the isoparametric curve $\boldsymbol{q}^*(\alpha,t_0)$, $\alpha \in (-\pi,\infty)$ is not twice differentiable.

proof. Since

$$\lim_{\alpha\to 0+}\frac{D_\alpha Z_2^2(\alpha,t) - D_\alpha Z_2^2(0,t)}{\alpha} = -\frac{t^2(1-t^2)}{6}$$
$$\lim_{\alpha\to 0-}\frac{D_\alpha Z_2^2(\alpha,t) - D_\alpha Z_2^2(0,t)}{\alpha} = \frac{t^2(1-t^2)}{6},$$

the second derivative of $Z_2^2(\alpha,t)$ with respect to α does not exist. Thus the isoparametric curve $\boldsymbol{q}^*(\alpha,t_0)$, $\alpha \in (-\pi,\infty)$ is not twice differentiable.

Remark. Even if the isoparametric curve $\boldsymbol{q}^*(\alpha,1/2)$ is a straight line segment $\overline{\boldsymbol{m}\boldsymbol{q}_1}$, it not twice differentiable.

4. Conclusion

In this paper, we found some geometric properties of the isoparametric curve of quadratic F-Bézier curves. We showed that the isoparametric curve is tangent to the control polygon at the end point of the curve, or a straight line, and proved that the isoparametric curves are C^1 continuous but not twice differentiable.

In future works, we have plans to find the necessary and sufficient condition for the cubic F-Bézier curve to be a conic section, and to obtain some geometric properties of the isoparametric curve of cubic F-Bézier curves.

Acknowledgements

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