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Analysis of Symmetric and Periodic Open Boundary Problem by Coupling of FEM and Fourier Series

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Most electrical machines like motor, generator and transformer are symmetric in terms of magnetic field distribution and mechanical structure. In order to analyze these problems effectively, many coupling techniques have been introduced. This paper deals with a coupling scheme for open boundary problem of symmetric and periodic structure. It couples an analytical solution of Fourier series expansion with the standard finite element method. The analytical solution is derived for the magnetic field in the outside of the boundary, and the finite element method is for the magnetic field in the inside with source current and magnetic materials. The main advantage of the proposed method is that it retains sparsity and symmetry of system matrix like the standard FEM and it can also be easily applied to symmetric and periodic problems. Also, unknowns of finite elements at the boundary are coupled with Fourier series coefficients. The boundary conditions are used to derive a coupled system equation expressed in matrix form. The proposed algorithm is validated using a test model of a bush bar for the power supply. And the each result is compared with analytical solution respectively.

Keywords : analytical solution, finite element method, Fourier series expansion, open boundary, symmetric and periodic structure

1. Introduction

The finite element method is a well-known numerical method to analyze the electromagnetic phenomena. Its advantage is that it can be easily applied to electromagnetic problems with complex geometry and nonlinear material. However, it is not suitable to open boundary problems since it requires finite element discretization even for the infinite region. Also, it is not easy and uncertain to determine the outer boundary of field region for accurate solution. To solve this problem, several methods have been proposed such as boundary element method, infinite element method, hybrid harmonic FEM, ballooning, and others [1-8].

In this paper, the interior region with the source and the material is represented by the FEM and the exterior region is represented using an analytical form of the Fourier series multiplied by radius power, which is a general solution of Laplace equation. The two expressions are coupled on the boundary using continuity condition of magnetic field [9]. Especially, in this paper the numerical techniques for the problem with symmetry or periodicity in structure and source distributions are presented. Many applications don't have to be analyzed for whole region such as electric rotary machine and electrically balanced structure apparatus. When the Fourier series is expanded, the FEM and analytical solutions are coupled considering the characteristics of odd and even function of magnetic potential.

The proposed method retains the sparsity and symmetric of system matrix like the standard FEM. This paper suggests an application method of infinite boundary conditions when it has symmetric structure in Neumann or Dirichlet conditions. So, as an analysis region is rotated, the waveform of solutions is symmetric in whole region. Therefore, only a part of model with symmetry structure is analyzed. This method increases the space efficiency of analysis region. And it can describe the inner region of boundary more accurately. In addition, computer memory and solving time can be saved by analyzing a half or a quarter of whole region. To validate usefulness of the proposed algorithm, we apply the method to a problem with an exact solution, and then compared its numerical result with the exact one.

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2. Numerical Algorithm

Fig. 1 shows the model to be analyzed by coupling analytical solution and FEM for open boundary problem. On the open boundary Γ, just outside the complex body (FE Region), the two kinds of solutions will be coupled.

Using the numerical procedure for finite element analysis, the governing equation is represented into the matrix form [8],

$$\begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{B} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{Bmatrix} \tag{1}$$

$\mathbf{A}_1$: Potentials of nodes in inner region

$\mathbf{A}_2$: Potentials of nodes on open boundary Γ

where, **A** is the magnetic vector potentials of nodes, **s** the system matrix and **F** the forcing vector. The subscripts 1 and 2 stand for the inner region and the boundary, respectively. Also, **B**, the second term of left side in (1), is derived from boundary integral term and expressed as

$$B_j = -\int_\Gamma N_j H_\theta \, d\Gamma \tag{2}$$

j : Boundary node number (j = 1, 2, ..., n_b)

where, n_b is the total number of nodes on boundary Γ, N_j the shape function related to node j, $H_\theta(=\nu\partial A/\partial\rho)$ the tangential component of magnetic field intensity and ν magnetic reluctivity. As the potential function **A** on Γ is expanded by Fourier series [10], the coefficients are as follows

$$C_n = \frac{1}{\pi}\sum_{j=0}^{n_b-1}\int_{\theta_j}^{\theta_{j+1}} A(a,\theta)\cos n\theta d\theta \tag{3}$$

$$S_n = \frac{1}{\pi}\sum_{j=0}^{n_b-1}\int_{\theta_j}^{\theta_{j+1}} A(a,\theta)\sin n\theta d\theta . \tag{4}$$

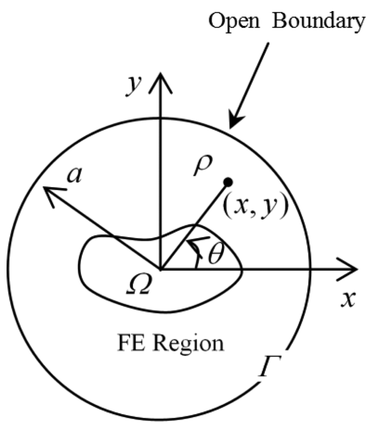


Fig. 1. Schematic diagram of open boundary problem.

The coefficients C_n and S_n expanded by Fourier series are components of $\mathbf{A}_3$

$$\mathbf{A}_3 = \{C_1, S_1, C_2, S_2, .., C_{n_h}, S_{n_h}\} \tag{5}$$

n_h: the number of harmonics

From (3) and (4), $\mathbf{A}_3$ can be written as follows

$$\mathbf{A}_3 = \mathbf{T}\mathbf{A}_2 \tag{6}$$

$$T_{ij} = \frac{1}{\pi n}\begin{cases} U_{nj} & i{=}odd \\ V_{nj} & i{=}even \end{cases} \tag{7}$$

$$U_{nj} = \frac{1}{n}\left[\frac{\cos n\theta_j - \cos n\theta_{j-1}}{\Delta\theta_{j-1}} - \frac{\cos n\theta_{j+1} - \cos n\theta_j}{\Delta\theta_j}\right] \tag{8}$$

$$V_{nj} = \frac{1}{n}\left[\frac{\sin n\theta_j - \sin n\theta_{j-1}}{\Delta\theta_{j-1}} - \frac{\sin n\theta_{j+1} - \sin n\theta_j}{\Delta\theta_j}\right] \tag{9}$$

where, i is the number from 1 to $2n_h$, j from 1 to n_b and n the integer of $(i+1)/2$.

Using the magnetic field intensity and coefficients of Fourier series, the boundary integral term is expressed as

$$B_j = -\frac{1}{\mu_0}\sum_{n=1}^{n_b}(U_{nj}C_n + V_{nj}S_n) \tag{10}$$

And again, boundary integral can be expressed by $\mathbf{A}_3$

$$\mathbf{B} = \mathbf{K}\mathbf{A}_3 \tag{11}$$

$$K_{ij} = -\frac{1}{\mu_0}\begin{cases} U_{ni} & i{=}odd,\ n = (j+1)/2 \\ V_{ni} & i{=}even,\ n = j/2 \end{cases} \tag{12}$$

Thus, from (1), (6) and (11), we can obtain the final system matrix equation.

$$\begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} & \mathbf{0} \\ \mathbf{S}_{21} & \mathbf{S}_{22} & \mathbf{K} \\ \mathbf{0} & \mathbf{T} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{0} \end{Bmatrix} \tag{13}$$

where **I** is the unit matrix.

By multiplying $\pi n\cdot\left(-\frac{1}{\mu_0}\right)$ to third equation, we can make the system matrix retain the symmetry and the sparsity like standard FEM.

3. Application Technique of Boundary Conditions

Fig. 2 represents a part of full region when the geometry and current distributions are symmetric and periodic. One is a half model with symmetry between left and right side with Dirichlet condition, the other a quarter with Dirichlet and Neumann conditions

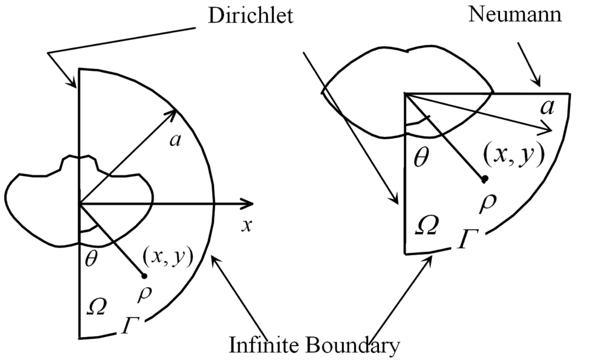


Fig. 2. Boundary conditions of symmetrical and periodic structure.

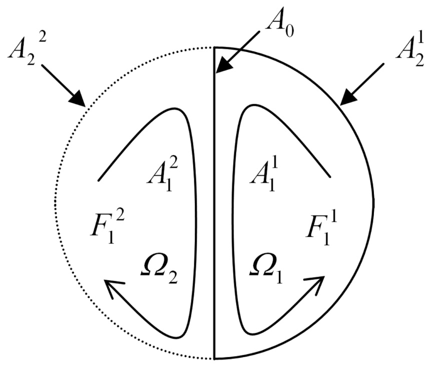


Fig. 4. Node ordering of symmetrical structure (half model).

Fig. 3 shows the distributions of the magnetic vector potential along the circumference at $\rho = a$. In half model, if numbers of nodes were ordered like the arrowed curve in Fig. 4, the system matrices of each region are equal. The superscripts 1 and 2 stand for the region Ω_1 and Ω_2, respectively.

Since the current distributions of each region are symmetric but have different sign, the forcing vectors satisfy following relationship

$$\mathbf{F}_1^1 = -\mathbf{F}_1^2 \tag{14}$$

And, the flux distributions are also symmetric and this means that the magnetic vector potentials of boundary nodes are

$$\mathbf{A}_0 = \mathbf{0} \text{ (Dirichlet boundary condition).} \tag{15}$$

From the above, the relationships between potential sets for each region are

$$\mathbf{A}_1^1 = -\mathbf{A}_1^2,\ \mathbf{A}_2^1 = -\mathbf{A}_2^2 \tag{16}$$

In case of half model, consequently, the system matrix (13) can be reduced as follows.

$$\begin{bmatrix} \mathbf{S}_{11}^1 & \mathbf{S}_{12}^1 & \mathbf{0} \\ \mathbf{S}_{21}^1 & \mathbf{S}_{22}^1 & \mathbf{K}^1 \\ \mathbf{0} & \mathbf{T}^1 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_1^1 \\ \mathbf{A}_2^1 \\ \mathbf{A}_3^1 \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_1^1 \\ \mathbf{F}_2^1 \\ \mathbf{0} \end{Bmatrix} \tag{17}$$

The potential function along the open boundary has sine waveform like symmetry in θ, and has a form of

$$A(\theta) = -A(2\pi - \theta). \tag{18}$$

Therefore, the coefficients of Fourier series are calculated as follows

$$C_n = 0 \tag{19}$$

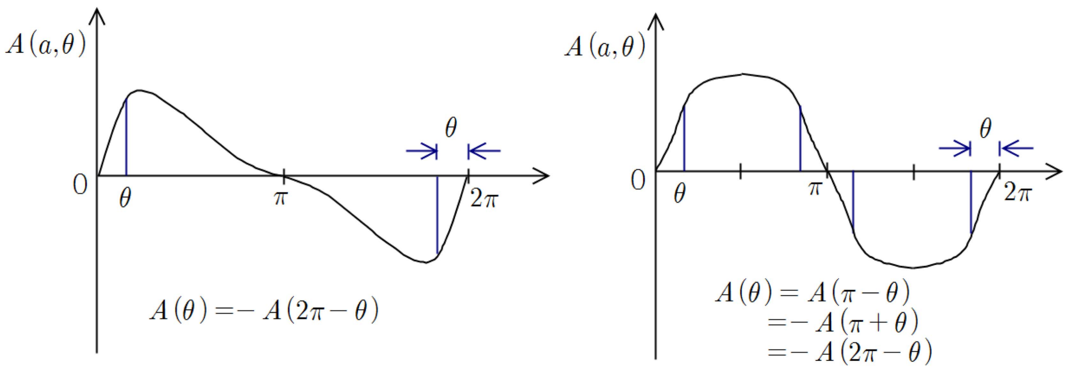
$$S_n = \frac{2}{\pi}\int_0^{\pi} A(a,\ \theta)\sin n\theta d\theta \tag{20}$$

The coefficients of cosine terms are zero, so the $\mathbf{A}_3^1$ is just defined the components of S_n.

$$\mathbf{A}_3^1 = \{S_1, S_2, \ldots, S_{n_h}\} \tag{21}$$

From (19) and (20), $\mathbf{A}_3^1$ can be written as follows

$$\mathbf{A}_3^1 = \mathbf{T}^1\mathbf{A}_2^1 \tag{22}$$



(a) (b)

Fig. 3. (Color online) Potential distributions of symmetrical and periodic structure.

$$T^1_{ij} = \frac{2}{\pi n} V^1_{nj} \tag{23}$$

$$V^1_{n1} = \frac{1}{n}\frac{\sin n\theta_1 - \sin n\theta_2}{\Delta\theta_1} - \cos\theta_1 \tag{24}$$

$$V^1_{nnb} = \frac{1}{n}\frac{\sin n\theta_{n_b} - \sin n\theta_{n_b-1}}{\Delta\theta_{j-1}} + \cos\theta_{n_b} \tag{25}$$

$$V^1_{nj} = \frac{1}{n}\left[\frac{\sin n\theta_j - \sin n\theta_{j-1}}{\Delta\theta_{j-1}} - \frac{\sin n\theta_{j+1} - \sin n\theta_j}{\Delta\theta_j}\right] \tag{26}$$

$$0 = \theta_1 < \theta_2 < \cdots < \theta_{n_b} = \pi \tag{27}$$

where, i is a number from 1 to n_h, j from 1 to n_b and n is equal to i. Using (24)-(26), boundary integral term is expressed as

$$B^1_j = -\frac{1}{\mu_0}\sum_{j=1}^{n_h} V^1_{nj} S_n . \tag{28}$$

And again, Eq. (28) can be expressed with $\mathbf{A}^1_3$.

$$\mathbf{B}^1 = \mathbf{K}^1\mathbf{A}^1_3 \tag{29}$$

$$K^1_{ij} = -\frac{1}{\mu_0} V^1_{ni} \quad n = j \tag{30}$$

Now, with the above (23) and (30), the system equation for a half model, (17), is completed. Also, in case of a quarter model, the system matrix is reduced as (17). And then the expressions of $\mathbf{T}^1$, $\mathbf{K}^1$ and $\mathbf{A}^1_3$ are similar to those of in half model.

The potential pattern has the half-wave symmetry as shown in Fig. 5 and following equations are satisfied.

$$A(\theta) = A(\pi - \theta) = -A(\pi + \theta) = -A(2\pi - \theta) \tag{31}$$

So, the coefficients of Fourier series are as follows

$$C_n = 0 \tag{32}$$

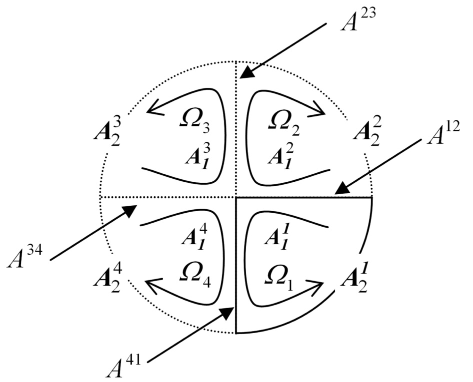


Fig. 5. Node ordering of periodic structure (quarter model).

$$S_n = \begin{cases} \frac{4}{\pi}\int_0^{\pi/2} A(a, \theta)\sin n\theta d\theta , & n = odd \\ 0 , & n = even \end{cases} \tag{33}$$

Therefore, $\mathbf{T}^1$, $\mathbf{K}^1$ and $\mathbf{A}^1_3$ of a quarter model are expressed as

$$T^1_{ij} = \frac{4}{\pi n} V^1_{nj}, \ n = 2i - 1 \tag{34}$$

$$K^1_{ij} = -\frac{1}{\mu_0} V^1_{ni}, \ n = 2i - 1 \tag{35}$$

$$\mathbf{A}^1_3 = \{S_1, S_2, S_5, ..., S_{n_h}\} \tag{36}$$

By applying (34), (35) and (36) to system matrix of Ω_1, a quarter model is also expressed.

4. Numerical Results

To validate the algorithm, a model that has an analytical solution is adopted as in Fig. 6, which is a bus bar for the power supply to the super conductive magnet. The magnetic field is calculated by the coupling method for full, half and quarter region. In Fig. 6, the width of conductor is $w = 0.1261$ (m), the height $h = 0.2606$ (m), the distance between centers of conductors $c = 0.5437$ (m) and current densities of each bars is 5 (A/mm^2).

Fig. 7 represents the flux distribution of full model simulated by coupling scheme. And, the flux distribution patterns of a half and a quarter model are shown in Fig. 8. The both results are reasonable.

The magnetic vector potentials and the errors along the test path are shown in Fig. 9. The radius and the angle of the test path are $r = 0.3$ (m) and from $\theta = 0°$ to $\theta = 90°$,

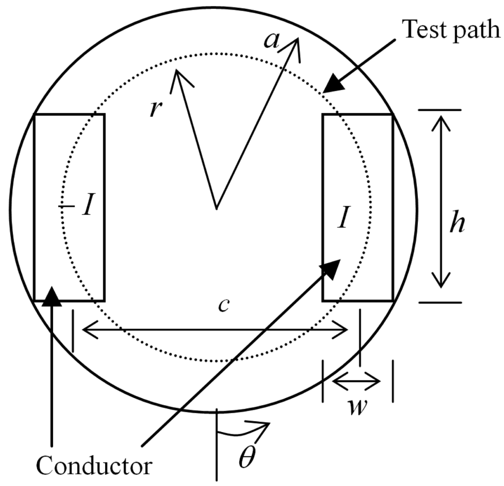


Fig. 6. Example model (Bus bar).

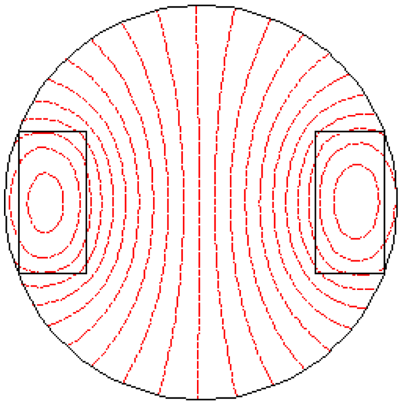


Fig. 7. (Color online) Flux distribution (full model).

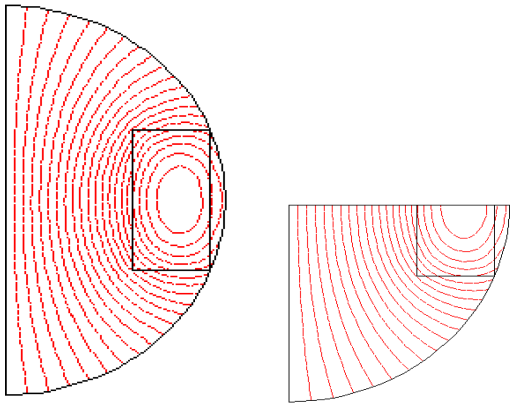


Fig. 8. (Color online) Flux distribution (half and quarter model).

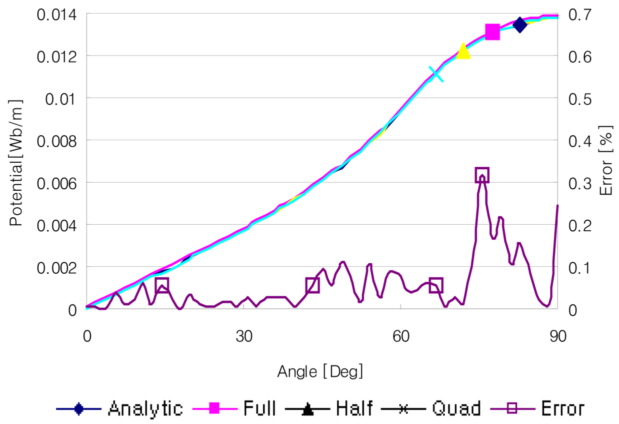


Fig. 9. (Color online) Potential comparison along the test path.

respectively. The error is defined as the percent difference between analytical solution and the result of a quarter model. All the potential curves look like coincide and the maximum error is 0.3 (%).

5. Conclusion

In electromagnetic field analysis, this paper suggests application methods of open boundary conditions when the problem of interest has symmetry and periodic in structure and in source. The outer (far) region of open boundary is calculated by analytical method and inner (near) region include current and magnetic material is analyzed by FEM. In this case, the solution sets are symmetric or periodic through the whole region. Therefore, analysis of only a part of model with symmetric structure is enough. Thus the proposed method can reduce the effort to enter the input data, necessary computer memory and computation time greatly. Proposed algorithm also can apply to the others as same approach if the models are 6th, 8th and 10th periodic problem. To prove the usefulness of algorithm, the example having analytical solution is adopted. The results are compared with those of each models *i.e.* a full, a half and a quarter model.

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