

ON THE SOLUTIONS OF EQUATIONS OVER NILPOTENT GROUPS OF CLASS 2

SEONG KUN KIM

ABSTRACT. In this paper, we study equations over nilpotent groups of class 2. We show that there are some overgroups which contains solutions of equations with exponent sum 1 over nilpotent groups of class 2. As known, equations over a field has a solution in an extension field which contains a copy of the given field. But it is not easy to find that a solution of equations over groups. In many cases, even if equations over groups has a solution, the overgroup is not concrete but very abstract. Here we find the concrete overgroups in case of nilpotent groups.

1. Introduction

For a given group H and the variable x , the *equation over the group H* is given by as follow.

$$w(x) = x^{n_1} h_1 x^{n_2} h_2 \cdots x^{n_k} h_k = 1,$$

where n_1, n_2, \dots, n_k are non-zero integer, k is a positive integer and h_1, h_2, \dots, h_k are taken from the group H . H is called the *coefficient group*. We say that the sum $\sum_{i=1}^k h_i$ is called the exponent sum of the equation, denoted $\exp_x(w(x))$. One can find that every equation over a group does not have a solution in H itself. So one can ask whether there are some groups which contains an isomorphic copy of a given group and a solution of the equation. For example, if the orders of g and h are distinct in a group H then there isn't an overgroup which contains a solution of the equation $xgx^{-1}h = 1$. In this case, the exponent sum of the given equation is zero so that one should restrict the equations or the coefficient groups.

There are two big problems on the equations over groups. One is to find the solutions of the equations with the non-zero exponent sum over any groups, the other is to find the solutions of any equations over any torsion free groups. Now we scan the several known results. First of all, the equation has a solution if all exponents are positive. Secondly, if the coefficient groups are locally residually finite or locally indicable then the equation with the non-zero exponent sum is

Received May 15, 2013; Accepted June 5, 2013.

2000 *Mathematics Subject Classification*. Primary 20F05.

Key words and phrases. equations over groups, nilpotent groups.

solvable. The equation with the exponent sum 1 over any torsion-free group is solvable and there are a lot of solvable specific equations. To find the solutions of equations over torsion-free groups, it is also known that one can consider the equations with the exponent sum zero.

In this paper, we consider the equation with exponent sum 1 over the nilpotent groups of class 2. In fact, the nilpotent group of class 2 should be locally residually finite. So the equation is solvable but we give another solution group using Levin methods.

2. Some preliminaries

In this section, we review some basic facts for equations over groups. Let H be a nilpotent group of class 2, i.e., $[H, H] \subseteq Z(H)$, where $Z(H)$ is the center of the group H . As we know that $H/Z(H)$ is abelian. Let $w(x) \in H * \langle x \rangle$ be an equation such that $\exp_x(w(x)) \neq 0$.

In this section, we will discuss more precisely the first question in the previous section. Let H be a group and x be an unknown variable. An *equation* over H is of the form

$$w(x) = x^{\varepsilon_1} h_1 x^{\varepsilon_2} h_2 \cdots x^{\varepsilon_n} h_n$$

where $w(x)$ is a cyclically reduced word in $H * F(x)$, $F(x)$ is the infinite cyclic group generated by x , each coefficient is in H and $\varepsilon_i = \pm 1$ for each i , n is a positive integer. The sum $\sum_i \varepsilon_i$ is called the *exponent sum*.

Therefore, we have a relative presentation $\langle H, x : w(x) \rangle$ with adding one more generator and one more relator. We will consider the relative presentation $\langle H, x : w(x) \rangle$ as the group $\frac{H * F(x)}{N(w(x))}$ presented by $\langle H, x : w(x) \rangle$ in this section.

The equation $w(x) = 1$ is said to have a *solution* in some overgroup of H or to be *solvable* in an overgroup of H if there is a group H^* containing an exact copy of the given group H as a subgroup and an element $h^* \in H^*$ such that $w(h^*)$ is the identity in H^* . This is equivalent to saying that the natural map

$$H \longrightarrow \langle H, x : w(x) \rangle$$

is injective.

This problem was originally announced by B. H. Neumann who considered the equation $w(x) = x^n h^{-1}$ for a given group H and an element $h \in H$ and asked whether every element of H has an n -th root in some overgroup. The solution is in the free product with amalgamation, $H \underset{\langle c^n = h \rangle}{*} \langle c \rangle$, where $\langle c \rangle$ is a cyclic group of order n times the order of h . Of course, $\langle c \rangle$ is the infinite cyclic group if the order of $h \in H$ is infinite. More generally, F. Levin showed the following theorem, see [10].

Theorem 2.1. *Let H be an arbitrary group and $w(x) = xh_1xh_2 \cdots xh_n$ be an equation where $h_i \in H$ for all i . Then $w(x)$ has a solution in some overgroup.*

Sketch of proof. Consider the following diagram

$$\begin{array}{ccc}
 H & \longrightarrow & \langle H, x : xh_1xh_2 \cdots xh_n \rangle \\
 & \searrow i & \downarrow f \\
 & & \prod_{i=1}^n H \rtimes C_n
 \end{array}$$

where $\prod_{i=1}^n H \rtimes C_n$ is the semidirect product of n -copies of H with C_n , a cyclic group of order n generated by c and the action by c permutes the n -copies of H in $\prod_{i=1}^n H \rtimes C_n$, i.e., $c(g_1, g_2, \dots, g_n) = (g_2, g_3, \dots, g_n, g_1) \cdot c$. The map f and i send $h \in H$ to the diagonal entry $(h, h, \dots, h) \cdot 1$, and f sends x to $(h_1^{-1}, h_2^{-1}, \dots, h_n^{-1}) \cdot c^{-1}$. Then we can verify that $f(w(x)) = 1$ so that it is a homomorphism and the diagram commutes. On the other hand, the map i from H to $\prod_{i=1}^n H \rtimes C_n$ given by $h \mapsto (h, h, \dots, h) \cdot 1$ is certainly injective so that the natural map from H to $\langle H, x : xh_1xh_2 \cdots xh_n \rangle$ is injective, too. \square

In the above theorem, the equation $w(x) = xh_1xh_2 \cdots xh_n$ has only positive exponents. Any equation with negative exponents is solved by changing to the variable $t = x^{-1}$. Naturally, one can ask about an equation that has both positive and negative exponents. Unfortunately, it is verified that not every equation is solved over an arbitrary group.

Example 3.1 Let H be a group. Assume that the element $a \in H$ has order 2 and $b \in H$ has order 3. Then the equation $w(x) = xax^{-1}b^{-1}$ can not be solved because of the following reason. If it can be solved then a and b with different orders are conjugate in a group, this is impossible. However, if a and b have the same order then the natural map $H \rightarrow \langle H, x : xax^{-1}b^{-1} \rangle$ is always injective and the group $\langle H, x : xax^{-1}b^{-1} \rangle$ is said to be an **HNN**-extension of H .

Thus, we should realize that some restrictions on groups or equations are necessary. The best known restrictions are the Kervaire-Laundebach question and Levin question.

- (a) **Kervaire-Laundebach question:** If the exponent sum of a given equation does not equal zero, then is the equation solvable in some overgroup ?
- (b) **Levin question:** Is any single equation over any torsion free group solvable ?

It is known that Kervaire-Laundebach question has a positive answer if H is locally residually finite [4] or locally indicable [6]. Also, Klyachko [9] has shown that any equation with exponent sum ± 1 over torsion free groups has a solution. Furthermore, many authors have given partial solutions with restrictions on equations and used various techniques by using the geometric methods. However, both questions remain open in full generality.

I would like to finish this section with some observations from Levin’s paper.

Theorem 2.2. *Let H be a group and $w(x) = x^{\varepsilon_1}h_1x^{\varepsilon_2}h_2 \cdots x^{\varepsilon_n}h_n$ be an equation where $h_i \in H$ and $\varepsilon_i \in \{\pm 1\}$ for all i . Suppose that the sum of exponents is nonzero and the product $h_1h_2 \cdots h_n$ is in the center of the subgroup generated by $\{h_1, h_2, \dots, h_n\}$, i.e., $h_1h_2 \cdots h_n$ commutes with h_i for all i . Then $w(x) = 1$ has a solution in some overgroup.*

Proof. Since the sum of exponents is nonzero, let the exponent sum $\sum_i \varepsilon_i = m > 0$ without loss of generality. If the sum of exponents is negative, one can use the change of variable $t = x^{-1}$. Now, think of another equation associated to the exponent sum of $w(x)$ as follow

$$w'(x) = x^m h_1 h_2 \cdots h_n.$$

By Levin’s paper, we can construct a commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & \langle H, x : x^m h_1 h_2 \cdots h_n \rangle \\ & \searrow & \downarrow f \\ & & \prod_{i=1}^m H \rtimes C_m \end{array}$$

such that $f(w'(x)) = 1$. Let c be a generator of C_m . The key observation is that $f(x)f(h_i) = f(h_i)f(x)$ for all h_i , because $f(x) = (1, 1, \dots, 1, (h_1h_2 \cdots h_n)^{-1}) \cdot c^{-1}$, $h_1h_2 \cdots h_n$ commutes with all h_i and c commutes with $f(h)$ for all h in H . We therefore have $f(w(x)) = f(w'(x)) = 1$, and then the natural map

$$H \longrightarrow \langle H, x : w(x) \rangle$$

is an injective homomorphism and the group $\prod_{i=1}^m H \rtimes C_m$ is also a solution group for the equation $w(x)$. This proves the theorem. □

Corollary 2.3. *Equations with nonzero exponent sum over abelian groups are solvable.*

This also follows from the Gerstenharber and Rothaus result because abelian groups are locally residually finite, [4].

Lemma 2.4. *Let H be a group and $w(x) = x^{m_1}h_1x^{m_2}h_2 \cdots x^{m_k}h_k$ with the non-zero exponent sum. If there is an element $h \in H$ such that*

$$h^{m_1}h_1h^{m_2}h_2 \cdots h^{m_k}h_k$$

is an element of the center of H then $w(x)$ is solvable in some overgroup.

Proof. Consider an equation $w(tg)$ obtained from $w(x)$ by replacing x to tg where g is such an element. Since $\exp_t(w(tg)) \neq 0$ and the product of coefficients in $w(tg)$ is in the center by assumption, $w(tg)$ is really solvable. So we can apply the lemma in my thesis. This completes the proof. □

Theorem 2.5. *Let H be a nilpotent group of class 2 and*

$$w(x) = x^{m_1} h_1 x^{m_2} h_2 \cdots x^{m_k} h_k$$

an equation with $\exp_x(w) = m \neq 0$. If there is an element $h \in H$ such that $g_1 g_2 \cdots g_k g^M$ is in the center, then the equation is $w(x)$ is solvable.

Corollary 2.6. *With the above assumption, the equation $w(x)$ is solvable if the exponent sum is 1.*

There are many references and results of equations over groups [1, 3, 9] It was suggested that the relation of C^* -algebra of foliation and a C^* -algebra obtained from a deformation quantization of a torus can be regarded as a mirror symmetry. From a physical point of view, it was argued that those two C^* -algebras are related by the T-duality, which is equivalent to the mirror symmetry on tori. Thus, our results and its Corollaries, may be considered as a mathematical interpretation.

References

- [1] Aaron S. Clark, *Disk diagrams, tessellations of the 2-sphere, and equations over groups*, Comm. Algebra **39** 2011(8), 2981-3020.
- [2] M. Edjvet, *Equations over groups and a theorem of Higman, Neumann, and Neumann*, Proc. Lond. Math. Soc. **62** (1991), 563-589.
- [3] S. M. Gersten, *Nonsingular equations of small weight over groups*, in: *Combinatorial group theory and topology* (Alta, Utah, 1984), Ann. of Math. Stud. **111**, Princeton Univ. Press, (1987), 121-124.
- [4] M. Gerstenhaber and O. S. Rothaus, *The solution of sets of equations in groups*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 1531-1533.
- [5] G. Higman, *The units of groups rings*, Proc. London Math. Soc. (2) **46** (1940), 231-248.
- [6] J. Howie, *On pairs of 2-complexes and systems of equations groups*, J. Reine Angew. Math. **324**, 165-174 (1981).
- [7] J. Howie, *The solution of length three equations over groups*, Proc. Edinb. Math. Soc. **26** (1983), 89-96.
- [8] S. V. Ivanov and A. A. Klyachko, *Solving equations of length at most six over torsion-free groups*, J. Group Theory **3** (2000), 329-337
- [9] A. Klyachko, *Funny property of sphere and equations over groups*, Comm. in Alg. **21** (7), 2555-2575, (1993)
- [10] F. Levin, *Solutions of equations over groups*, Bull. Amer. Math. Soc. **68**, 603-604 (1962).
- [11] B. H. Neumann, *Adjunction of elements to groups*, J. London Math. Soc. **24**, 247-254 (1949).

SEONG KUN KIM
 SCHOOL OF GENERAL STUDIES
 KANGWON NATIONAL UNIVERSITY
 SAMCHEOK 245-711, KOREA
 E-mail address: kimseong@kangwon.ac.kr