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ON DECOMPOSITIONS OF THE COMPLETE EQUIPARTITE GRAPHS $K_{km(2t)}$ INTO GREGARIOUS m-CYCLES

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Abstract. For an even integer m at least 4 and any positive integer t, it is shown that the complete equipartite graph $K_{km(2t)}$ can be decomposed into edge-disjoint gregarious m-cycles for any positive integer k under the condition satisfying $\frac{(m-1)^2+3}{4m} < k$. Here it will be called a *gregarious cycle* if the cycle has at most one vertex from each partite set.

1. Introduction

A complete multipartite graph with partite sets of order $a_1, a_2, \ldots, a_n$ will be denoted by $K(a_1, a_2, \ldots, a_n)$ so that any two vertices in the same partite set have no edge joining them while any two vertices in different partite sets have one edge joining them. If all the partite sets have the same size of order m then we refer to the complete multipartite graph as *equipartite*, and use the notation $K_{n(m)}$ to denote this graph with n parts of size m.

The problem of the existence of edge-disjoint decompositions of complete graphs and complete multipartite graphs into cycles of a fixed length have been considered in a number of different ways. After a series of developments, the problem of decomposing a complete graph of odd order, and also a complete graph of even order minus 1-factor into cycles of a fixed length has been completely solved (see [1], [11] and [12]). But it is still an open problem when $K_{n(m)}$ has an edge-disjoint decomposition.

We give our definition of gregarious cycles clear. We will call a cycle of a multipartite graph *gregarious* if at most one vertex of the cycle comes from any particular partite set. In fact, Billington and and Hoffman ([2]) first introduced the concept of the gregarious cycle for the case of a tripartite graph, and the notion of gregarious cycles has been modified in various ways in the later papers ([2], [4], [8], [13]) to meet developing problems. Now, we will say that a graph is γ_m*-decomposable* if it is decomposable into gregarious m-cycles, and a decomposition of a graph into gregarious m-cycles will be called a γ_m*-decomposition* of the graph.

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Šajna ([12]) showed that the complete graph $K_{n(2)}$ has a decomposition into m-cycles if and only if m divides the number of edges. However, the cycles in the decomposition were not necessarily gregarious. It seems that the requirement of gregariousness makes the problem much more complicated and difficult. Billington and Hoffman ([3]) and Cho and et el. ([8]) independently produced gregarious 4-cycle decompositions for certain complete multipartite graphs. Cho and Gould ([9]) also showed that gregarious 6-cycle decomposition is possible for $K_{n(2t)}$ for all $t \geq 1$ if $n \geq 6$. One can get some other results and comments of 6- and 8-cycle gregarious cycles decompositions ([5], [7]). Also, Cho and Kim ([10]) determined that the complete equipartite graph $K_{km+1(2t)}$ can be γ_m-decomposable for all integers k, t and even integer m with at least 4.

In this paper, we will show that the complete equipartite graph $K_{km(2t)}$ is decomposable into gregarious m-cycles for any positive integers k, t and any even integer m with at least 4 satisfying $\frac{(m-1)^2+3}{4m} < k$.

Now, we state the main theorem of the paper.

Theorem 1.1. *Let k and t be any positive integers and m be an even integer with $m \geq 4$. Then the graph $K_{km(2t)}$ is γ_m-decomposable whenever $\frac{(m-1)^2+3}{4m} < k$.*

In fact, Theorem 1.1 can be obtained as a corollary to the following theorem, which is a special case of Theorem 1.1. So we will focus on proving the next theorem in the subsequent sections.

Theorem 1.2. *Let k be any positive integer and m be an even integer with $m \geq 4$. Then the graph $K_{km(2)}$ is γ_m-decomposable whenever $\frac{(m-1)^2+3}{4m} < k$.*

Proof of Theorem 1.1 We adopt the method used in [6] and [8]. Replace each vertex a of $K_{km(2)}$ by t new vertices and label them $a_1, a_2, \ldots, a_t$. We now join the vertex a_i to the vertex b_j for all $i, j = 1, 2, \ldots, t$ if ab was an edge in $K_{km(2)}$. In other terminology, the resulting graph is known as the *composition of $\overline{K}_t$ by $K_{km(2)}$* or the *wreath product of $\overline{K}_t$ by $K_{km(2)}$*. Obviously, this new graph is $K_{km(2t)}$. Now, by Theorem 1.2, we have a γ_m-decomposition Φ of $K_{km(2)}$. If $\lambda : a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ is a gregarious m-cycle in Φ, then

$$\lambda_{ij} : a_i^{(1)}, a_j^{(2)}, a_i^{(3)}, a_j^{(4)}, \ldots, a_i^{(m-1)}, a_j^{(m)} \qquad (i = 1, 2, \ldots, t,\ j = 1, 2, \ldots, t).$$

are t^2 edge-disjoint gregarious m-cycles of $K_{km(2t)}$. The collection of all such cycles of $K_{km(2t)}$ obtained from each cycle in Φ constitutes a γ_m-decomposition of $K_{km(2t)}$. $\square$

In section 2, we introduce feasible and semi-feasible sequences of differences of numbers in $\mathbb{Z}_{km}$ and explain the method of producing gregarious m-cycles from feasible and semi-feasible sequences. In section 3, we will prove Theorem

1.2 by producing appropriate feasible and semi-feasible sequences and generating gregarious m-cycles.

2. Cycles from feasible sequences of differences

Most of definitions and concepts in this section come from the earlier papers ([7], [8] and [9]).

For $K_{n(2)}$, let the partite sets be $A_0 = \{0, \overline{0}\}$, $A_1 = \{1, \overline{1}\}$, … and $A_{n-1} = \{n-1, \overline{n-1}\}$. Thus, the elements in $\mathbb{Z}_n$ are used as indices of the partite sets and as vertices of the graph as well. An edge between a vertex in A_i and another vertex in A_j is called an *edge of distance* d if $i-j = \pm d$ for some d with $0 < d \leq \frac{n}{2}$, where the arithmetic is done in $\mathbb{Z}_n$. In particular, if $d = \frac{n}{2}$ then the edges of distance d are called the *diagonal edges*. For example, the edges $0\overline{4}$, 73, $\overline{7}\,\overline{2}$ and $\overline{8}3$ are all edges of distance 4 in $K_{9(2)}$, and the edges $4\,\overline{9}$ and $\overline{0}\,\overline{5}$ are diagonal edges of $K_{10(2)}$.

Put $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm\frac{n-1}{2}\}$ if n is odd and $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm\frac{n-2}{2}, \frac{n}{2}\}$ if n is even. Then, $\mathcal{D}_n$ is a complete set of differences of two distinct numbers in $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. A sequence $\rho = (r_1, r_2, \ldots, r_m)$ of differences in $\mathcal{D}_n$ is called a *feasible sequence*, or an *f-sequence* for simplicity, if

(i) $\sum_{i=1}^{m} r_i = 0$, that is, the total sum of the terms of the sequence is zero, and

(ii) $\sum_{i=p}^{q} r_i \neq 0$ for all p, q with $1 < p$ or $q < m$, that is, any proper partial sum of consecutive entries is nonzero,

where the arithmetic is done in $\mathbb{Z}_n$.

Let $\rho = (r_1, r_2, \ldots, r_m)$ be a sequence of differences of $\mathcal{D}_n$. The *sequence of initial sums*, or the *s-sequence* for short, of ρ is the sequence

$$\sigma_\rho = (s_0, s_1, s_2, \ldots, s_{m-1})$$

of elements in $\mathbb{Z}_n$, where $s_0 = 0$ and $s_i = \sum_{j=1}^{i} r_j$ for $i = 1, 2, \ldots, m-1$. Note that, $s_i = s_{i-1} + r_i$ for each $i = 1, 2, \ldots, m-1$ and $s_{m-1} + r_m = s_0$.

In this paper, with the above notation, the sequence σ_ρ represents the sequence of partite sets which a m-cycle traverses, and the feasibility of ρ guarantees that the cycle is proper and gregarious. Now, the following lemma is trivial from the definitions.

Lemma 2.1. *Let $\sigma_\rho = (s_0, s_1, s_2, \ldots, s_{m-1})$ be the s-sequence of a sequence $\rho = (r_1, r_2, \ldots, r_m)$ of differences in $\mathcal{D}_n$. Then ρ is an f-sequence if and only if $\sum_{i=1}^{m} r_i = 0$ and all entries of σ_ρ are mutually distinct.*

Let ϕ^+ and ϕ^- be mappings of $\mathbb{Z}_n$ into $\cup_{i=0}^{n-1} A_i$ defined by $\phi^+(i) = i$ and $\phi^-(i) = \overline{i}$ for all i in $\mathbb{Z}_n$. A *flag* is a sequence $\phi^* = (\phi_0, \phi_1, \ldots, \phi_{m-1})$ where $\phi_i = \phi^+$ or ϕ^-, $i = 0, 1, \ldots, m-1$. Given such a flag ϕ^*, we also use the same notation ϕ^* to denote the mapping defined by $\phi^*(s_0, s_1, \ldots, s_{m-1}) =$

$\langle \phi_0(s_0), \phi_1(s_1), \ldots, \phi_{m-1}(s_{m-1})\rangle$ for every sequence $(s_0, s_1, \ldots, s_{m-1})$ of distinct elements in $\mathbb{Z}_n$. Note that $\phi^*(s_0, s_1, \ldots, s_{m-1})$ is a gregarious m-cycle.

Let $\tau : \mathbb{Z}_n \to \mathbb{Z}_n$ be the mapping defined by $\tau(i) = i+1$ for all i in $\mathbb{Z}_n$. Then, $\tau^k(i) = i+k$ for all i, k in $\mathbb{Z}_n$ and τ^n is the identity mapping. We can extend each τ^k to a mapping $\tau_*^k : \mathbb{Z}_n^m \to \mathbb{Z}_n^m$ by defining

$$\tau_*^k(s_0, s_1, \ldots, s_{m-1}) = (\tau^k(s_0), \tau^k(s_1), \ldots, \tau^k(s_{m-1})).$$

Now, if we are given a pair (ρ, ϕ^*) consisting of an f-sequence and a flag, we can produce a class $\{\phi^*(\tau_*^k(\sigma_\rho)) \mid k \in \mathbb{Z}_n\}$ of gregarious m-cycles. For example, if $\rho = (r_1, r_2, \ldots, r_6)$ for $m = 6$ and $\phi^* = (\phi^+, \phi^-, \phi^-, \phi^+, \phi^+, \phi^-)$, then $\sigma_\rho = (s_0, s_1, s_2, \ldots, s_5)$ and the gregarious 6-cycles in the class are:

$$\begin{array}{ccllllll}
\phi^*(\tau_*^0(\sigma_\rho)) & = & \langle\, 0, & \overline{s_1}, & \overline{s_2}, & s_3, & s_4, & \overline{s_5}\,\rangle, \\
\phi^*(\tau_*^1(\sigma_\rho)) & = & \langle\, 1, & \overline{s_1+1}, & \overline{s_2+1}, & s_3+1, & s_4+1, & \overline{s_5+1}\,\rangle, \\
\phi^*(\tau_*^2(\sigma_\rho)) & = & \langle\, 2, & \overline{s_1+2}, & \overline{s_2+2}, & s_3+2, & s_4+2, & \overline{s_5+2}\,\rangle, \\
\vdots & \vdots & & & \vdots & & & \\
\phi^*(\tau_*^k(\sigma_\rho)) & = & \langle\, k, & \overline{s_1+k}, & \overline{s_2+k}, & s_3+k, & s_4+k, & \overline{s_5+k}\,\rangle, \\
\vdots & \vdots & & & \vdots & & & \\
\phi^*(\tau_*^{n-1}(\sigma_\rho)) & = & \langle\, n-1, & \overline{s_1-1}, & \overline{s_2-1}, & s_3-1, & s_4-1, & \overline{s_5-1}\,\rangle.
\end{array}$$

Note that every column on the right-hand side has one vertex from every partite set. Thus, each edge of the form $p\,\overline{q}$ appears as the first edge of a γ_6-cycle above if $q - p = s_1 = r_1$. Each edge of the form $\overline{p}\,\overline{q}$ appears as the second edge of the gregarious 6-cycle above if $q - p = s_2 - s_1 = r_2$. Similarly, each edge of the form $\overline{p}\,q$ with $q - p = r_3$, of the form $p\,q$ with $q - p = r_4$, of the form $p\,\overline{q}$ with $q - p = r_5$, and of the form $\overline{p}\,q$ with $q - p = r_6$, appears in the gregarious 6-cycles above.

Depending on the number n, we sometimes need modified notions of difference sets and feasible sequences as well. Let $\mathbb{Z}_{n-1}^* = \{\infty\} \cup \mathbb{Z}_{n-1} = \{\infty, 0, 1, \ldots, n-2\}$. Then, the set $\mathcal{E}_n = \{\pm\infty, \pm 1, \pm 2, \ldots, \pm\frac{n-2}{2}\}$ when n is even, or $\mathcal{E}_n = \{\pm\infty, \pm 1, \pm 2, \ldots, \pm\frac{n-3}{2}, \frac{n-1}{2}\}$ when n is odd, is a complete set of differences of two distinct numbers in $\mathbb{Z}_{n-1}^*$.

An f-sequence of differences in $\mathcal{E}_n \backslash \{\pm\infty\}$ is the same as the f-sequence above except that arithmetic in taken $\mathbb{Z}_{n-1}^*$. By convention, $a \pm \infty = \infty, \infty \pm a = \infty$ for $a \in \mathbb{Z}_{n-1}$, and $\infty \pm \infty = 0$. Let an edge involving ∞ be an edge of *infinite distance*. For example, $\infty\overline{3}$, $6\overline{\infty}$ are edges of infinite distance. A *semi-feasible sequence*, or an *sf-sequence*, is a sequence $\rho = (r_1, r_2, \ldots, r_{m-2}, -\infty, \infty)$ of differences in $\mathcal{E}_n$ such that $r_1, r_2, \ldots, r_{m-2}$ belong to $\mathbb{Z}_{n-1}$ and any proper partial sum of consecutive entries is nonzero. The sequence σ_ρ of initial sums of ρ is defined in the same way as above. Thus, $\sigma_\rho = (0, e_1, e_2, \ldots, e_{m-2}, \infty)$, where $e_i = \sum_{j=1}^{i} r_j$ for $i = 1, 2, \ldots, m-2$. Also, a flag ϕ^* is the same as above. The translation τ is the permutation $(0, 1, \ldots, n-2)(\infty)$ on $\mathbb{Z}_{n-1}^*$. Thus, $\tau(i) = i + 1$ for all i in $\mathbb{Z}_{n-1}$ and τ fixes ∞.

As in lemma 2.1, it can be seen that a sequence $\rho = (r_1, r_2, \ldots, r_{m-2}, -\infty, \infty)$ is an sf-sequence if and only if entries of its sequence $\sigma_\rho = (0, e_1, \ldots, e_{m-2}, \infty)$ of initial sums are mutually distinct. From a sequence of initial sums of an f-sequence or an sf-sequence, we construct starter cycles and generate classes of gregarious m-cycles in the same way above.

This procedure is the method we will use to obtain a γ_m-decomposition of $K_{n(2)}$. The main problem then is how to choose pairs of f-sequences, sf-sequences and flags so that, in the gregarious m-cycles produced by these pairs, each of the edges $p\,q$, $\overline{p}\,q$, $p\,\overline{q}$ and $\overline{p}\,\overline{q}$ with $q - p = d$ appears exactly once for every distance d with $1 \le d \le \frac{n}{2}$. Note that we sometimes need to produce a class with only $\frac{n}{2}$ gregarious m-cycles.

3. The Proof of Theorem 1.2.

In this section, $n = km$, m and k will be a positive integer with m even ,$m \ge 4$ and $\frac{(m-1)^2+3}{4m} < k$. The number of edges in $K_{km(2)}$ is $2km(km-1) = 2km(n-1)$, and we will produce $2k(n-1)$ edge-disjoint gregarious m-cycles in $2k$ classes, each class containing $n-1$ members.

Let $m = 2t$. Consider $\mathbb{Z}^*_{n-1} = \infty \cup \mathbb{Z}_{n-1}$ and partition $\mathcal{E}_{km}$ into k subsets $E_0 = \{\pm\infty, \pm 1, \pm 2, \ldots, \pm(\frac{m}{2}-1)\}$ and $E_r = \{\pm(rt), \pm(rt+1), \ldots, \pm(rt+t-1)\}$ for $r = 1, 2, \ldots, k-1$.

Let $\rho^{(0)} = (1, 2, \ldots, \frac{m}{2}-2, \frac{m}{2}-1, 1, 2, \ldots, \frac{m}{2}-2, \frac{m}{2}-1, \infty, -\infty)$. Then the sequence of initial sums of $\rho^{(0)}$ is $\eta^{(0)} = \big(0, 1, 3, \ldots, (\frac{m}{2}-1)\frac{m}{2}, \infty\big)$. Since $\frac{(m-1)^2+3}{4m} < k$, all elements of $\eta^{(0)}$ are different in $\mathbb{Z}^*_{n-1}$ so that it is sf-sequence.

Now, we produce two starter gregarious m-cycles from sf-sequence $\eta^{(0)}$. Give two special flags $\phi_1^* = (\phi_{1i}), \phi_2^* = (\phi_{2j})$ for $i, j = 0, 1, \ldots, m-1$, and let $\phi_1^*(\eta^{(0)}) = C_0^{(0)}$ and $\phi_2^*(\eta^{(0)}) = D_0^{(0)}$ as follows. First of all, put

$$\phi_{10} = \phi^+, \phi_{1(m-2)} = \phi^-, \phi_{1(m-1)} = \phi^-,$$

$$\phi_{20} = \phi^-, \phi_{2(m-2)} = \phi^+, \phi_{2(m-1)} = \phi^+.$$

If $e_{(i+1)} - e_i = l$ and $e_{(i+\frac{m}{2})} - e_{(i+\frac{m}{2}-1)} = l$ in $\eta^{(0)}$ then give $\phi_{1i} = \phi_{1(i+1)} = \phi^+$ and $\phi_{1(i+\frac{m}{2}-1)} = \phi_{1(i+\frac{m}{2})} = \phi^-$ for the differences of $l = 1, 5, 9, \ldots$ with $l < \frac{m}{2}-1$ in $C_0^{(0)}$ and give $\phi_{1i} = \phi_{1(i+1)} = \phi^-$ and $\phi_{1(i+\frac{m}{2}-1)} = \phi_{1(i+\frac{m}{2})} = \phi^+$ for $l = 3, 7, \ldots$ with $l < \frac{m}{2}-1$ in $C_0^{(0)}$. By the same way, if $e_{(i+1)} - e_i = l$ and $e_{(i+\frac{m}{2})} - e_{(i+\frac{m}{2}-1)} = l$ in $\eta^{(0)}$ then give $\phi_{2i} = \phi_{2(i+1)} = \phi^+$ and $\phi_{2(i+\frac{m}{2}-1)} = \phi_{2(i+\frac{m}{2})} = \phi^-$ for the differences of $l = 2, 6, 10, \ldots$ with $l < \frac{m}{2}-1$ in $D_0^{(0)}$ and give $\phi_{2i} = \phi_{2(i+1)} = \phi^-$ and $\phi_{2(i+\frac{m}{2}-1)} = \phi_{2(i+\frac{m}{2})} = \phi^+$ for $l = 4, 8, \ldots$ with $l < \frac{m}{2}-1$ in $D_0^{(0)}$. After doing this procedure, one can find that there are three flags undefined to produce $C_0^{(0)}$ and $D_0^{(0)}$. Consider the last difference $l = \frac{m}{2}-2$ at the above. Then it is easy to define the rest of flags for exactly once appearance of edges $pq, \overline{p}q, p\overline{q}, \overline{pq}$ with the same difference(or distance) l.

For instance, see examples 3.1 and 3.2 in the last of this section. So we produce two starter cycles $C_0^{(0)}$ and $D_0^{(0)}$ as follows.

$$\begin{aligned} C_0^{(0)} &: 0, e_1, \overline{e_2}, \ldots, \overline{e_{2t-2}}, \overline{\infty}, \\ D_0^{(0)} &: \overline{0}, e_1, e_2, \ldots, e_{2t-2}, \infty. \end{aligned}$$

To generate the rest of starter cycles, we divide the proof into two cases depending on whether m is divisible by 4 or not.

Case (1). Suppose m is divisible by 4 and put $m = 2t$. So, t is even and $\frac{km}{2} = kt$.

Fix an r and consider the f-sequence

$$\begin{aligned} \rho^{(r)} = \big(&rt, -(rt{+}1),\ rt{+}2,\ \ldots\ ,\ -(rt{+}(t{-}3)),\ rt{+}(t{-}2),\ -(rt{+}(t-1)), \\ &-(rt{+}(t{-}2)),\ rt{+}(t{-}3),\ -(rt{+}(t{-}4)),\ \ldots\ ,\ rt{+}1,\ -rt,\ rt{+}(t{-}1)\big). \end{aligned}$$

Since each element of E_r appears once, the total sum of entries of $\rho^{(r)}$ is zero. Let $\eta^{(r)} = (0, e_1^{(r)}, e_2^{(r)}, \ldots, e_{2t-1}^{(r)})$ be the sequence of initial sums of $\rho^{(r)}$. For $i = 1, 3, 5, \ldots, t{-}1$,

$$\begin{aligned} e_i^{(r)} &= \Big\{\sum_{j=1}^{i-1}(-1)^{j-1}(rt{+}(j{-}1))\Big\} + (rt{+}(i{-}1)) \\ &= \Big\{\sum_{j=1}^{i-1}(-1)^{j-1}(j{-}1)\Big\} + (rt{+}(i{-}1)) \\ &= -\frac{i{-}1}{2} + rt + i{-}1 = rt + \frac{i{-}1}{2}, \end{aligned}$$

and they are mutually distinct and constitute the interval $I_a = \{s \mid rt \le s \le rt{+}\frac{t-2}{2}\}$. For $i = 2, 4, 6, \ldots, 2t{-}2$,

$$\begin{aligned} e_i^{(r)} &= \sum_{j=1}^{t}(-1)^{j-1}(rt{+}(j{-}1)) + \sum_{j=t+1}^{i}(-1)^{j}(rt{+}(2t{-}(j{+}1))) \\ &= \sum_{j=1}^{t}(-1)^{j-1}(j-1) + \sum_{j=t+1}^{i}(-1)^{j}(2t{-}(j{+}1)) \\ &= -\frac{t}{2} - \frac{i{-}t}{2} = -\frac{i}{2} = 2kt - \frac{i{+}2}{2}, \end{aligned}$$

and they are mutually distinct and constitute the interval $I_b = \{s \mid (2k-1)t \leq s \leq 2kt-2\}$. For $i = t+1, t+3, \ldots, 2t-1$, we have

$$\begin{aligned} e_i^{(r)} &= \sum_{j=1}^{t}(-1)^{j-1}(rt+(j-1)) + \sum_{j=t+1}^{i}(-1)^j(rt+(2t-(j+1))) \\ &= -\frac{t}{2} + \Big\{ \sum_{j=t+1}^{i-1}(-1)^j(rt+(2t-(j+1)))\Big\} - (rt+(2t-i-1)) \\ &= -\frac{t}{2} - \frac{(i-1)-t}{2} - rt - (2t-i-1) = -(r+2)t + \frac{i+3}{2} \\ &= (2k-r-2)t + \frac{i+1}{2}, \end{aligned}$$

and they are mutually distinct and constitute the interval $I_c = \{s \mid (2k-r-\frac{3}{2})t+1 \leq s \leq (2k-r-1)t\}$. Since $1 \leq k$ and $1 \leq r \leq k-1$, all the interval are subintervals of the interval $\{s \mid 0 < s \leq 2kt-2\}$. Furthermore, we have $a < c < b$ for all $a \in I_a$, $b \in I_b$ and $c \in I_c$. Thus, all the entries of $\eta^{(r)}$ are mutually distinct. By Lemma 2.1, $\rho^{(r)}$ is an f-sequence. For $i = 1, 2, \ldots, 2t-1$, temporarily use e_i for $e_i^{(r)}$ for simplicity. Now, we take two special flags $\phi_1^* = (\phi_{1i})$ and $\phi_2^* = (\phi_{2j}), i, j = 0, 1, 2, \ldots, 2t-1$ as follows. $\phi_{1i} = \phi^-$ for $i = t+1, t+3, \ldots, 2t-1$ and $\phi_{1i} = \phi^+$ for the others i. $\phi_{2j} = \phi^+$ for $j = t+1, t+3, \ldots, 2t-1$ and $\phi_{2j} = \phi^-$ for the others j. Then, we produce two starter cycles from $\eta^{(r)}$ as follows:

$$\begin{aligned} C_0^{(r)} &: 0, e_1, e_2, e_3, \ldots, e_{t-2}, e_{t-1}, e_t, \overline{e_{t+1}}, e_{t+2}, \overline{e_{t+3}}, e_{t+4}, \ldots, \overline{e_{2t-3}}, e_{2t-2}, \overline{e_{2t-1}}, \\ D_0^{(r)} &: \overline{0}, \overline{e_1}, \overline{e_2}, \overline{e_3}, \ldots, \overline{e_{t-2}}, \overline{e_{t-1}}, \overline{e_t}, e_{t+1}, \overline{e_{t+2}}, e_{t+3}, \overline{e_{t+4}}, \ldots, e_{2t-3}, \overline{e_{2t-2}}, e_{2t-1}. \end{aligned}$$

Finally, by applying the translations τ_*^k we generate a class of $km-1$ gregarious m-cycles with each of the above starter cycles. Then, it is easily seen that each edge of distance d appears exactly once in the cycles for $d = rt,\ rt+1, \ldots,\ rt+(t-2),\ rt+(t-1)$. For example, if $j - i = rt + 3$, then the edges ij and $\bar{i}j$ appear in some cycles generated by $C_0^{(r)}$ at the places corresponding to e_3e_4 and $\overline{e_{2t-5}}e_{2t-4}$, respectively, while the edges $\bar{i}\,\bar{j}$ and $i\bar{j}$ appear in some cycles generated by $D_0^{(r)}$ at the places corresponding to $\overline{e_3}\,\overline{e_4}$ and $e_{2t-5}\overline{e_{2t-4}}$, respectively.

Perform the above process for each $r = 1, 2, \ldots, k-1$, and we obtain $2k$ classes of gregarious m-cycles, each class containing $km-1$ cycles. In these cycles, each edge of every nonzero distance appears exactly once. Therefore, these cycles constitute a decomposition of $K_{km(2)}$ into gregarious m-cycles.

Case(2). Suppose m is not divisible by 4 and put $m = 2t$. So, t is odd and $\frac{km}{2} = kt$. We proceed almost the same way as in Case (1), except the pattern of feasible sequences and that of starter cycles. Fix an r and consider the f-sequence

$$\rho^{(r)} = \big(rt,\ -(rt{+}1),\ rt{+}2,\ \dots\ ,\ rt{+}(t{-}3),\ -(rt{+}(t{-}2)),\ rt{+}(t{-}1),$$
$$rt{+}(t{-}2),\ -(rt{+}(t{-}3)),\ rt{+}(t{-}4),\ \dots\ ,\ rt{+}1, -rt,\ -(rt{+}(t-1))\big).$$

Since each element of E_r appears once, the total sum of entries of $\rho^{(r)}$ is zero. Let $\eta^{(r)} = (0, e_1^{(r)}, e_2^{(r)}, \dots, e_{2t-1}^{(r)})$ be the sequence of initial sums of $\rho^{(r)}$. Computing each entry of $\eta^{(r)}$ as in Case (1), we obtain that

$$e_i^{(r)} = \begin{cases} rt + \dfrac{i-1}{2} & \text{for } i = 1, 3, \dots, t, t{+}2, \dots, 2t{-}1 \quad \text{(a)} \\ -\dfrac{i}{2} = 2kt{-}\dfrac{i+2}{2} & \text{for } i = 2, 4, 6, \dots, t{-}1, \quad \text{(b)} \\ 2(r{+}1)t{-}\dfrac{i+4}{2} & \text{for } i = t{+}1, t{+}3, \dots, 2t{-}2, \quad \text{(c)} \end{cases}$$

They are all nonzero and and the intervals for the numbers in cases (a), (b) and (c) are

$$I_a = \{s \mid rt \le s \le rt{+}(t{-}1)\},$$
$$I_b = \{s \mid (2k{-}\frac{1}{2})t{-}\frac{1}{2} \le s \le 2kt{-}2\},$$
$$I_c = \{s \mid (2r{+}1)t{-}1 \le s \le (2r{+}\frac{3}{2})t{-}\frac{5}{2}\},$$

respectively. Since $1 \le k$ and $1 \le r \le k{-}1$, we have $a < c < b$ for all $a \in I_a$, $b \in I_b$ and $c \in I_c$. Thus, all the entries of $\eta^{(r)}$ are mutually distinct. By Lemma 2.1, $\rho^{(r)}$ is an f-sequence.

For $i = 1, 2, \dots, 2t{-}1$, temporarily use e_i for $e_i^{(r)}$ for simplicity. Now, we take two special flags $\phi_1^* = (\phi_{1i})$ and $\phi_2^* = (\phi_{2j}), i, j = 0, 1, 2, \dots, 2t-1$ as follows. $\phi_{1i} = \phi^-$ for $i = t+1, t+3, \dots, 2t-2$ and $2t-1$, $\phi_{1i} = \phi^+$ for the others i. $\phi_{2j} = \phi^+$ for $j = 1, 3, \dots, t-2$ and $2t-1$, $\phi_{2j} = \phi^-$ for the others j. Now, we produce two starter cycles out of $\eta^{(r)}$ as follows.

$$C_0^{(r)} \ : \ 0, e_1, e_2, e_3, \dots, e_{t-4}, e_{t-3}, e_{t-2}, e_{t-1}, e_t, \overline{e_{t+1}}, e_{t+2}, \overline{e_{t+3}}, \dots, e_{2t-3}, \overline{e_{2t-2}}, \overline{e_{2t-1}},$$
$$D_0^{(r)} \ : \ \overline{0}, e_1, \overline{e_2}, e_3, \dots, e_{t-4}, \overline{e_{t-3}}, e_{t-2}, \overline{e_{t-1}}, \overline{e_t}, \overline{e_{t+1}}, \overline{e_{t+2}}, \overline{e_{t+3}}, \dots, \overline{e_{2t-3}}, \overline{e_{2t-2}}, e_{2t-1}.$$

As before, it can be easily shown that each of edges $ij, i\overline{j}, \overline{i}j$ and $\overline{ij}$ of distance d appears exactly once in theses cycles for $d = rt,\ rt{+}1, \dots,\ rt{+}(t{-}2),\ rt{+}(t{-}1)$. With each of these starter cycles, by applying the translations τ_*^k we generate a class of $km{-}1$ gregarious m-cycles,

Perform the above process for each $r = 1, \dots, k{-}1$, and we obtain a decomposition of $K_{km(2)}$ into gregarious m-cycles as in Case (1). This completes the proof of Theorem 1.2.

Example 3.1. Let $m = 6$ and $k = 2$. We have

$$\mathcal{E}_{mk} = \mathcal{E}_{12} = \{\pm\infty, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}.$$

According to the method above, put

$$\rho^{(0)} = (1, 2, 1, 2, \infty, -\infty), \qquad \rho^{(1)} = (3, -4, 5, 4, -3, -5),$$

and the corresponding sequences of initial sums are

$$\eta^{(0)} = (0, 1, 3, 4, 6, \infty), \qquad \eta^{(1)} = (0, 3, 10, 4, 8, 5).$$

The cycles for the γ_6-decomposition is as follows :

$$\begin{array}{llll}
C_0^{(0)}: & 0, 1, \overline{3}, \overline{4}, \overline{6}, \overline{\infty} & D_0^{(0)}: & \overline{0}, 1, 3, \overline{4}, 6, \infty \\
C_1^{(0)}: & 1, 2, \overline{4}, \overline{5}, \overline{7}, \overline{\infty} & D_1^{(0)}: & \overline{1}, 2, 4, \overline{5}, 7, \infty \\
C_2^{(0)}: & 2, 3, \overline{5}, \overline{6}, \overline{8}, \overline{\infty} & D_2^{(0)}: & \overline{2}, 3, 5, \overline{6}, 8, \infty \\
C_3^{(0)}: & 3, 4, \overline{6}, \overline{7}, \overline{9}, \overline{\infty} & D_3^{(0)}: & \overline{3}, 4, 6, \overline{7}, 9, \infty \\
C_4^{(0)}: & 4, 5, \overline{7}, \overline{8}, \overline{10}, \overline{\infty} & D_4^{(0)}: & \overline{4}, 5, 7, \overline{8}, 10, \infty \\
C_5^{(0)}: & 5, 6, \overline{8}, \overline{9}, \overline{0}, \overline{\infty} & D_5^{(0)}: & \overline{5}, 6, 8, \overline{9}, 0, \infty \\
C_6^{(0)}: & 6, 7, \overline{9}, \overline{10}, \overline{1}, \overline{\infty} & D_6^{(0)}: & \overline{6}, 7, 9, \overline{10}, 1, \infty \\
C_7^{(0)}: & 7, 8, \overline{10}, \overline{0}, \overline{2}, \overline{\infty} & D_7^{(0)}: & \overline{7}, 8, 10, \overline{0}, 2, \infty \\
C_8^{(0)}: & 8, 9, \overline{0}, \overline{1}, \overline{3}, \overline{\infty} & D_8^{(0)}: & \overline{8}, 9, 0, \overline{1}, 3, \infty \\
C_9^{(0)}: & 9, 10, \overline{1}, \overline{2}, \overline{4}, \overline{\infty} & D_9^{(0)}: & \overline{9}, 10, 1, \overline{2}, 4, \infty \\
C_{10}^{(0)}: & 10, 0, \overline{2}, \overline{3}, \overline{5}, \overline{\infty} & D_{10}^{(0)}: & \overline{10}, 0, 2, \overline{3}, 5, \infty
\end{array}$$

$$\begin{array}{llll}
C_0^{(1)}: & 0, 3, 10, 4, \overline{8}, \overline{5} & D_0^{(1)}: & \overline{0}, 3, \overline{10}, \overline{4}, \overline{8}, 5 \\
C_1^{(1)}: & 1, 4, 0, 5, \overline{9}, \overline{6} & D_1^{(1)}: & \overline{1}, 4, \overline{0}, \overline{5}, \overline{9}, 6 \\
C_2^{(1)}: & 2, 5, 1, 6, \overline{10}, \overline{7} & D_2^{(1)}: & \overline{2}, 5, \overline{1}, \overline{6}, \overline{10}, 7 \\
C_3^{(1)}: & 3, 6, 2, 7, \overline{0}, \overline{8} & D_3^{(1)}: & \overline{3}, 6, \overline{2}, \overline{7}, \overline{0}, 8 \\
C_4^{(1)}: & 4, 7, 3, 8, \overline{1}, \overline{9} & D_4^{(1)}: & \overline{4}, 7, \overline{3}, \overline{8}, \overline{1}, 9 \\
C_5^{(1)}: & 5, 8, 4, 9, \overline{2}, \overline{10} & D_5^{(1)}: & \overline{5}, 8, \overline{4}, \overline{9}, \overline{2}, 10 \\
C_6^{(1)}: & 6, 9, 5, 10, \overline{3}, \overline{0} & D_6^{(1)}: & \overline{6}, 9, \overline{5}, \overline{10}, \overline{3}, 0 \\
C_7^{(1)}: & 7, 10, 6, 0, \overline{4}, \overline{1} & D_7^{(1)}: & \overline{7}, 10, \overline{6}, \overline{0}, \overline{4}, 1 \\
C_8^{(1)}: & 8, 0, 7, 1, \overline{5}, \overline{2} & D_8^{(1)}: & \overline{8}, 0, \overline{7}, \overline{1}, \overline{5}, 2 \\
C_9^{(1)}: & 9, 1, 8, 2, \overline{6}, \overline{3} & D_9^{(1)}: & \overline{9}, 1, \overline{8}, \overline{2}, \overline{6}, 3 \\
C_{10}^{(1)}: & 10, 2, 9, 3, \overline{7}, \overline{4} & D_{10}^{(1)}: & \overline{10}, 2, \overline{9}, \overline{3}, \overline{7}, 4
\end{array}$$

Example 3.2. Let $m = 8$ and $k = 3$. We have

$$\mathcal{E}_{km} = \mathcal{E}_{24} = \{\pm\infty, \pm 1, \pm 2, \ldots, \pm 11\}.$$

By the procedure above, we produce an sf-sequence and f-sequences

$$\rho^{(0)} = (1, 2, 3, 1, 2, 3, \infty, -\infty), \qquad \rho^{(1)} = (4, -5, 6, -7, -6, 5, -4, 7)$$
$$\text{and} \qquad \rho^{(2)} = (8, -9, 10, -11, -10, 9, -8, 11).$$

The corresponding sequences of initial sums are

$$\eta^{(0)} = (0, 1, 3, 6, 7, 9, 12, \infty), \qquad \eta^{(1)} = (0, 4, 22, 5, 21, 15, 20, 16)$$
$$\text{and} \qquad \eta^{(2)} = (0, 8, 22, 9, 21, 11, 20, 12),$$

respectively. The starter cycles obtained by the procedure are

$$\begin{array}{llll} C_0^{(0)}: & 0, 1, \overline{3}, \overline{6}, \overline{7}, 9, \overline{12}, \overline{\infty}, & D_0^{(0)}: & \overline{0}, 1, 3, 6, \overline{7}, \overline{9}, 12, \infty, \\ C_0^{(1)}: & 0, 4, 22, 5, 21, \overline{15}, 20, \overline{16}, & D_0^{(1)}: & \overline{0}, \overline{4}, \overline{22}, \overline{5}, \overline{21}, 15, \overline{20}, 16, \\ C_0^{(2)}: & 0, 8, 22, 9, 21, \overline{11}, 20, \overline{12}, & D_0^{(2)}: & \overline{0}, \overline{8}, \overline{22}, \overline{9}, \overline{21}, 11, \overline{20}, 12. \end{array}$$

Finally, the six classes of 23 gregarious 8-cycles generated by these starter cycles, which we omit from here, form a decomposition of $K_{24(2)}$ into gregarious 8-cycles.

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