

Variance Swap Pricing with a Regime-Switching Market Environment

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ABSTRACT

In this paper we provide a valuation formula for a variance swap with regime switching. A variance swap is a forward contract on variance, the square of realized volatility of the underlying asset. We assume that the volatility of underlying asset is governed by Markov regime-switching process with finite states. We find that the proposed model can provide ease of calculation and be superior to the models currently available.

Keywords: Variance Swap, Regime-Switching

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1. INTRODUCTION

Many practitioners consider vega exposures to be an important and tricky factor among the Greeks of their position in financial markets. The vega measures the sensitivity of the value of an investor's portfolio to a small change in volatility. Practitioners have a consensus that a considerable profit or loss could be caused by a little change in the volatility of the underlying asset. Even though there are different types of traders in financial markets, most of the market participants emphasize the importance of volatility trading (See Demeterfi *et al.*, 1999).

For trading volatility or variance, many traders use various contracts. In Chicago Board Options Exchange (CBOE), VIX futures and options are the most representative products for trading volatility. However these products are unable to resolve the needs of various market participants. So they use volatility swaps and variance swaps which are traded in the over-the-counter (OTC) market. These are the most actively traded products for trading vega risk in the OTC market. In Korea Exchange (KRX), there are not volatility products. However a volatility index of KOSPI (VKOSPI) calculation has been published in KRX, and it is expected that VKOSPI derivatives are listed in the near future.

A volatility swap is a forward contract on the realized volatility over the life of the contract. And a variance swap is a similar contract on variance, the square of volatility of underlying asset. Demeterfi *et al.* (1999) explain the properties of variance swap and they derive a pricing formula for theoretical fair value of variance swap by replicating a variance swap. They have replicated a variance swap by portfolios of vanilla options. After that many researchers use the stochastic volatility models (SVMs) for obtaining analytical or numerical results.

Swishchuk (2004), Zhu and Lian (2011) propose the analytical method for pricing the volatility swap based on Heston SVM by using discrete sampling times. While Swishchuk (2004) uses probabilistic approach method, Zhu and Lian (2011) approach the problem by solving the stochastic differential equation.

Elliott and Lian (2012) suggest the valuation of the variance swaps and the volatility swaps with the dynamics of the regime switching Heston SVM. They assume that the expected return and the long-term mean of variance of the stock depend on the state of regime process. But it is difficult to derive the forward characteristic function, which involves solving partial differential equations for calculating the expectation.

When SVMs are applied to the valuation of volatil-

ity derivatives, it remains difficult to estimate the model parameter and to calculate the value of derivatives. As pointed by Ang and Bekaert (2002), a regime-switching model has the empirical importance in equity markets. And estimation methods for regime-switching models have been investigated by many researchers. Therefore a regime-switching model provides ease of calculation and reflects the phenomenon of discrete changes in event.

Because of the square relationship between volatility and variance, the value of a volatility swap should be closely correlated to that of a variance swap. Therefore, we focus on valuing variance swaps based on regime-switching stochastic volatility model which has finite states in this paper.

2. THE MODEL

We consider a financial market to be complete: there are no taxes, no transaction costs, no short-sale restrictions, and no difference between borrowing and lending rates. And we assume that there is a filtered complete probability space $(\Omega, \mathcal{F}, \{F_t\}, P)$. The filtration $\{F_t\}$ satisfies the usual conditions and is generated by a 1-dimensional Brownian motion B_t and a Markov chain X_t . A Markov chain X_t is independent of B_t . We suppose that the underlying asset evolves according to the geometric Brownian motions of the form

$$dS_t = rS_t dt + \sigma(t)S_t dB_t,$$

under the risk-neutral measure P , where r is a risk-free interest rate and $\sigma(t)$ is the volatility of asset price changes according to a continuous-time Markov chain X_t . For convenience, we regard the underlying asset as the stock in this paper.

A Markov chain X_t is a Markov process with a finite (or countable) state space U consisting of consecutive nonnegative integers. Define the transition probability function $P_{ij}(\cdot)$ from state i to state j by

$$p_{ij}(t) = P\{X_{t+s} = j | X_s = i\}, \quad i, j \in U, \quad t, s \geq 0.$$

The matrix of transition probability functions $P(t) = (P_{ij}(t))$ is called the transition matrix of the Markov chain X_t . For more details, see Heyman and Sobel (1982).

We assume that the volatility $\sigma(t)$ depends on the state of the Markov chain X_t , that is $\sigma(t) = \sigma_{X_t}$, where X_t can have a value in $U = \{1, 2, \dots, N\}$ at time t . And σ_i means i -state volatility for each $i \in U$. In other words, the volatility $\sigma(t)$ can take a value in $\{\sigma_1, \sigma_2, \dots, \sigma_N\}$ at time t .

In the following example, we illustrate the matrix of transition probability functions of a two state regime-switching process.

Example 2.1: We consider a regime-switching with state space $U = \{H, L\}$ in this example. Let λ_j be a rate to

jump from state i to state j . Then the transition probability function $P_{ij}(t)$ is given by

$$P_{ij}(t) = \frac{\lambda_j}{\lambda_j + \lambda_{ji}} - \frac{\lambda_j}{\lambda_j + \lambda_{ji}} e^{-(\lambda_j + \lambda_{ji})t},$$

where $i, j = \{H, L\}$ and $t \geq 0$. $P_{ij}(t)$ is obtained by solving the forward Kolmogorov equation. Since the derivation of $P_{ij}(t)$ is well known, we omit them here.

In this case the matrix of transition probability functions $P(t)$ has the form

$$P(t) = (P_{ij}(t)) = \begin{pmatrix} \frac{\lambda_{21}}{\lambda_{12} + \lambda_{21}} + \frac{\lambda_{12}}{\lambda_{12} + \lambda_{21}} e^{-(\lambda_{12} + \lambda_{21})t} & \frac{\lambda_{12}}{\lambda_{12} + \lambda_{21}} - \frac{\lambda_{12}}{\lambda_{12} + \lambda_{21}} e^{-(\lambda_{12} + \lambda_{21})t} \\ \frac{\lambda_{21}}{\lambda_{12} + \lambda_{21}} - \frac{\lambda_{21}}{\lambda_{12} + \lambda_{21}} e^{-(\lambda_{12} + \lambda_{21})t} & \frac{\lambda_{12}}{\lambda_{12} + \lambda_{21}} + \frac{\lambda_{21}}{\lambda_{12} + \lambda_{21}} e^{-(\lambda_{12} + \lambda_{21})t} \end{pmatrix},$$

where $\lambda_{ij} \geq 0$ and $t \geq 0$. Clearly, $P(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3. THE VALUATION OF A VARIANCE SWAP

A variance swap is a forward contract on the annualized variance. And a variance is the square of the volatility of the stock. The volatility is a measure for variation of the stock over time. The payoff of a variance swap at the maturity date T is equal to

$$A \cdot (\sigma_R^2(T) - K), \quad (1)$$

where $\sigma_R^2(T)$ is the realized stock variance (quoted in annual terms) over the life of the contract,

$$\sigma_R^2(T) = \frac{1}{T} \int_0^T \sigma^2(s) ds, \quad (2)$$

K is the prefixed variance strike and A is the vega notional amount of the swap per annualized volatility point squared. And we assume that the volatility $\sigma(t)$ changes according to a regime-switching process with a finite state space U .

The value of a forward contract V on future realized variance with strike price K is the expected present value of the future payoff in the risk-neutral world:

$$V(T) = e^{-rT} A \cdot E[\sigma_R^2(T) - K], \quad (3)$$

where $E[\cdot]$ is the expectation under risk-neutral measure. The following theorem gives us a valuation, $V(T)$.

Theorem 3.1: Assuming the current time is t , let $V_i(t, T; K, A)$ be the value of a variance swap in initial state i

with maturity T , fixed vega amount A and strike variance K , then it satisfies

$$V_i(t, T; K, A) = e^{-r(T-t)} \frac{A}{T-t} \left\{ \sum_{j \in U} \sigma_j^2 \int_t^T P_{ij}(\tau) d\tau - K \right\} \quad (4)$$

Proof: By the equation (2), the value of variance swap, equation (3) is transferred to

$$\begin{aligned} V_i(t, T; K, A) &= e^{-r(T-t)} A E \left[\frac{1}{T-t} \int_t^T \sigma^2(s) ds - K \mid \sigma(t) = \sigma_i \right] \\ &= e^{-r(T-t)} \frac{A}{T-t} E \left[\int_t^T \sigma^2(s) ds - K \mid \sigma(t) = \sigma_i \right] \\ &= e^{-r(T-t)} \frac{A}{T-t} \left\{ \int_t^T E[\sigma^2(s) \mid \sigma(t) = \sigma_i] ds - K \right\}, \end{aligned}$$

where the starting state of volatility is i -state, σ_i . Clearly, we could apply Fubini's theorem, we obtain the last equality of above equations.

Then we need to know only $E[\sigma^2(s) \mid \sigma(t) = \sigma_i]$ for gaining the equation (4), value of variance swap. Since we define the volatility $\sigma(t)$ as depending on the state of the Markov chain,

$$E[\sigma^2(s) \mid \sigma(t) = \sigma_i] = \sum_{j \in U} \sigma_j^2 P_{ij}(\tau),$$

where $\tau = s - t$.

Hence the value of variance swap $V_i(t, T; K, A)$ is

$$e^{-r(T-t)} \frac{A}{T-t} \left\{ \sum_{j \in U} \sigma_j^2 \int_t^T P_{ij}(\tau) d\tau - K \right\}. \quad \square$$

Theorem 3.1 says that a fair swap value can be computed by the matrix of transition probability of the Markov chain. For SVMs, the parameter estimation is known to be difficult. However, it is easy to obtain parameters of regime-switching model as described by Hardy (2001). Since the transition probability of the Markov chain is induced by the intensity of Poisson process, Theorem 3.1 shows that the value of a variance swap is easily calculated for comparison with the other SVMs.

In order to demonstrate the calculation of variance swaps, we present the following Table 1. To price a variance swap, it is enough that we calculate the conditional expectation of future variance. Table 1 present the expectations of variance in the different periods. Following the Example 2.1, we assume that there are two regime-states, high and low volatility states. And we use the parameters obtained by the maximum likelihood estimation (MLE, Hardy, 2001) for KOSPI 200 index during the 2008~2009 in Table 1. In other words, the parameters in columns 4~7 are obtained by the MLE method for the closing prices of KOSPI 200 index. And v_H, v_L in columns 8, 9 represent the conditional expectations of future variance,

$$v_i = E \left[\int_0^T \sigma^2(s) ds \mid \sigma(0) = \sigma_i \right], \quad i \in \{H, L\}$$

over time T . Here "H" and "L" mean a high-state, and a low-state, respectively.

Table 1. The Conditional Expectations of Variance in 2008-2009.

year	Term	No. of Obs.	Regime-Switching Environments					
			σ_H	σ_L	P_{LH}	P_{HL}	v_H	v_L
2008	1 yr.	266	68.20	24.29	16.23	51.32	2567	1249
	1 st half	121	19.10	17.41	4.75	24.96	179	167
	2 nd half	146	70.37	27.12	7.32	11.29	2044	648
2009	1 yr.	235	29.19	11.13	24.34	17.50	725	301
	1 st half	107	24.98	4.73	41.46	1.99	300	261
	2 nd half	129	18.68	15.60	1.83	42.24	130	124

Note) σ_H : high regime volatility, σ_L : low regime volatility, P_{LH} : transition probability from low state to high state, P_{HL} : transition probability from high state to low state, v_H : conditional expectation with initial high state, v_L : conditional expectation with initial low state, 1yr.-a 1-year period, 1st half -the first half of the year, 2nd half-the second half of the year.

4. CONCLUSION

In this paper, we have valued a variance swap when a volatility of underlying asset is modeled as a Markov regime switching process with a finite state. These results can be used to price a volatility swap under the same environment.

It has been well known that stochastic volatility models are useful. Many researchers have tried to apply SVMs to valuing derivatives. When SVM has been applied to a valuation problem, it remains difficult to have analytic formula for derivatives. Our regime-switching models reduce the complexity of computations and improve the accuracy close to the real volatility.

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