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A Risk-Averse Insider and Asset Pricing in Continuous Time

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ABSTRACT

This paper derives an equilibrium asset price when there exist three kinds of traders in financial market: a risk-averse informed trader, noise traders, and risk neutral market makers. This paper is an extended version of Kyle's (1985, Econometrica) continuous time model by introducing insider's risk aversion. We obtain not only the equilibrium asset pricing and market depth parameter but also insider's value function and optimal insider's trading strategy explicitly. The comparative static shows that the market depth (the reciprocal of market pressure) increases with time and volatility of noise traders' trading.

Keywords: Risk-Aversion, Informed Trader, Asset Pricing, Market Depth, Dynamic Programming Principle

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1. INTRODUCTION

Many kinds of traders who have different information exist in the financial market. Each trader has different motives for participating in asset trading and some traders who have monopoly power with their private information may crucially affect the equilibrium asset pricing. This feature exists in the most real financial markets such as equity and derivative markets. So it is important to include the asymmetric information structure in asset pricing and different kinds of traders should be incorporated in the model.

The literatures of asset pricing under asymmetric information have been developed from 1980s and now regard as one of the most important strand for understanding the equilibrium asset price. Kyle (1985) is a pioneer to model the sequential action under asymmetric information. His model consists of two stages. In the first stage, a single informed trader and noise traders submit their market order simultaneously and then the market makers set the equilibrium price given the market order. Since the market makers set the price after receiving total market order of informed trader and noise trader, the sequential model is called signaling model in which the market makers estimate the liquidation value of the stock from the market order. This model is compared with the screening model in which the market makers set the price first. This model is first considered by Glosten (1989), In that paper market makers offer the menu of contracts in the first stage. Then the informed trader chooses his demand function to maximize his profits.

The Kyle model is extended into various aspects. Back (1992) develops continuous time model in the general setting and Back and Pedersen (1998) includes long-lived information with time varying volatility of noise trading. The risk aversion of informed trader is included in Subrahmanyam (1991) and extended into discrete time setting by Holden and Subrahmanyam (1994). Baruch (2002) studies continuous time model with a single risk-averse informed trader. His model is an extended version of Back (1992) and Hoden and Subrahmanyam (1994).

The model developed in this paper is similar to Baruch (2002). However, we apply dynamic programming principle to derive the equilibrium outcomes so that the market depth, optimal demand function for informed trader, and his value function are obtained explicitly. The equilibrium outcomes derived in this paper give similar implications with previous studies and additional comparative statics are also given. In summary, when there is a single risk-averse informed trader, while the market depth increases as his risk aversion increases near the end of trading, the parameter decreases with risk aversion in the beginning of the trading. Moreover the market depth also increases with time and the volatility of the noise trading.

This paper is organized as follows. In section 2, we describe the financial market and the equilibrium definition and outcomes are given in section 3. Then the comparative statics results are represented in section 4. Section 5 concludes with a summary and future research.

2. THE FINANCIAL MARKET

We consider the continuous market microstructure in which there are three kinds of traders: a single risk-averse informed trader, noise traders, and risk neutral market makers. The informed trader has private information of the liquidation value of a stock so he trades because of the informational reason. The noise traders' trading motive is outside market and market makers set the equilibrium price to clear the market orders. All market participants trade or set market price continuously so that this model is neither signaling nor screening model.

There is one tractable stock in the market and let $\tilde{v}$ be the *Ex post* liquidation value of the risky asset, which is a normal random variable with mean p_0 and variance Σ. When the liquidation value $\tilde{v}$ realizes, the informed trader receives the realized full information v and he uses his informational power to maximize his expected utility (or expected profits when he is risk neutral). The noise traders' trading at time t is denoted by z_t which is assumed to evolve

$$dz_t = \sigma dB_t$$

where B_t is a standard Brownian motion and σ represents a constant volatility of noise trading. If we denote the informed trader's cumulative trading quantity and the equilibrium price at time t as x_t and p_t respectively, the informed trader's trading strategy is given by

$$dx_t = \beta_t(\tilde{v} - p_t)dt,$$

where β_t is an informed trader's sensitivity of trade. Thus the informed trader's remaining profits Π_t is given by

$$\Pi_t = \int_t^1 (v - p_s)dx_s = \int_t^1 \beta_s (v - p_s)^2 ds$$

So an informed trader who has CARA utility function with a coefficient of risk aversion γ is defined by

$$U(\Pi_t(x, p)) = -e^{-\gamma \int_t^1 d\Pi_s}.$$

There are many competitive risk neutral market makers in the market. They average zero profits and their role is to set the equilibrium asset price by the market efficient condition. Therefore, the sketch of trading progress is as follows: the informed trader and noise traders submit their market orders and market makers set the equilibrium asset price to clear the market orders after receiving all market order, which is called order flow ($dw_t = dx_t + dz_t$). Since our model is continuous time model, the price should also be contained in the informed trader's information and he can deduce the order flow from the market price.

While participating market, the informed trader's objective is to maximize his expected utility by using his private information and the market prices the market makers set. To make profits as high as possible, he should pretend not to have any information as noise traders. So he may distribute his private information during the trading period.

3. EQUILIBRIUM

The competition among risk neutral market makers induces that the equilibrium asset price equals to the market makers' optimal forecasting of the liquidation value of a stock from receiving the order flow, i.e.,

$$p_t = \mathbf{E}[\tilde{v} | w_s, 0 < s \le t]. \tag{1}$$

Therefore the equilibrium is defined by a pair (x, p), which satisfies the following two conditions:

i) *(profit maximization) given the equilibrium price p, the informed trader's strategy x maximizes*

$$\mathbf{E}[U(\Pi_t(x, p)) | \boldsymbol{F}_t]$$

where $\boldsymbol{F}_t$ is the σ-algebra generated by $\tilde{v} = v$ and $\{p_s : 0 < s \le t\}$

ii) *(market efficiency) given informed trader's trading strategy x, the equilibrium price rule is competitive so that it is set by (1)*

From (1), the pricing rule is given by

$$p_t = p_0 + \int_0^t \lambda_s dw_s,$$

where λ_t is some deterministic function which is defined as the market pressure in Kyle (1985). Therefore given $x_t = \int_0^t \beta_s(\tilde{v} - p_s)ds$, the linear pricing rule evolves

$$dp_t = \lambda_t dw_t = \lambda_t \beta_t (v - p_t)dt + \lambda_t \sigma dB_t. \tag{2}$$

The reciprocal of market pressure $1/\lambda_t$ is called 'market depth' since it represents the required order flow to change prices at a given amount. Since the liquidation value v and the volatility of noise trade volume σ are given exogenously, the equilibrium is equal to find a pair (β_t, λ_t) as a natural consequence. Compared to Baruch (2002), since he considers elastic noise demand function the variance of noise trading quantity is not constant any more. This is the main difference and main factor that makes it hard to obtain the informed trader's value function explicitly. We also adopt different approach to get the market pressure λ_t and the sensitivity β_t of the informed trader's strategy.

The informed trader's problem can be rewritten as subject to the pricing rule p_t given in (2) with two boundary conditions $V(1, p_0) = -1$ and $\int_0^1 \lambda_s^2\sigma^2 ds = \Sigma$.

$$V(t, P_t) = \max_{\beta_t} \mathbf{E}\left[-e^{-\gamma\int_t^1 d\Pi s}\middle|\boldsymbol{F}_t\right]$$
$$= \max_{\beta t} E\left[-e^{-\gamma\int (v-p_s)dx_s}\middle|\boldsymbol{F}_t\right] \quad (3)$$

These two boundary conditions are natural. The first condition is trivial condition from the definition of the utility function and the second condition means that the information is fully revealed after end of the trading. Since the monopolist risk-averse informed trader wants to maximize his expected utility function, it is optimal to use his informational advantage until the end of the trading as much as possible.

Then the informed trader's value function can be derived by the dynamic programming principle and we have the Hamilton-Jacobi-Bellman (HJB) equation in the next proposition.

Proposition 3.1: *The HJB equation of the informed trader's value function (3) is determined by*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial p^2}\lambda_t^2\sigma^2$$
$$+\max_{\beta_t}\left[\left\{\lambda_t(v-p_t)\frac{\partial V}{\partial p} - \gamma V(v-p_t)^2\right\}\beta_t\right] = 0,$$

with two conditions $V(1, p_0) = -1$ *and* $\int_0^1 \lambda_s^2\sigma^2 ds = \Sigma.$

Proof: Let β_t^* be an optimal value of the problem after some fixed time τ and define the informed trader's sensitivity as

$$\beta = \begin{cases} \beta s, & if\ t \le s \le \tau \\ \beta_t^*, & if\ \tau \le s \le 1, \end{cases}$$

Then the value function becomes

$$\max_{\beta t} \mathbf{E}\left[-e^{-\gamma\int_t^\tau (v-p_s)dx_s}\middle|\boldsymbol{F}_t\right]$$
$$= \max_{\beta t} \mathbf{E}\left[V(\tau, p_\tau)\cdot - e^{-\gamma\int_t^\tau (v-p_s)dx_s}\middle|\boldsymbol{F}_t\right]$$
$$= \max_{\beta t} \mathbf{E}[V(\tau, p_\tau)\cdot e^{-\gamma(H_\tau - H_t)}|\boldsymbol{F}_t]$$
$$= \max_{\beta t} \mathbf{E}[V(\tau, p_\tau)\cdot \phi_\tau|\boldsymbol{F}_t] \le V(t, p_t),$$

where $dH_t := (v-p_t)2\beta_t dt,\ \phi_\tau := e^{-\gamma(H\tau - Ht)}$.

If we define $q_\tau = V(\tau, p_\tau)\phi_\tau$, then the Ito's formula gives

$$dq_\tau = \phi_\tau dV_\tau + V_\tau d\phi_\tau + < d\phi_\tau,\ dV_\tau > \quad (4)$$

and

$$q_\tau - q_t = \int_t^\tau \phi_s dV_s + \int_t^\tau V_s d\phi_s. \quad (5)$$

The third term in (4) should be vanished because the function ϕ_t is deterministic. Furthermore, the differential form of the value function is given by

$$dV_s = \frac{\partial V}{\partial s}ds + \frac{\partial V}{\partial p}dp + \frac{1}{2}\frac{\partial^2 V}{\partial p^2}(\lambda_s\sigma)^2 ds$$
$$= \left\{\frac{\partial V}{\partial s} + \lambda_s\beta_s(v-p_s)\frac{\partial V}{\partial p} + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(\lambda_s\sigma)^2\right\}ds$$
$$+ \lambda_s\sigma\frac{\partial V}{\partial p}dB_s.$$

Therefore, we can rewrite the relation in (5) as

$$q_\tau = V_t + \int_t^\tau \phi_s\left\{\frac{\partial V}{\partial s} + \lambda_s\beta_s(v-p_s)\frac{\partial V}{\partial p} + \frac{1}{2}\frac{\partial^2 V}{\partial p^2}(\lambda_s\sigma)^2\right\}ds$$
$$+\int_t^\tau \phi_s\lambda_s\sigma\frac{\partial V}{\partial p}dB_s - \int_t^\tau \gamma V\phi_s(v-p_s)^2\beta_s ds.$$

If we take a conditional expectation on both sides with the filtration $\boldsymbol{F}_t$, the previous equation represented by

$$\mathbf{E}[q_\tau|\boldsymbol{F}_t] - V_t = \mathbf{E}\left[\int_t^\tau \phi s\{HJB\}ds\middle|\boldsymbol{F}_t\right],$$

and the dynamic programming principle gives the HJB equation. This completes the proof. ■

It is easily checked that the optimal sensitivity β_t^* of informed trader's strategy is arbitrary. This means there are many optima for the informed trader and this is the same results with Back (1992) in which the informed trader is risk neutral. Even though there are many optima, we can derive a unique equilibrium in this case. The optimal sensitivity parameter in HJB equation is derived from the informed trader's problem only but the market efficient condition, which is the second condition of equilibrium, gives other implications for the optimal parameters. In Proposition 3.1, the HJB equation is linear in β_t^* so we have two differential equations with boundary conditions. These equations determine the market pressure λ_t^* and the informed trader's value function.

Theorem 3.1: *The informed trader's value function which satisfies the HJB equation in (3) is obtained by*

$$V(t, p_t) = C_2\sqrt{\sigma_2\gamma t + C_1} \cdot e^{-\frac{1}{2}\sigma^2\gamma^2 v^2 t + \gamma(\sigma^2\gamma t + C_1)(vp_t - \frac{1}{2}p_t^2)}$$

where C_1 *and* C_2 *are given by*

$$C_1 = \frac{-\gamma\sigma^2 + \sigma^2\sqrt{\gamma^2 + 1/\Sigma^2}}{2},$$
$$C_2 = -\frac{1}{\sqrt{\sigma^2\gamma + C_1}} \cdot e^{-\frac{1}{2}C_1\gamma v^2}.$$

The price pressure parameter is also obtained from

$$\lambda_t = \frac{1}{\sigma^2\gamma t + C_1}. \tag{6}$$

Proof: The optimal β_t of the HJB equation in Proposition 3.1 implies the following differential equations whose solutions are the value function and market pressure λ_t.

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial p^2}\lambda_t^2\sigma^2 = 0 \\ \lambda_t(v - p_t)\dfrac{\partial V}{\partial p} - \gamma V(v - p_t)^2 = 0 \end{cases}$$

The second equation is an ordinary differential equation (ODE) which has a solution of the form

$$\mathrm{V}(\mathrm{t}, \mathrm{p_t}) = g(t)e^{\frac{\gamma}{\lambda_t}vpt\frac{1}{2}\frac{\gamma}{\lambda_t}p_t^2},$$

where g(t) is a deterministic function. If we substitute this $\mathrm{V}(\mathrm{t}, \mathrm{p_t})$ into the second equation, the following system of ODE is derived

$$\begin{cases} \dfrac{1}{2}\sigma^2\gamma^2 - \dfrac{1}{2}\gamma\left(\dfrac{1}{\lambda_t}\right)' = 0 \\ \gamma\left(\dfrac{1}{\lambda_t}\right)' v - \sigma^2\gamma^2 v = 0 \\ g'(t) - \dfrac{1}{2}\sigma^2\gamma(\lambda_t - \gamma v^2)g(t) = 0, \end{cases} \tag{7}$$

The first and second equations in (6) are same equations which determine the market pressure parameter explicitly as (6). From the third ODE, the deterministic function g(t) can be induced by

$$\mathrm{g(t)} = \mathrm{C_2}\left|\sigma^2\gamma t + C_1\right|^{\frac{1}{2}} \cdot e^{-\frac{1}{2}\sigma^2\gamma^2 v^2 t},$$

where C_2 is also an arbitrary constant. Then the coefficients C_1 and C_2 are solutions to the equations induced by the two conditions stated in Proposition 3.1

$$C_2\sqrt{\sigma^2\gamma + C_1} \cdot e^{\frac{1}{2}\gamma C_1 v^2} = -1,$$
$$C_1^2 + \sigma^2\gamma C_1 - \frac{\sigma^2}{\Sigma} = 0.$$

The coefficient C_1 has two real solutions but the positivity of market pressure λ_1 at time 1 makes C_1 uniquely. ■

When the informed trader is risk neutral the market pressure is constant expressed by $\sqrt{\Sigma}/\sigma$. In fact, this value is the exact same with original Kyle's continuous time model. This means that informed trader's cost of trading is constant over the trading period so that he should use his private information at a constant rate.

Furthermore, the explicit market pressure actually has the similar form with that of Baruch (2002). He derives the system of differential equation for market pressure and conditional variance Σ_t of $\tilde{v}$ given the order flow until time t. Since he considers the elastic noise demand the volatility of noise trading is not constant any more. So the market pressure has different values corresponding to the form of elastic noise demand. It is easily checked that, however, the solution to ODE for market pressure has the same form of the market pressure (6) when the volatility of the noise trading is constant. In addition, he figures out that the market pressure's behavior, which is decreasing in time, is independent of the elasticity of the noise demand function and we also have the similar characteristic for the market depth. Contrast to time variable, the behavior of market pressure is not clear for the variation of risk aversion and volatility of noise trading. The more specific results are given in the next section.

The explicit value function can also give some interesting implications. The informed trader's expected profits at time t is deduced from the value function and the impact of informed trader's risk taking behavior on his profits can be analyzed. But it is hard to give explicit explanations because the variation of risk aversion affects the market pressure and the equilibrium price at the same time.

Although we have an explicit value function, the optimal strategy of informed trader is not determined uniquely as mentioned in Proposition 3.1. The market efficient condition of equilibrium conditions leads to another expression for the market pressure and it determines the informed trader's demand function uniquely.

Theorem 3.2: *When the monopolist insider has a CARA utility function with risk aversion γ, his instantaneous optimal demand is given by*

$$dx_t = \beta_t^*(v - p_t)dt,$$

where the optimal sensitivity is derived by

$$\beta_t^* = \frac{\sigma^2\gamma + C_1}{1-t}$$

Furthermore, the conditional variance Σ_t is also given by

$$\Sigma_t = \left(\frac{\sigma^2}{\sigma^2\gamma + C_1}\right) \cdot \left(\frac{1-t}{\sigma^2\gamma t + C_1}\right)$$

where the constant coefficient is determined in Theorem 3.1.

Proof: Since the conditional variance should have the same value with the volatility of the price (2) it can be calculated by

$$\Sigma_t = \int_t^1 \lambda_s^2 \sigma^2 ds = \int_t^1 \left(\frac{1}{\sigma^2\gamma t + C_1}\right)^2 \cdot \sigma^2 ds.$$

Moreover, the market efficient condition makes the equilibrium price be written as

$$p_t = \mathbf{E}\,[\tilde{v}|ws, 0 < s \le t] = p_0 + \int_0^t \lambda_s dw_s.$$

Here the market pressure should have the same value with the value derived from filtering theorem and it is determined from

$$\lambda_t = \frac{\beta_t \Sigma_t}{\sigma^2}.$$

Therefore we have the optimal sensitivity explicitly. ■

It can be easily verified that $\Sigma_0 = \Sigma$ and $\Sigma_1 = 0$. The first condition is natural, and the second condition means that it is optimal for the informed trader to use his private information fully just before the public announcement. This is because that the conditional variance Σ_t measures the information which has not been revealed in the price market makers set. The full information revelation by the end of the trading is the property which is independent of informed trader's risk averseness as Kyle (1985) and Baruch (2002).

The fact that the optimal sensitivity of informed trader's demand is increasing with time means the informational reveal speed is up prior to the public announcement. So the equilibrium price converges to the liquidation value of the stock and the convergence speed also increases with time. Moreover, since near the public announcement the market pressure decreases with informed trader's risk aversion the informed trader with higher risk aversion trades more near the end of the trading.

4. COMPARATIVE STATICS

The equilibrium outcomes obtained in previous section are affected by the risk aversion of the informed trader. To the asset pricing under asymmetric information, the informational advantage and informed trader's usage are the important factor to understanding the equilibrium price. When he is risk averse he should consider the risk taking behavior as well. In addition to pretending he is noise trader, he wants to maximize his expect utility but he is more cautious than risk neutral trader.

In this section, we examine the impact of risk aversion on the equilibrium parameters such as market pressure, informed trader's sensitivity, and conditional variance which is related to the unrevealed information. Especially we focus on the market depth which is defined as the ability of the market to absorb the trading quantities without large price impact. The reciprocal of the market depth is the market pressure λ_t and as mentioned, it represents the cost of trading to informed trader and the degree of information revelation of equilibrium price market makers set. In other words, the market depth represents the market characteristic related to the price.

Proposition 4.1: *Let the reciprocal of market pressure be market depth ($1/\lambda_t$). Then the following comparative results hold:*

(i) $1/\lambda_0$ *decreases as risk aversion* γ *increases;*
(ii) $1/\lambda_1$ *increases as risk aversion* γ *increases;*
(iii) $1/\lambda_t$ *increases as volatility of noise trading increases;*
(iv) $1/\lambda_t$ *increases with time.*

Proof: Omitted.

The first two results are intuitive. If the informed trader is more risk averse at the beginning of the trading, he may act more aggressively. Then his private information would be incorporate into the equilibrium price and the price becomes more volatile. On the other hand, as the informed trader takes more risk, the unveiled information reduces near the end of trading because of the intense release in the beginning.

The third relation is also intuitive. When the noise trading is more volatile, the fundamental volatility of the equilibrium price is also high. So the informed trader has more room to act strategically. The last result means that the ability to absorb the market order increases with time so that the informed trader trades more actively to get the more profits or expected utility.

The consequence of the informed trader's optimal sensitivity parameter β_t^* is automatically obtained. The risk aversion and time affect same way to the market depth. This is natural because the deeper market depth the more informed trader's trading volume.

5. CONCLUSIONS

The asset pricing model when the market consists of single risk-averse informed trader, noise traders and market makers is considered in continuous time. This paper is the extension version of Kyle (1985) by incorporating a risk averseness of an informed trader.

Compared to the original Kyle's continuous time model, our equilibrium results give some additional implications. The risk averseness of the monopolist informed trader induces the time varying market pressure and more trading volume near the end of the trading. This would be the similar to the results in Baruch (2002) but we apply the different approach to get the explicit solution and obtain the informed trader's value function and the optimal strategy explicitly, which are not derived in Baruch (2002). Furthermore, we have comparative statics results in the other aspects such as volatility of the noise trading.

This work can be extended into various directions. If there are many informed traders in the market, they should concern with the other informed traders' behavior. More strategic results may be induced. In addition, the market makers' trading motive can be changed into profit maximizing while they set prices. Then the loss of the noise traders is distributed to informed traders and market makers. This gives more realistic feature when financial institutions trade with some private information. Lastly, our model can also be applied to specific financial assets like derivatives when market participants have different informational motives.

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