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Noninformative priors for the ratio of parameters of two Maxwell distributions

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Abstract

We develop noninformative priors for a ratio of parameters of two Maxwell distributions which is used to check the equality of two Maxwell distributions. Specially, we focus on developing probability matching priors and Jeffreys' prior for objective Bayesian inferences. The probability matching priors, under which the probability of the Bayesian credible interval matches the frequentist probability asymptotically, are developed. The posterior propriety under the developed priors will be shown. Some simulations are performed for identifying the usefulness of proposed priors in objective Bayesian inference.

Keywords: Matching prior, Maxwell distribution, ratio of parameters, reference prior.

1. Introduction

Maxwell distribution is useful to study the relationship between the velocity of particle, temperature and mass. This distribution can be applied in the area of reliability, because the support of the distribution is non-negative. The statistical usage of Maxwell distribution was investigated by Breitenberger (1963) and Bingham and Mardia (1978).

In reliability area, Bekkera and Rouxa (2005) proposed maximum likelihood estimator, Bayes estimator and empirical Bayes estimator of reliability function. Their proposed estimators were compared through simulation study. Krishna and Malik (2009) compared Bayes estimator and maximum likelihood estimator of reliability function through simulation study.

We propose statistical inference for the ratio of two parameters of Maxwell distributions. There can be found a little study for this issue in statistical literature. The ratio of parameter has a meaning when one want to check the equality of two Maxwell distributions. Kang *et al.* (2012b) proposed likelihood based inference for the ratio of parameters. They developed signed log-likelihood and modified signed log-likelihood statistics for testing the ratio.

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In this study, we develop objective priors for the ratio of parameters. The use of subjective priors is difficult when there exists only a little prior information for the parameter. Even when the prior information is abundant, the use of the subjective priors is criticized about assumption of prior distribution or hyper parameters. These situations make us consider noninformative priors. For inference of the ratio of two parameters, the noninformative priors of the ratio of two normal variances were developed by Kim *et al.* (2009). Also Kang *et al.* (2012a) derived the noninformative priors of the ratio of two scale parameters in half logistic distributions.

Among the noninformative priors, there exist kinds of objective priors satisfying some objective criteria. Jeffreys' prior, reference prior and probability matching prior are typical objective priors. Our interest is concentrated on developing the probability matching priors.

Probability matching priors initiated by Welch and Peers (1963) and developed by many authors play an important role in objective Bayesian analysis. Datta and Mukerjee (2004) summarized probability matching priors completely in their monograph. They introduced and treated various probability matching priors in many statistical models. Following Datta and Mukerjee (2004), there are several probability matching priors which satisfy certain criteria like quantile, distribution function, highest posterior distribution, other credible regions and prediction. Among them, we want to confine our interest in developing probability matching priors of quantile.

The outline of this paper is as follows. In Section 2, we introduce brief idea of probability matching priors and develop probability matching priors of the ratio. In Section 3, we prove the posterior propriety under the proposed prior and compute marginal posterior density of parameter of interest. In Section 4, simulated frequentist coverage probabilities of the proposed prior are shown. Also, a real data example is given. Section 5 devotes some conclusions.

2. Probability matching priors

The probability density function (p.d.f.) of Maxwell distribution with parameter θ is given by

$$f(x|\theta) = \frac{4}{\sqrt{\pi}}\theta^{3/2}x^2\exp\{-\theta x^2\}, x>0, \theta>0.$$

Let $X_1, X_2, \cdots, X_m$ be a random sample of size m from Maxwell distribution with parameter θ_1 and $\mathbf{X} = (X_1, X_2, \cdots, X_m)$. Let $Y_1, Y_2, \cdots, Y_n$ be a random sample of size n from Maxwell distribution with parameter θ_2 and $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$. Assume that X_i and Y_j are independent each other. Denote $\mathbf{x} = (x_1, \cdots, x_m)$ as observations of $\mathbf{X}$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ as observations of $\mathbf{Y}$.

Then the likelihood function of θ_1 and θ_2 under observations $\mathbf{x}$ and $\mathbf{y}$ is

$$f(\theta_1, \theta_2|\mathbf{x}, \mathbf{y}) = \left(\frac{4}{\sqrt{\pi}}\right)^N \left(\prod_{i=1}^m x_i^2\right)\left(\prod_{j=1}^n y_j^2\right)\theta_1^{\frac{3m}{2}}\theta_2^{\frac{3n}{2}}\exp\left\{-\theta_1\sum_{i=1}^m x_i^2 - \theta_2\sum_{j=1}^n y_j^2\right\}, \quad (2.1)$$

where $N = m+n$. Letting $t_1 = \sum_{i=1}^m x_i^2$ and $t_2 = \sum_{j=1}^n y_j^2$. The log-likelihood function of θ_1 and θ_2 is given by

$$\mathcal{L}(\theta_1, \theta_2) = N\log(\frac{4}{\sqrt{\pi}}) + \frac{3m}{2}\log(\theta_1) + 2\sum_{i=1}^m \log(x_i) - \theta_1 t_1 + \frac{3n}{2}\log(\theta_2) + 2\sum_{j=1}^n \log(y_j) - \theta_2 t_2. \quad (2.2)$$

The information matrix of θ_1 and θ_2 is

$$I(\theta_1, \theta_2) = \begin{pmatrix} \frac{3m}{2\theta_1^2} & 0 \\ 0 & \frac{3n}{2\theta_2^2} \end{pmatrix}.$$

We want to develop noninformative priors for the ratio of two parameters for objective Bayesian inference. Let

$$\omega = (\omega_1, \omega_2)$$

where $\omega_1 = \theta_1/\theta_2$ is the parameter of interest and $\omega_2 = \theta_1^m \theta_2^n$ is a nuisance parameter.

Under this reparametrization, the log-likelihood function of ω is given by

$$\mathcal{L}(\omega) = \frac{3}{2} log(\omega_2) - t_1 \omega_1^{n/N} \omega_2^{1/N} - t_2 \omega_1^{-m/N} \omega_2^{1/N} + c, \tag{2.3}$$

where c is a constant which vanishes when one applies the derivative with respect to ω_1 or ω_2. Denote, for $i, j, k = 1, 2$,

$$l_i = \frac{\partial \mathcal{L}(\omega)}{\partial \omega_i}, l_{ij} = \frac{\partial^2 \mathcal{L}(\omega)}{\partial \omega_i \partial \omega_j} \text{ and } l_{ijk} = \frac{\partial \mathcal{L}(\omega)}{\partial \omega_i \partial \omega_j \partial \omega_k}.$$

And let, for $i, j, k = 1, 2$,

$$L_{ij} = E\left[l_{ij}\right], L_{ijk} = E\left[l_{ijk}\right], L_{i,jk} = E\left[l_i l_{jk}\right] \text{ and } L_{i,j,k} = E\left[l_i l_j l_k\right].$$

Then, we can have that

$$L_{11} = -\frac{2}{\sqrt{\pi}} \Gamma(\frac{5}{2}) \omega_1^{-2}, L_{22} = -(\frac{1}{N} + \frac{1}{2}) \omega_2^{-2}$$

and

$$L_{21} = L_{12} = 0.$$

Conclusively, under the above reparametrization, the information matrix of ω_1 and ω_2 is given by

$$I(\omega) = \begin{pmatrix} \frac{2}{\sqrt{\pi}} \Gamma(\frac{5}{2}) \omega_1^{-2} & 0 \\ 0 & (\frac{1}{N} + \frac{1}{2}) \omega_2^{-2} \end{pmatrix}.$$

So, we know that ω_1 is orthogonal to ω_2 in the sense of Cox and Reid (1987).

From the above information matrix, the Jeffreys' prior is given by

$$\pi_J(\omega_1, \omega_2) \propto \frac{1}{\omega_1 \omega_2}, \omega_1, \omega_2 > 0. \tag{2.4}$$

From now on, we develop probability matching priors of quantile when the parameter of interest is the ratio of parameters. The quantile matching prior is that the quantile of posterior distribution under quantile matching prior matches the frequentist probability in an asymptotic sense. Let $\omega_1(\alpha; \mathbf{X}, \mathbf{Y})$ be the α quatile of the posterior under the prior π. Then the probability matching prior satisfies the following equation.

$$P^\pi\left(\omega_1 \leq \omega_1(\alpha; \mathbf{X}, \mathbf{Y}) | \mathbf{X}, \mathbf{Y}\right) = P_\omega\left(\omega_1 \leq \omega_1(\alpha; \mathbf{X}, \mathbf{Y}) | \omega\right) = \alpha + o(n^{-k}), \tag{2.5}$$

where P^{π} is a probability measure on posterior distribution under prior π and P_{ω} is a frequentist's probability measure. When π satisfies the equation (2.5) with $k = 1/2$, π is called as the first order probability matching prior. And the second order probability matching prior satisfies the equation (2.5) with the order $k = 1$.

The first order probability matching priors of quantile are characterized by

$$\pi_1(\omega_1, \omega_2) \propto \omega_1^{-1} d(\omega_2), \omega_1, \omega_2 > 0, \tag{2.6}$$

where $d(\cdot)$ is an arbitrary function which is differentiable in its arguments. If we take $d(\omega_2) = 1/\omega_2$, then we can easily verify that Jeffreys' prior (2.4) is the first order probability matching prior.

Among the first order probability matching priors, the second order probability matching priors satisfy the following additional differential equation about $d(\omega_2)$,

$$\frac{1}{6} d(\omega_2) \frac{\partial}{\partial \omega_1} \left\{ I_{11}^{-3/2} L_{1,1,1} \right\} + \frac{\partial}{\partial \omega_2} \left\{ I_{11}^{-1/2} L_{112} I^{22} d(\omega_2) \right\} = 0, \tag{2.7}$$

where I_{ij} is the (i,j)-th element of information matrix and I^{ij} is the (i,j)-th element of the inverse of information matrix.

In our case, $I_{11} = c_0 \omega_1^{-2}$ and $I_{22} = c_1 \omega_2^{-2}$, and we have found that

$$L_{1,1,1} = c_2 \omega_1^{-3},$$

and

$$L_{112} = c_3 \omega_1^{-2} \omega_2^{-1},$$

where c's with subscript i, c_i, for all integer i, is a constant that does not depend on any parameters.

Now, the first term in (2.7) is

$$I_{11}^{-3/2} L_{1,1,1} = \left(c_0 \omega_1^{-2} \right)^{-3/2} c_2 \omega_1^{-3} = c_4,$$

This fact leads the first term in (2.7) to 0. The second term in (2.7) is

$$I_{11}^{-1/2} L_{112} I^{22} = \left(c_0 \omega_1^{-2} \right)^{-1/2} c_3 \omega_1^{-2} \omega_2^{-1} c_1^{-1} \omega_2^2 = c_5 \omega_1^{-1} \omega_2.$$

So, the equation (2.7) reduces to

$$\frac{\partial}{\partial \omega_2} \left\{ \omega_1^{-1} \omega_2 d(\omega_2) \right\} = 0.$$

A solution of the above equation is

$$d(\omega_2) = \omega_2^{-1}.$$

Hence the second order probability matching prior is given by

$$\pi_2(\omega_1, \omega_2) \propto \frac{1}{\omega_1 \omega_2}, \omega_1, \omega_2 > 0. \tag{2.8}$$

Actually, the Jeffreys' prior is the second order probability matching prior.

When the parameter of interest is ω_1, the reference prior of Berger and Bernardo (1989, 1992) is given by

$$\pi_R(\omega) \propto \frac{1}{\omega_1 \omega_2}, \omega_1, \omega_2 > 0.$$

It can be obtained easily by the orthogonality of parameters and the work of Datta and Ghosh (1995).

3. Posterior propriety under the proposed prior

This section devotes the propriety of the posterior distribution and finds the marginal posterior distribution of the parameter of interest in an explicit form. Since the proposed prior is improper, one must prove the propriety of posterior under improper prior. We will consider the general form of the following prior.

$$\pi_g(\omega) \propto \omega_1^{-1}\omega_2^{-a}, \omega_1 > 0, \omega_2 > 0, a \geq 0. \tag{3.1}$$

When $a = 0$, $\pi_g(\omega)$ is the first order matching prior given in (2.6) and when $a = 1$, it is the prior which satisfies the second order matching criterion.

Theorem 3.1 Under the prior (3.1), the joint posterior probability density function of ω_1 and ω_2 is given by

$$p(\omega_1, \omega_2|t_1, t_2) = \frac{\omega_1^{-1}\omega_2^{\frac{3}{2}-a}e^{-\omega_2^{\frac{1}{N}}(\omega_1^{\frac{n}{N}}t_1+\omega_2^{-\frac{m}{N}}t_2)}t_1^{m(\frac{5}{2}-a)}t_2^{n(\frac{5}{2}-a)}}{N\Gamma\left(m(\frac{5}{2}-a)\right)\Gamma\left(n(\frac{5}{2}-a)\right)}, \omega_1, \omega_2 > 0, \tag{3.2}$$

and is proper, if $5/2 > a$.

Proof: When we combine the prior (3.1) and the likelihood $e^{\mathcal{L}(\omega)}$, the joint posterior is proportional to

$$p(\omega|t_1, t_2) \propto \omega_1^{-1}\omega_2^{\frac{3}{2}-a}e^{-\omega_2^{\frac{1}{N}}(\omega_1^{\frac{n}{N}}t_1+\omega_1^{-\frac{m}{N}}t_2)}.$$

The normalizing constant of the above joint posterior can be obtained as follows. An integration of joint posterior with respect to ω_2 results in

$$\begin{aligned}\int_0^\infty\int_0^\infty p(\omega_1, \omega_2|t_1, t_2)d\omega_1 d\omega_2 &= N \times \Gamma\left(N(\frac{5}{2}-a)\right)\\ &\int_0^\infty \omega_1^{-1}\left[\omega_1^{\frac{n}{N}}t_1 + \omega_1^{-\frac{m}{N}}t_2\right]^{-N(\frac{5}{2}-a)}d\omega_1.\end{aligned}$$

Letting

$$z = \frac{\omega_1}{\omega_1 + t_2/t_1},$$

the integration with respect to ω_1 changes to Beta function. And this completes the proof. □

The marginal posterior probability density function for the parameter of interest, ω_1, is given by

$$p(\omega_1|t_1, t_2) = \frac{(t_1^m t_2^n)^{\frac{5}{2}-a}\Gamma\left(N(\frac{5}{2}-a)\right)}{\Gamma\left(m(\frac{5}{2}-a)\right)\Gamma\left(n(\frac{5}{2}-a)\right)}\omega_1^{-1}\left[\omega_1^{\frac{n}{N}}t_1 + \omega_1^{-\frac{m}{N}}t_2\right]^{-N(\frac{5}{2}-a)}, \omega_1 > 0. \tag{3.3}$$

Based on the equation (3.3), one can make Bayesian inference for ω_1.

4. Simulation and data analysis

4.1. Simulation

We want to evaluate our results by observing the frequentist coverage probability of credible intervals based on priors π_1 and π_2.

Let $\omega_1(\alpha; \mathbf{X}, \mathbf{Y})$ be the posterior α-quantile of ω_1 given $\mathbf{X}$ and $\mathbf{Y}$. Then this quantile must satisfies (2.5), asymptotically. Since the event $\{\omega_1 < \omega_1(\alpha; \mathbf{X}, \mathbf{Y})\}$ is the same as $F^{\pi}(\omega_1) < F^{\pi}(\omega_1(\alpha; \mathbf{X}, \mathbf{Y})) = \alpha$, where $F^{\pi}(\cdot)$ is a distribution function of the marginal posterior distribution of ω_1 which can be obtained by (3.3). Our results can be justified if the relative frequency of $F^{\pi}(\omega_1) < \alpha$ is close to α. This relative frequency is supplied in Table 4.1 with some selected values of θ_1, θ_2, m and n. In this table, π_1 denotes the first order probability matching prior and π_2 denotes the second order probability matching prior. The relative frequency is computed with $\alpha = 0.05, 0.95$ and we perform 10,000 replications.

In this Table 4.1, we can observe that the second probability matching prior π_2 meets the nominal level 0.05(0.95) even in small sample size, as we expected. But, the first order probability matching prior does not match the nominal level. This result does not depend on the values of θ_1 and θ_2. Conclusively, π_2 performs well.

Table 4.1 The estimated coverage probabilities of posterior quantiles

θ_1	θ_2	ω_1	m	n	π_1		π_2	
					0.05	0.95	0.05	0.95
0.1	0.1	1	2	2	0.0835	0.9172	0.0540	0.9479
			3	5	0.0794	0.9250	0.0496	0.9483
			5	3	0.0763	0.9206	0.0490	0.9488
			10	10	0.0776	0.9259	0.0513	0.9537
0.1	1	0.1	2	2	0.0767	0.9190	0.0492	0.9487
			3	5	0.0775	0.9249	0.0495	0.9501
			5	3	0.0766	0.9225	0.0515	0.9531
			10	10	0.0780	0.9245	0.0501	0.9546
0.1	2	0.05	2	2	0.0750	0.9198	0.0467	0.9503
			3	5	0.0848	0.9202	0.0530	0.9495
			5	3	0.0794	0.9234	0.0546	0.9508
			10	10	0.0806	0.9201	0.0523	0.9507
1	0.1	10	2	2	0.0782	0.9144	0.0509	0.9463
			3	5	0.0748	0.9287	0.0467	0.9530
			5	3	0.0763	0.9199	0.0493	0.9487
			10	10	0.0794	0.9248	0.0539	0.9481
1	1	1	2	2	0.0767	0.9246	0.0501	0.9508
			3	5	0.0831	0.9229	0.0522	0.9502
			5	3	0.0722	0.9199	0.0509	0.9492
			10	10	0.0749	0.9234	0.0486	0.9521
1	2	0.5	2	2	0.0776	0.9219	0.0509	0.9515
			3	5	0.0807	0.9235	0.0498	0.9469
			5	3	0.0772	0.9175	0.0508	0.9490
			10	10	0.0782	0.9218	0.0490	0.9516
2	0.1	20	2	2	0.0803	0.9200	0.0501	0.9502
			3	5	0.0784	0.9190	0.0481	0.9471
			5	3	0.0802	0.9234	0.0513	0.9486
			10	10	0.0821	0.9234	0.0523	0.9495
2	1	2	2	2	0.0801	0.9229	0.0509	0.9520
			3	5	0.0822	0.9220	0.0505	0.9501
			5	3	0.0771	0.9183	0.0489	0.9477
			10	10	0.0779	0.9232	0.0505	0.9505
2	2	1	2	2	0.0798	0.9195	0.0504	0.9470
			3	5	0.0793	0.9236	0.0504	0.9512
			5	3	0.0754	0.9178	0.0488	0.9469
			10	10	0.0814	0.9246	0.0506	0.9516

4.2. Data analysis

The following data analyzed in Kazmi *et al.* (2012) is the burning velocity (cm/sec) of 55 chemical materials. They analyzed this data assuming Maxwell distribution. We divide these data into two groups randomly. Then our parameter of interest, ω_1 will be close to 1. The data is given below :

30	38	40	42	44	45	46	46	46	50	51	56	56	58	60
61	61	64	66	67	68	68	71	80	82	87	89	166		
31	40	40	41	41	41	43	44	44	46	46	46	46	46	47
47	48	48	48	52	54	54	55	58	82	108	312			

In this data, $m = 28$, $n = 27$, $t_1 = 125196$ and $t_2 = 167492$. As a note, the maximum likelihood estimate of parameter of interest is $\widehat{\omega}_1 = \frac{\widehat{\theta}_1}{\widehat{\theta}_2} = \frac{mt_2}{nt_1} = 1.387388$ and using the developed priors π_1 and π_2, the Bayes estimates under the squared error loss function are given by $\widetilde{\omega}_{\pi_1} = 1.408251$ and $\widetilde{\omega}_{\pi_2} = 1.422512$, respectively. It seems that the point estimates do not differ seriously.

Based on the study of Kang *et al.* (2012b), the 95% confidence interval for ω_1 based on signed log-likelihood statistic is given by (0.900, 2.141) and based on modified signed log-likelihood statistic is given by (0.899, 2.145). The 95% Bayesian credible interval for ω_1 under the prior π_1 is (0.992, 1.942) and Bayesian credible interval for ω_1 under the prior π_2 is (0.899, 2.145) which is the same interval as the modified signed log-likelihood statistic. All the intervals in the above contain the value $\omega_1 = 1$ as we expected. The length of credible interval based on π_1 is shorter than that of π_2. But we have seen in Table 1 that the coverage probability based on π_1 didn't match the nominal level.

Conclusively, in terms of coverage probability, the use of π_2 is recommended.

5. Concluding remarks

In this study, we developed noninformative priors for the ratio of parameters of two Maxwell distributions. We suggested Jeffreys' prior, reference prior and probability matching prior when the parameter of interest is the ratio of two parameters. We proved that except the first order probability matching prior, the rest of priors are identical and also gave the propriety of posterior. Through simulation and data analysis, we recommend the second order matching prior for Bayesian inference in this study.

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