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HILBERT 2-CLASS FIELD TOWERS OF IMAGINARY QUADRATIC FUNCTION FIELDS

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Abstract. In this paper we study the infiniteness of Hilbert 2-class field towers of imaginary quadratic function fields over $\mathbb{F}_q(T)$, where q is a power of an odd prime number.

1. Introduction and statement of the results

For a number field F, let $F_0^{(2)} = F$ and $F_{n+1}^{(2)}$ be the Hilbert 2-class field of $F_n^{(2)}$ for $n \geq 0$. Then the sequence of fields

$$F = F_0^{(2)} \subset F_1^{(2)} \subset \cdots \subset F_n^{(2)} \subset \cdots$$

is called the *Hilbert* 2*-class field tower* of F and we say that F has an infinite Hilbert 2-class field tower if $F_n^{(2)} \neq F_{n+1}^{(2)}$ for all $n \geq 0$. Assume that F is an imaginary quadratic number field. Let $r_2(\mathcal{C}l_F)$ denote the 2-rank of class group $\mathcal{C}l_F$ of F. By Golod-Shafarevich's Theorem, F has an infinite Hilbert 2-class field tower if $r_2(\mathcal{C}l_F) \geq 5$. It has been conjectured by Martinet [6] that the Hilbert 2-class field tower of F is infinite if $r_2(\mathcal{C}l_F) \geq 4$. Let $r_4(\mathcal{C}l_F) = r_2(\mathcal{C}l_F^2)$ be the 4-rank of $\mathcal{C}l_F$. It has been shown by Koch [5] and Hajir [3, 4] that F has an infinite Hilbert 2-class field tower if $r_4(\mathcal{C}l_F) \geq 3$. In [2], Gerth has proved that a positive proportion of the imaginary quadratic number fields F with $r_2(\mathcal{C}l_F) = r$ have infinite Hilbert 2-class field towers and $r_4(\mathcal{C}l_F) = s$ for $r = 3, 1 \leq s \leq 3$ and for $r = 4, 0 \leq s \leq 4$.

Let $k = \mathbb{F}_q(T)$ be a rational function field over the finite field $\mathbb{F}_q$, $\infty = (1/T)$ and $\mathbb{A} = \mathbb{F}_q[T]$. For a finite separable extension F of k, write $\mathcal{O}_F$ for the integral closure of $\mathbb{A}$ in F and H_F for the Hilbert class field of F with respect to $\mathcal{O}_F$ (see [7]). Let ℓ be a prime number. Let $F_0^{(\ell)} = F$ and $F_{n+1}^{(\ell)}$ be the Hilbert ℓ-class field of $F_n^{(\ell)}$ for $n \geq 0$, i.e., $F_{n+1}^{(\ell)}$ is the maximal extension of $F_n^{(\ell)}$ inside

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$H_{F_n^{(\ell)}}$ whose degree over $F_n^{(\ell)}$ is a power of ℓ. The sequence of fields

$$F = F_0^{(\ell)} \subset F_1^{(\ell)} \subset \cdots \subset F_n^{(\ell)} \subset \cdots$$

is called the *Hilbert ℓ-class field tower of F*. We say that the Hilbert ℓ-class field tower of F is infinite if $F_n^{(\ell)} \neq F_{n+1}^{(\ell)}$ for each $n \geq 0$. Let $\mathcal{C}l_F$ and $\mathcal{O}_F^*$ be the ideal class group and the group of units of $\mathcal{O}_F$, respectively. For any multiplicative abelian group A, write $r_\ell(A) = \dim_{\mathbb{F}_\ell}(A/A^\ell)$ for the ℓ-rank of A. The following theorem is a function field analog of Golod-Shafarevich due to Schoof.

Theorem 1.1 (Schoof [9]). *Let F be a finite separable extension of k. Then the Hilbert ℓ-class field tower of F is infinite if*

$$r_\ell(\mathcal{C}l_F) \geq 2 + 2\sqrt{r_\ell(\mathcal{O}_F^*) + 1}.$$

Assume that q is odd. Let F be an *imaginary quadratic function field* over k, i.e., F is a quadratic extension of k in which ∞ is ramified. Fix a generator γ of $\mathbb{F}_q^*$. Let $\mathcal{P}$ be the set of monic irreducible polynomials in $\mathbb{A}$. Then F can be written as $F = k(\sqrt{D})$ with $D = \gamma^a P_1 \cdots P_t$, $a \in \{0, 1\}$, $P_i \in \mathcal{P}$ for $1 \leq i \leq t$ and $2 \nmid \deg D$. Here, D is uniquely determined by F and write $D_F = D$. Let s_F be the number of monic irreducible divisors P_i of D_F of odd degree. Since $\deg D_F$ is odd, s_F is a positive odd integer. We will assume that $\deg P_i$ is odd for $1 \leq i \leq s_F$ and $\deg P_i$ is even for $s_F + 1 \leq i \leq t$. For $0 \neq N \in \mathbb{A}$, write $\omega(N)$ for the number of monic irreducible divisors of N. By genus theory ([1, Corollary 3.5]), we have $r_2(\mathcal{C}l_F) = \omega(D_F) - 1$. Since $\mathcal{O}_F^* = \mathbb{F}_q^*$ and $r_2(\mathcal{O}_F^*) = 1$, by Theorem 1.1, we see that F has an infinite Hilbert 2-class field tower if $\omega(D_F) \geq 6$. The following theorem is a function field analogue of Koch and Hajir's one.

Theorem 1.2. *Let F be an imaginary quadratic function field over k. Let s_F be the number of distinct monic irreducible divisors of D_F of odd degree. If $r_4(\mathcal{C}l_F) \geq 3$, then the Hilbert 2-class field tower of F is infinite, except the cases that $q \equiv 1 \bmod 4$, $\omega(D_F) = 4$ and $s_F = 3$. In this exceptional case, if D_F has monic irreducible divisors P and Q such that $\deg P$ is divisible by 4 and $(\frac{P}{Q})_4 = 1$, then the Hilbert 2-class field tower of F is infinite, where $(\frac{*}{*})_4$ is the 4-th power residue symbol.*

For any positive integers n, r and integer s with $0 \leq s \leq r$, let $X_{r;n}$ be the set of imaginary quadratic function fields F with $r_2(\mathcal{C}l_F) = r$ and $\deg(D_F) = n$, $X_{r,s;n}$ be the subset of $X_{r;n}$ consisting of $F \in X_{r;n}$ with $r_4(\mathcal{C}l_F) = s$ and $X_{r,s;n}^*$ be the subset of $X_{r,s;n}$ consisting of $F \in X_{r,s;n}$ having an infinite Hilbert 2-class field tower. We define a density $\delta_{r,s}^*$ by

$$\delta_{r,s}^* = \liminf_{\substack{n\to\infty \\ n:\text{odd}}} \frac{|X_{r,s;n}^*|}{|X_{r;n}|}.$$

Then we have:

Theorem 1.3. *$\delta^*_{3,s} \geq 2^{-9}$ for $1 \leq s \leq 3$ and $\delta^*_{4,s} \geq 2^{-14}$ for $0 \leq s \leq 4$.*

Remark 1.4. Theorem 1.3 means that a positive proportion of imaginary quadratic function fields F with $r_2(\mathcal{C}l_F) = r$ have infinite Hilbert 2-class field towers and $r_4(\mathcal{C}l_F) = s$ for $r = 3, 1 \leq s \leq 3$ and for $r = 4, 0 \leq s \leq 4$.

2. Preliminaries

2.1. Martinet's inequality

Let E and K be finite geometric separable extensions of k such that E/K is a cyclic extension of degree ℓ with $\Delta = Gal(E/K)$, where ℓ is a prime number not dividing q. Let $\mathcal{O}_E$ be the integral closure of $\mathbb{A}$ in E and $\mathcal{O}_E^*$ be the group of units of $\mathcal{O}_E$. Then $H^0(\Delta, \mathcal{O}_E^*)$ and $H^1(\Delta, \mathcal{O}_E^*)$ are elementary abelian ℓ-groups with

$$\frac{|H^0(\Delta, \mathcal{O}_E^*)|}{|H^1(\Delta, \mathcal{O}_E^*)|} = \ell^{-1} \prod_{\mathfrak{p}_\infty \in S_\infty(K)} |\Delta_{\mathfrak{p}_\infty}|,$$

where $S_\infty(K)$ is the set of primes of K lying above ∞ and $\Delta_{\mathfrak{p}_\infty}$ denotes the decomposition group of $\mathfrak{p}_\infty$ in Δ. Following the arguments in [6, §2], we get the following proposition.

Proposition 2.1. *Let E/K be as above. Let $\gamma_{E/K}$ be the number of prime ideals of $\mathcal{O}_K$ that ramify in E and $\rho_{E/K}$ be the number of primes $\mathfrak{p}_\infty$ in $S_\infty(K)$ that ramify or inert in E. Then the Hilbert ℓ-class field tower of E is infinite if*

$$\gamma_{E/K} \geq |S_\infty(K)| - \rho_{E/K} + 3 + 2\sqrt{\ell|S_\infty(K)| + (1-\ell)\rho_{E/K} + 1}. \tag{2.1}$$

The inequality (2.1) is called the *Martinet's inequality.* Now, by using Proposition 2.1, we give some sufficient conditions for an imaginary quadratic function field F to have infinite Hilbert 2-class field tower. Let $(\frac{*}{*})$ denote the quadratic residue symbol in $\mathbb{A}$.

Corollary 2.2. *Let F be an imaginary quadratic function field over k. If there exists a nonconstant divisor D' of D_F such that either D' or D_F/D' is monic of even degree and $(\frac{D'}{P_i}) = 1$ for monic irreducible divisors P_i $(1 \leq i \leq 4)$ of D_F, then F has an infinite Hilbert 2-class field tower.*

Proof. Let $K = k(\sqrt{D'})$ and $E = KF$. Since either D' or D_F/D' is monic of even degree, the infinite prime $\mathfrak{p}_\infty$ of F splits in E. It is easy to see that any finite primes of F is unramified in E. Hence E is contained in $F_1^{(2)}$. Since P_1, P_2, P_3 and P_4 split in K, we have $\gamma_{E/K} \geq 8$. We also have $|S_\infty(K)| = \rho_{E/K} = 2$ or $(|S_\infty(K)|, \rho_{E/K}) = (1, 0)$ according as D' is monic of even degree or D_F/D' is monic of even degree. By Proposition 2.1 on E/K, we see that E has an infinite Hilbert 2-class field tower. Since E is contained in $F_1^{(2)}$, F also has an infinite Hilbert 2-class field tower. □

Corollary 2.3. *Let F be an imaginary quadratic function field over k. If D_F has two distinct nonconstant monic divisors D_1 and D_2 of even degrees satisfying $(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ for monic irreducible divisors P_i $(i = 1, 2)$ of D_F, then F has an infinite Hilbert 2-class field tower.*

Proof. Let $K = k(\sqrt{D_1}, \sqrt{D_2})$ and $E = KF$. Since ∞ splits completely in K, the infinite prime $\mathfrak{p}_\infty$ of F splits in E. It is easy to see that any finite primes of F is unramified in E. Hence E is contained in $F_1^{(2)}$. By applying Proposition 2.1 on E/K with $\gamma_{E/K} \geq 8$ and $|S_\infty(K)| = \rho_{E/K} = 4$, we see that E has an infinite Hilbert 2-class field tower. Since E is contained in $F_1^{(2)}$, F also has an infinite Hilbert 2-class field tower. □

Corollary 2.4. *Let F be an imaginary quadratic function field over k. If D_F has two distinct nonconstant monic divisors D_1 and D_2 of even degrees satisfying $(\frac{D_1}{P_1}) = (\frac{D_2}{P_1}) = 1$ for monic irreducible divisor P_1 of D_F and D_F has two monic irreducible divisors P_2, P_3 which are different from P_1 and $P_i \nmid D_1D_2$ $(i = 2, 3)$, then F has an infinite Hilbert 2-class field tower.*

Proof. Let $K = k(\sqrt{D_1}, \sqrt{D_2})$ and $E = KF$. As in the proof of Corollary 2.3, we can show that E is contained in $F_1^{(2)}$. Since $(\frac{D_1}{P_1}) = (\frac{D_2}{P_1}) = 1$, P_1 splits completely in K. Each P_i $(i = 2, 3)$ splits in at least one of $k(\sqrt{D_1}), k(\sqrt{D_2})$ and $k(\sqrt{D_1D_2})$. Hence we have $\gamma_{E/K} \geq 8$ and $|S_\infty(K)| = \rho_{E/K} = 4$. Then by applying Proposition 2.1, E has an infinite Hilbert 2-class field tower. Since E is contained in $F_1^{(2)}$, F also has an infinite Hilbert 2-class field tower. □

2.2. Rédei-matrix and 4-rank of class group

Let F be an imaginary quadratic function field over k with $D_F = \gamma^a P_1 \cdots P_t$. Let $d_i \in \mathbb{F}_2$ be defined by $d_i \equiv \deg P_i \bmod 2$ for $1 \leq i \leq t$. We will separate two cases:

- (CASE A) $q \equiv 1 \bmod 4$ with $a = 0$ or $q \equiv 3 \bmod 4$ with $a = 1$,
- (CASE B) $q \equiv 1 \bmod 4$ with $a = 1$ or $q \equiv 3 \bmod 4$ with $a = 0$.

We associate a $t \times t$ matrix $R_F = (e_{ij})_{1 \leq i,j \leq t}$ over $\mathbb{F}_2$ to F, where $e_{ij} \in \mathbb{F}_2$ is defined by $(-1)^{e_{ij}} = (\frac{P_i}{P_j})$ for $1 \leq i \neq j \leq t$ and the diagonal entries $e_{ii} \in \mathbb{F}_2$ are defined to satisfy the relation

$$\sum_{j=1}^{t} e_{ij} = \begin{cases} 0 & (\text{CASE A}), \\ d_i & (\text{CASE B}). \end{cases}$$

Let s_F be the number of monic irreducible divisors P_i of D_F of odd degree. Since $\deg D_F$ is odd, s_F is an odd integer. We will assume that $\deg P_i$ is odd for $1 \leq i \leq s_F$. If $q \equiv 3 \bmod 4$ and $1 \leq j \leq s_F$, we have

$$\sum_{i=1}^{t} e_{ij} = e_{jj} + \sum_{1 \leq i \leq s_F, i \neq j} (e_{ji} + 1) + \sum_{i=s_F+1}^{t} e_{ji} = \sum_{i=1}^{t} e_{ji}$$

since s_F is odd. Otherwise, since $e_{ij} = e_{ji}$, we trivially have the same equality. Hence, the diagonal entries e_{ii} of R_F also satisfy the relation

$$\sum_{i=1}^{t} e_{ij} = \begin{cases} 0 & (\text{CASE A}), \\ d_j & (\text{CASE B}). \end{cases}$$

Proposition 2.5. *For an imaginary quadratic function field F over k, we have*

$$r_4(\mathcal{C}l_F) = \begin{cases} t - 1 - \operatorname{rank} R_F & (\text{CASE A}), \\ t - \operatorname{rank} R_F & (\text{CASE B}). \end{cases}$$

Proof. Let M_F be the $(t+1) \times (t+1)$ matrix over $\mathbb{F}_2$ given by

$$M_F = \begin{pmatrix} e_{11} & \cdots & e_{1t} & (a+\epsilon)d_1 \\ \vdots & & \vdots & \vdots \\ e_{t1} & \cdots & e_{tt} & (a+\epsilon)d_t \\ d_1 & \cdots & d_t & 1 \end{pmatrix},$$

where $\epsilon = 0$ if $q \equiv 1 \bmod 4$, $\epsilon = 1$ if $q \equiv 3 \bmod 4$ and the entries $e_{ii} \in \mathbb{F}_2$ are defined to satisfy $(a+\epsilon)d_i + \sum_{j=1}^{t} e_{ij} = 0$. Then $r_4(\mathcal{C}l_F)$ satisfies the following equality ([1, Corollary 3.8]):

$$r_4(\mathcal{C}l_F) = t - \operatorname{rank} M_F. \tag{2.2}$$

In (CASE A), we have $a + \epsilon = 0$ in $\mathbb{F}_2$. Hence we can see that $\operatorname{rank} M_F = \operatorname{rank} R_F + 1$ and the entries e_{ii} satisfy $\sum_{j=1}^{t} e_{ij} = 0$. By (2.2), we have $r_4(\mathcal{C}l_F) = t - 1 - \operatorname{rank} R_F$.

Now, we consider (CASE B). Then $a + \epsilon = 1$ in $\mathbb{F}_2$, so

$$M_F = \begin{pmatrix} e_{11} & \cdots & e_{1t} & d_1 \\ \vdots & & \vdots & \vdots \\ e_{t1} & \cdots & e_{tt} & d_t \\ d_1 & \cdots & d_t & 1 \end{pmatrix}$$

and the entries e_{ii} satisfy $d_i + \sum_{j=1}^{t} e_{ij} = 0$. By adding first t columns to the last column on M_F, we can see that $\operatorname{rank} M_F = \operatorname{rank} M'_F$, where

$$M'_F = \begin{pmatrix} e_{11} & \cdots & e_{1t} \\ \vdots & & \vdots \\ e_{t1} & \cdots & e_{tt} \\ d_1 & \cdots & d_t \end{pmatrix}.$$

By adding first t rows to the last row on M'_F, we can see that $\operatorname{rank} M'_F = \operatorname{rank} R_F$, so $r_4(\mathcal{C}l_F) = t - \operatorname{rank} R_F$ by (2.2). □

2.3. Some asymptotic results

Let $\mathcal{P}$ be the set of all monic irreducible polynomials in $\mathbb{A}$. For positive integers n and t, we write $\mathcal{P}(n)$ for the subset of $\mathcal{P}$ consisting of $P \in \mathcal{P}$ with $\deg P = n$, $\mathcal{P}(n,t)$ for the set of monic square free polynomials $N \in \mathbb{A}$ with $\deg N = n$ and $\omega(N) = t$ and $\mathcal{P}'(n,t)$ for the subset of $\mathcal{P}(n,t)$ consisting of $N = P_1 \cdots P_t \in \mathcal{P}(n,t)$ such that $\deg(P_i) \neq \deg(P_j)$ for $i \neq j$. As $n \to \infty$, we have

$$|\mathcal{P}(n,t)| = \frac{q^n (\log n)^{t-1}}{(t-1)!n} + O\Big(\frac{q^n(\log n)^{t-2}}{n}\Big), \tag{2.3}$$

$$|\mathcal{P}(n,t) \setminus \mathcal{P}'(n,t)| = o\Big(\frac{q^n(\log n)^{t-1}}{n}\Big). \tag{2.4}$$

The following two lemmas are due to Wittmann ([10, Lemmas 3.3 and 3.5]).

Lemma 2.6. *For $P_1, \ldots, P_u \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_u \in \{\pm 1\}$, as $n \to \infty$, we have*

$$\sum_{\substack{P \in \mathcal{P}(n) \\ 1 \le i \le u : (P_i/P) = \varepsilon_i}} 1 = 2^{-u}\frac{q^n}{n} + O\Big(\frac{q^{n/2}}{n}\Big).$$

Lemma 2.7. *Let n be a positive integer and $d_1, \ldots, d_t \in \{0,1\}$ satisfying $\sum_{i=1}^t d_i \equiv n \bmod 2$. Then, as $n \to \infty$, we have*

$$\sum_{\substack{0<n_1<\cdots<n_t \\ n_1 \equiv d_1(2), \ldots, n_t \equiv d_t(2) \\ n_1+\cdots+n_t = n}} \frac{1}{n_1 \cdots n_t} = 2^{-(t-1)}\frac{(\log n)^{t-1}}{(t-1)!n} + O\Big(\frac{(\log n)^{t-2}}{n}\Big).$$

For $d_1, \ldots, d_t \in \{0,1\}$ and $\varepsilon_{ij} \in \{\pm 1\}$ for $1 \le i < j \le t$, let

$$J_n(\{d_i\};\{\varepsilon_{ij}\}) = \sum_{\substack{0<n_1<\cdots<n_t \\ n_1 \equiv d_1(2), \ldots, n_t \equiv d_t(2) \\ n_1+\cdots+n_t = n}} \sum_{P_1 \in \mathcal{P}(n_1)} \sum_{\substack{P_2 \in \mathcal{P}(n_2) \\ (P_1/P_2) = \varepsilon_{12}}} \cdots \sum_{\substack{P_t \in \mathcal{P}(n_t) \\ \forall i<t : (P_i/P_t) = \varepsilon_{it}}} 1.$$

Lemma 2.8. *As $n \to \infty$, we have*

$$J_n(\{d_i\};\{\varepsilon_{ij}\}) = 2^{1-\frac{t^2+t}{2}}\frac{q^n(\log n)^{t-1}}{(t-1)!n} + O\Big(\frac{q^n(\log n)^{t-2}}{n}\Big).$$

Proof. Let $J = J_n(\{d_i\};\{\varepsilon_{ij}\})$. As $n \to \infty$, using Lemmas 2.6 and 2.7, we get

$$\begin{aligned} J &= \sum_{\substack{0<n_1<\cdots<n_t \\ n_1 \equiv d_1(2), \ldots, n_t \equiv d_t(2) \\ n_1+\cdots+n_t = n}} \prod_{i=1}^t \Big(2^{-(i-1)}\frac{q^{n_i}}{n_i} + O\Big(\frac{q^{n_i/2}}{n_i}\Big)\Big) \\ &= 2^{-\frac{(t^2-t)}{2}} q^n \sum_{\substack{0<n_1<\cdots<n_t \\ n_1 \equiv d_1(2), \ldots, n_t \equiv d_t(2) \\ n_1+\cdots+n_t = n}} \frac{1}{n_1 \cdots n_t} + O\Big(q^n \sum_{i=1}^t \sum_{\substack{0<n_1<\cdots<n_t \\ n_1+\cdots+n_t=n}} \frac{q^{-n_i/2}}{n_1 \cdots n_t}\Big) \\ &= 2^{1-\frac{t^2+t}{2}}\frac{q^n(\log n)^{t-1}}{(t-1)!n} + O\Big(\frac{q^n(\log n)^{t-2}}{n}\Big). \end{aligned}$$

□

For $N = P_1 \cdots P_t, N' = P'_1 \cdots P'_t \in \mathcal{P}'(n,t)$, we say that N and N' are equivalent if $\deg(P_i) \equiv \deg(P'_i) \bmod 2$ for $1 \le i \le t$ and $(\frac{P_i}{P_j}) = (\frac{P'_i}{P'_j})$ for $1 \le i < j \le t$. Write $\mathcal{N}(N)$ for the set of polynomials in $\mathcal{P}'(n,t)$ which are equivalent to N.

Proposition 2.9. *For any $N \in \mathcal{P}'(n,t)$, as $n \to \infty$, we have*

$$|\mathcal{N}(N)| = 2^{1-\frac{(t^2+t)}{2}} \frac{q^n(\log n)^{t-1}}{(t-1)!n} + O\Big(\frac{q^n(\log n)^{t-2}}{n}\Big).$$

Proof. Let $N = P_1 \cdots P_t \in \mathcal{P}'(n,t)$. Then we have $|\mathcal{N}(N)| = J_n(\{d_i\};\{\varepsilon_{ij}\})$, where $d_1,\ldots,d_t \in \{0,1\}$ satisfying $\deg P_i \equiv d_i \bmod 2$ for $1 \le i \le t$ and $\varepsilon_{ij} = (\frac{P_i}{P_j})$ for $1 \le i < j \le t$. Now, the result follows immediately from Lemma 2.8. □

3. Proof of Theorem 1.2

Let F be an imaginary quadratic function field with $D_F = \gamma^a P_1 \cdots P_t$. By Theorem 1.1, the Hilbert 2-class field tower of F is infinite if $r_2(\mathcal{C}l_F) \ge 5$. Hence, it remains to consider the cases $(r_2(\mathcal{C}l_F), r_4(\mathcal{C}l_F)) = (3,3), (4,3)$ or $(4,4)$. Recall that s_F denotes the number of monic irreducible divisors P_i of D_F of odd degree. Since $\deg D_F$ is odd, s_F is an odd integer. We will assume that $\deg P_i$ is odd for $1 \le i \le s_F$. Write $\vec{R}_i$ for the i-th row vector of R_F and $\vec{0}$ for the zero row vector. In (CASE B), we always have $\vec{R}_i \ne \vec{0}$ for $1 \le i \le s_F$ since $e_{i1} + \cdots + e_{it} = 1$. In the following proof, we will consider the cases that $\operatorname{rank} R_F = 0, 1$ or 2.

- If $\operatorname{rank} R_F = 0$, then $R_F = O$, so we have $(\frac{P_i}{P_j}) = 1$ for all $1 \le i \ne j \le t$.
- If $\operatorname{rank} R_F = 1$, then any nonzero row of R_F forms a basis for the row space of R_F. Especially, $\{\vec{R}_1\}$ is always a basis for the row space of R_F in (CASE B).
- If $\operatorname{rank} R_F = 2$, then any two distinct nonzero rows of R_F forms a basis for the row space of R_F.

3.1. Case $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F) = 3$ with $D_F = \gamma^a P_1P_2P_3P_4$

We first consider (CASE A). By Proposition 2.5, we have $\operatorname{rank} R_F = 0$, i.e., $R_F = O$. If $s_F = 1$, we have $(\frac{P_3}{P_i}) = (\frac{P_4}{P_i}) = 1$ for $i \in \{1,2\}$, so F has an infinite Hilbert 2-class field tower by Corollary 2.3. If $s_F = 3$, we have $q \equiv 1 \bmod 4$ since $(\frac{P_1}{P_2}) = (\frac{P_2}{P_1}) = 1$. In this case, we suppose that $\deg P_4$ is divisible by 4 and $(\frac{P_4}{P_1})_4 = 1$. Then ∞ and P_1 splits completely in $K = k(\sqrt[4]{P_4})$. Since $(\frac{P_4}{P_2}) = (\frac{P_4}{P_3}) = 1$, P_2 and P_3 split in $k(\sqrt{P_4})$. Put $E = KF$. Then E is contained in $F_1^{(2)}$. By applying Proposition 2.1 on E/K with $\gamma_{E/K} \ge 8$ and $|S_\infty(K)| = \rho_{E/K} = 4$, we see that E has an infinite Hilbert 2-class field tower. Since E is contained in $F_1^{(2)}$, F also has an infinite Hilbert 2-class field tower.

Now, we consider (CASE B). By Proposition 2.5, we have rank $R_F = 1$. Since $\vec{R}_1 + \vec{R}_2 + \vec{R}_3 + \vec{R}_4 = \vec{e}$, where

$$\vec{e} = \begin{cases} (1\ 0\ 0\ 0) & \text{if } s_F = 1, \\ (1\ 1\ 1\ 0) & \text{if } s_F = 3, \end{cases}$$

we have $\vec{R}_i \in \{\vec{0}, \vec{e}\}$ for $1 \le i \le 4$. If $s_F = 1$, we have $\vec{R}_1 = \vec{e}$ and $\vec{R}_2 = \vec{R}_3 = \vec{R}_4 = \vec{0}$. Then we have $(\frac{P_3}{P_i}) = (\frac{P_4}{P_i}) = 1$ for $i \in \{1, 2\}$, so F has an infinite Hilbert 2-class field tower by Corollary 2.3. If $s_F = 3$, we have $\vec{R}_1 = \vec{R}_2 = \vec{R}_3 = \vec{e}$ and $\vec{R}_4 = 0$. We also have $q \equiv 1 \bmod 4$ since $(\frac{P_1}{P_2}) = (\frac{P_2}{P_1})$. In this case, as in above (CASE A), we can show that F also has an infinite Hilbert 2-class field tower.

3.2. Case $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F)) = 4$ with $D_F = \gamma^a P_1 P_2 P_3 P_4 P_5$

First consider (CASE A). By Proposition 2.5, we have rank $R_F = 0$, i.e., $R_F = O$. If $s_F \le 3$, then $(\frac{P_5}{P_i}) = 1$ for $1 \le i \le 4$, so F has an infinite Hilbert 2-class field tower by Corollary 2.2. If $s_F = 5$, for $D_1 = P_1P_2$ and $D_2 = P_1P_3$, we have $(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ for $i \in \{4, 5\}$, so F has an infinite Hilbert 2-class field tower by Corollary 2.3.

Consider (CASE B). By Proposition 2.5, we have rank $R_F = 1$. Since $\{\vec{R}_1\}$ is a basis for the row space of R_F and $\vec{R}_1 + \vec{R}_2 + \vec{R}_3 + \vec{R}_4 + \vec{R}_5 = \vec{f}$, where

$$\vec{f} = \begin{cases} (1\ 0\ 0\ 0\ 0) & \text{if } s_F = 1, \\ (1\ 1\ 1\ 0\ 0) & \text{if } s_F = 3, \\ (1\ 1\ 1\ 1\ 1) & \text{if } s_F = 5, \end{cases}$$

we have $\vec{R}_1 = \vec{f}$. If $s_F \le 3$, we have $\vec{R}_5 = \vec{0}$ since $e_{51} = e_{15} = 0$. Then $(\frac{P_5}{P_i}) = 1$ for $1 \le i \le 4$, so so F has an infinite Hilbert 2-class field tower by Corollary 2.2. If $s_F = 5$, we have $\vec{R}_i = \vec{f}$ for $1 \le i \le 5$. For $D_1 = P_1P_2$ and $D_2 = P_1P_3$, we have $(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ for $i \in \{4, 5\}$, so F has an infinite Hilbert 2-class field tower by Corollary 2.3.

3.3. Case $r_2(\mathcal{C}l_F) = 4$ and $r_4(\mathcal{C}l_F) = 3$ with $D_F = \gamma^a P_1 P_2 P_3 P_4 P_5$

First consider (CASE A). By Proposition 2.5, we have rank $R_F = 1$. Assume $s_F = 1$. If $\vec{R}_i = \vec{0}$ for some $2 \le i \le 5$, say $\vec{R}_5 = \vec{0}$, then $(\frac{P_5}{P_i}) = 1$ for $1 \le i \le 4$, so F has an infinite Hilbert 2-class field tower by Corollary 2.2. Otherwise, we have $\vec{R}_2 = \vec{R}_3 = \vec{R}_4 = \vec{R}_5 \ne \vec{0}$. For $D_1 = P_3P_4$ and $D_2 = P_3P_5$, we have $(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ for $i \in \{1, 2\}$, so F has an infinite Hilbert 2-class field tower by Corollary 2.3.

Assume $s_F = 3$. If $\vec{R}_4 = \vec{0}$ or $\vec{R}_5 = \vec{0}$, say $\vec{R}_5 = \vec{0}$, then $(\frac{P_5}{P_i}) = 1$ for $1 \le i \le 4$, so F has an infinite Hilbert 2-class field tower by Corollary 2.2. We may assume $\vec{R}_4 = \vec{R}_5 \ne \vec{0}$. If $\vec{R}_1 = \vec{R}_2 = \vec{R}_3$, for $D_1 = P_1P_2$ and $D_2 = P_1P_3$, we have $(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ for $i \in \{4,5\}$, so F has an infinite Hilbert 2-class field tower by Corollary 2.3. We may assume $\vec{R}_1 \neq \vec{R}_2 = \vec{R}_3$. Since $\vec{R}_1 + \vec{R}_2 + \vec{R}_3 + \vec{R}_4 + \vec{R}_5 = \vec{0}$, we have $\vec{R}_1 = \vec{0}$ and $\vec{R}_2 = \vec{R}_3 = \vec{R}_4 = \vec{R}_5 \neq \vec{0}$. Then we have $q \equiv 1 \bmod 4$ and $a = 0$. Since D_F/P_1 is monic of even degree and $(\frac{P_1}{P_i}) = 1$ for $2 \leq i \leq 5$, F has an infinite Hilbert 2-class field tower by Corollary 2.2.

If $s_F = 5$, then at least three rows of R_F are equal, say $\vec{R}_3 = \vec{R}_4 = \vec{R}_5$. For $D_1 = P_3P_4$ and $D_2 = P_3P_5$, we have $(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ for $i \in \{1,2\}$, so F has an infinite Hilbert 2-class field tower by Corollary 2.3.

Now we consider (CASE B). In this case we have $\operatorname{rank} R_F = 2$ by Proposition 2.5. Assume $s_F = 1$. If $\vec{R}_i = \vec{0}$ for some $2 \leq i \leq 5$, say $\vec{R}_5 = \vec{0}$, then $(\frac{P_5}{P_i}) = 1$ for $1 \leq i \leq 4$, so F has an infinite Hilbert 2-class field tower by Corollary 2.2. We may assume $\vec{R}_i \neq \vec{0}$ for $2 \leq i \leq 5$. It only need to consider the case that either $\{\vec{R}_1, \vec{f}\}$ or $\{\vec{R}_5, \vec{f}\}$ is a basis of the row space of R_F. In any case, since $\vec{R}_i \neq \vec{0}$ and $e_{i1} + e_{i2} + e_{i3} + e_{i4} + e_{i5} = 0$ for $2 \leq i \leq 5$, we have $\vec{R}_2 = \vec{R}_3 = \vec{R}_4 = \vec{R}_5$. For $D_1 = P_3P_4$ and $D_2 = P_3P_5$, we have $(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ for $i \in \{1,2\}$, so F has an infinite Hilbert 2-class field tower by Corollary 2.3.

Assume $s_F = 3$. If $\vec{R}_4 = \vec{0}$ or $\vec{R}_5 = \vec{0}$, say $\vec{R}_5 = \vec{0}$, then $(\frac{P_5}{P_i}) = 1$ for $1 \leq i \leq 4$, so F has an infinite Hilbert 2-class field tower by Corollary 2.2. We may assume $\vec{R}_i \neq \vec{0}$ for $i \in \{4,5\}$. Then $\{\vec{R}_1, \vec{R}_4\}$ forms a basis of the row space of R_F, so we have $\vec{R}_5 = \vec{R}_4$ and $\vec{R}_i \in \{\vec{R}_1, \vec{R}_1 + \vec{R}_4\}$ for $i \in \{2,3\}$. If $\vec{R}_1 = \vec{R}_2 = \vec{R}_3$, then, for $D_1 = P_1P_2$ and $D_2 = P_1P_3$, we have $(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ for $i \in \{4,5\}$, so F has an infinite Hilbert 2-class field tower by Corollary 2.3. By changing the role of P_1 and P_3 if it is necessary, we are reduced to the case $\vec{R}_1 = \vec{R}_2 \neq \vec{R}_3$. Since $\vec{R}_1 + \vec{R}_2 + \vec{R}_3 + \vec{R}_4 + \vec{R}_5 = \vec{f}$, we have $\vec{R}_3 = \vec{f}$ and $\vec{R}_1 = \vec{R}_2 = \vec{f} + \vec{R}_4$. By using the polynomial quadratic reciprocity law, we can see that $q \equiv 1 \bmod 4$ and $\vec{R}_4 = \vec{R}_5 = (1\ 1\ 0\ 1\ 1)$. Then $(\frac{P_4}{P_3}) = (\frac{P_5}{P_3}) = 1$, so F has an infinite Hilbert 2-class field tower by Corollary 2.4.

If $s_F = 5$, since $\vec{R}_i \neq \vec{0}$ and $e_{i1} + e_{i2} + e_{i3} + e_{i4} + e_{i5} = 1$ for $1 \leq i \leq 5$, at least three of rows of R_F are equal, say $\vec{R}_3 = \vec{R}_4 = \vec{R}_5$. For $D_1 = P_3P_4$ and $D_2 = P_3P_5$, we have $(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ for $i \in \{1,2\}$, so F has an infinite Hilbert 2-class field tower by Corollary 2.3.

4. Proof of Theorem 1.3

Let r, s be integers with $0 \leq s \leq r$ and n be a positive odd integer. We have

$$X_{r;n} = \left\{ k(\sqrt{\gamma^a N}) : a \in \{0,1\} \text{ and } N \in \mathcal{P}(n, r+1) \right\}.$$

Let $\bar{X}_{r;n}$ be the subset of $X_{r;n}$ consisting of $k(\sqrt{\gamma^a N}) \in X_{r;n}$ with $N \in \mathcal{P}'(n, r+1)$ and $\bar{X}^*_{r,s;n} = \bar{X}_{r;n} \cap X^*_{r,s;n}$. Then, by (2.4), we have

$$\delta^*_{r,s} = \liminf_{\substack{n\to\infty \\ n:\text{odd}}} \frac{|\bar{X}^*_{r,s;n}|}{|\bar{X}_{r;n}|}. \tag{4.1}$$

By (2.3) and (2.4), we have

$$(4.2)\qquad |\bar{X}_{r;n}| = \frac{2q^n(\log n)^r}{r!n} + O\Big(\frac{q^n(\log n)^{r-1}}{n}\Big) \quad \text{as } n\to\infty.$$

For any $N\in\mathcal{P}'(n,r+1)$, let

$$\mathcal{S}_a(N) = \Big\{k(\sqrt{\gamma^a N'}) : N'\in\mathcal{N}(N)\Big\} \quad (a=0,1).$$

Then $\mathcal{S}_0(N)\cup\mathcal{S}_1(N)$ is a subset of $\bar{X}_{r;n}$ and by Proposition 2.9, we have (4.3)

$$|\mathcal{S}_0(N)| = |\mathcal{S}_1(N)| = 2^{-\frac{r(r+3)}{2}}\frac{q^n(\log n)^r}{r!n} + O\Big(\frac{q^n(\log n)^{r-1}}{n}\Big) \quad \text{as } n\to\infty.$$

By (4.2) and (4.3), we have

$$(4.4)\qquad \lim_{\substack{n\to\infty\\ n:\text{odd}}}\frac{|\mathcal{S}_0(N)|}{|\bar{X}_{r;n}|} = \lim_{\substack{n\to\infty\\ n:\text{odd}}}\frac{|\mathcal{S}_1(N)|}{|\bar{X}_{r;n}|} = 2^{-\frac{(r+1)(r+2)}{2}}.$$

4.1. $\delta^*_{3,s}$

Consider $F_a = k(\sqrt{\gamma^a N})$ $(a=0,1)$, where $N=P_1P_2P_3P_4\in\mathcal{P}'(n,4)$ such that $2|\deg P_i$ for $1\le i\le 3$, $2\nmid\deg P_4$ and $(\frac{P_j}{P_i})=1$ for $i=1,2$ and $j=3,4$. Then $r_2(\mathcal{C}l_{F_a})=3$ and F_a has an infinite Hilbert 2-class field tower by Lemma 2.3. Moreover, every fields in $\mathcal{S}_a(N)$ also have infinite Hilbert 2-class field towers.

- *Case* $(\frac{P_2}{P_1}) = (\frac{P_4}{P_3}) = -1$. We have

$$R_{F_0} = \begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&0\end{pmatrix},\quad R_{F_1} = \begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$$

with $\operatorname{rank} R_{F_0}=3$ and $\operatorname{rank} R_{F_1}=2$, so $r_4(\mathcal{C}l_{F_0}) = r_4(\mathcal{C}l_{F_1}) = 1$. Hence $F_0,F_1\in\bar{X}^*_{3,1;n}$ and $\mathcal{S}_0(N)\cup\mathcal{S}_1(N)\subset\bar{X}^*_{3,1;n}$, so we have

$$\delta^*_{3,1} \ge \lim_{\substack{n\to\infty\\ n:\text{odd}}}\frac{|\mathcal{S}_0(N)|}{|\bar{X}_{3;n}|} + \lim_{\substack{n\to\infty\\ n:\text{odd}}}\frac{|\mathcal{S}_1(N)|}{|\bar{X}_{3;n}|} = 2^{-9}.$$

- *Case* $(\frac{P_2}{P_1}) = -1$ *and* $(\frac{P_4}{P_3}) = 1$. We have

$$R_{F_0} = \begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\quad R_{F_1} = \begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

with $\operatorname{rank} R_{F_0}=2$ and $\operatorname{rank} R_{F_0}=1$, so $r_4(\mathcal{C}l_{F_0}) = r_4(\mathcal{C}l_{F_1}) = 2$. Hence $F_0,F_1\in\bar{X}^*_{3,2;n}$ and $\mathcal{S}_0(N)\cup\mathcal{S}_1(N)\subset\bar{X}^*_{3,2;n}$, so we have

$$\delta^*_{3,2} \ge \lim_{\substack{n\to\infty\\ n:\text{odd}}}\frac{|\mathcal{S}_0(N)|}{|\bar{X}_{3;n}|} + \lim_{\substack{n\to\infty\\ n:\text{odd}}}\frac{|\mathcal{S}_1(N)|}{|\bar{X}_{3;n}|} = 2^{-9}.$$

• *Case* $(\frac{P_2}{P_1}) = (\frac{P_4}{P_3}) = 1$. We have

$$R_{F_0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_{F_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\operatorname{rank} R_{F_0} = 1$ and $\operatorname{rank} R_{F_1} = 0$, so $r_4(\mathcal{C}l_{F_0}) = r_4(\mathcal{C}l_{F_1}) = 3$. Hence $F_0, F_1 \in \bar{X}^*_{3,3;n}$ and $\mathcal{S}_0(N) \cup \mathcal{S}_1(N) \subset \bar{X}^*_{3,3;n}$, so we have

$$\delta^*_{3,3} \geq \lim_{\substack{n\to\infty \\ n:\text{odd}}} \frac{|\mathcal{S}_0(N)|}{|\bar{X}_{3;n}|} + \lim_{\substack{n\to\infty \\ n:\text{odd}}} \frac{|\mathcal{S}_1(N)|}{|\bar{X}_{3;n}|} = 2^{-9}.$$

4.2. $\delta^*_{4,s}$

Now consider $F_a = k(\sqrt{\gamma^a N})$ $(a = 0, 1)$, where $N = P_1P_2P_3P_4P_5 \in \mathcal{P}'(n, 5)$ such that $2|\deg P_i$ for $1 \leq i \leq 4$, $2 \nmid \deg P_5$ and $(\frac{P_j}{P_i}) = 1$ for $1 \leq i < j \leq 4$. Then $r_2(\mathcal{C}l_{F_a}) = 4$ and F_a has an infinite Hilbert 2-class field tower by Lemma 2.3, so every field in the set $\mathcal{S}_a(N)$ also has an infinite Hilbert 2-class field tower.

• *Case* $(\frac{P_5}{P_i}) = -1$ *for* $1 \leq i \leq 4$. We have

$$R_{F_0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad R_{F_0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

with $\operatorname{rank} R_{F_0} = 5$ and $\operatorname{rank} R_{F_1} = 4$, so $r_4(\mathcal{C}l_{F_0}) = r_4(\mathcal{C}l_{F_1}) = 0$. Hence $F_0, F_1 \in \bar{X}^*_{4,0;n}$ and $\mathcal{S}_0(N) \cup \mathcal{S}_1(N) \subset \bar{X}^*_{4,0;n}$, so we have

$$\delta^*_{4,0} \geq \lim_{\substack{n\to\infty \\ n:\text{odd}}} \frac{|\mathcal{S}_0(N)|}{|\bar{X}_{4;n}|} + \lim_{\substack{n\to\infty \\ n:\text{odd}}} \frac{|\mathcal{S}_1(N)|}{|\bar{X}_{4;n}|} = 2^{-14}.$$

• *Case* $(\frac{P_5}{P_i}) = 1$ *for* $1 \leq i \leq s < 4$ *and* $(\frac{P_5}{P_i}) = -1$ *for* $s+1 \leq i \leq 4$. We have

$$R_{F_0} = \begin{pmatrix} O_1 & O_2 \\ O_3 & M_0 \end{pmatrix}, \quad R_{F_1} = \begin{pmatrix} O_1 & O_2 \\ O_3 & M_1 \end{pmatrix}$$

where O_1 is the $s \times s$ zero matrix, O_2 is the $s \times (5-s)$ zero matrix, O_3 is the $(5-s) \times s$ zero matrix and M_0, M_1 are the $(5-s) \times (5-s)$ matrices given by

$$M_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & s+1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & s \end{pmatrix}.$$

Since $\operatorname{rank} R_{F_0} = \operatorname{rank} M_0 = 5 - s, \operatorname{rank} R_{F_1} = \operatorname{rank} M_1 = 4 - s$, we have $r_4(F_0) = r_4(F_1) = s$, $F_0, F_1 \in \bar{X}^*_{4,s;n}$ and $\mathcal{S}_0(N) \cup \mathcal{S}_1(N) \subset \bar{X}^*_{4,s;n}$. Hence

$$\delta^*_{4,s} \geq \lim_{\substack{n\to\infty \\ n:\mathrm{odd}}} \frac{|\mathcal{S}_0(N)|}{|\bar{X}_{4;n}|} + \lim_{\substack{n\to\infty \\ n:\mathrm{odd}}} \frac{|\mathcal{S}_1(N)|}{|\bar{X}_{4;n}|} = 2^{-14}.$$

- *Case* $(\frac{P_5}{P_i}) = 1$ *for* $1 \leq i \leq 4$. We have

$$R_{F_0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_{F_0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\operatorname{rank} R_{F_0} = 1$ and $\operatorname{rank} R_{F_1} = 0$, so $r_4(\mathcal{C}l_{F_0}) = r_4(\mathcal{C}l_{F_1}) = 4$. Hence $F_0, F_1 \in \bar{X}^*_{4,4;n}$ and $\mathcal{S}_0(N) \cup \mathcal{S}_1(N) \subset \bar{X}^*_{4,4;n}$, so we have

$$\delta^*_{4,4} \geq \lim_{\substack{n\to\infty \\ n:\mathrm{odd}}} \frac{|\mathcal{S}_0(N)|}{|\bar{X}_{4;n}|} + \lim_{\substack{n\to\infty \\ n:\mathrm{odd}}} \frac{|\mathcal{S}_1(N)|}{|\bar{X}_{4;n}|} = 2^{-14}.$$

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