

## A SEMI-RIEMANNIAN MANIFOLD OF QUASI-CONSTANT CURVATURE ADMITS SOME HALF LIGHTLIKE SUBMANIFOLDS

DAE HO JIN

**ABSTRACT.** In this paper, we study the curvature of a semi-Riemannian manifold  $\widetilde{M}$  of quasi-constant curvature admits some half lightlike submanifolds  $M$ . The main result is two characterization theorems for  $\widetilde{M}$  admits extended screen homothetic and statical half lightlike submanifolds  $M$  such that the curvature vector field of  $\widetilde{M}$  is tangent to  $M$ .

### 1. Introduction

Chen-Yano [1] introduced the notion of a *Riemannian manifold of quasi-constant curvature* as a Riemannian manifold  $(\widetilde{M}, \widetilde{g})$  equipped with the curvature tensor  $\widetilde{R}$  satisfying the following condition:

$$(1.1) \quad \begin{aligned} \widetilde{g}(\widetilde{R}(X, Y)Z, W) = & \alpha\{\widetilde{g}(Y, Z)\widetilde{g}(X, W) - \widetilde{g}(X, Z)\widetilde{g}(Y, W)\} \\ & + \beta\{\widetilde{g}(X, W)\theta(Y)\theta(Z) - \widetilde{g}(X, Z)\theta(Y)\theta(W) \\ & + \widetilde{g}(Y, Z)\theta(X)\theta(W) - \widetilde{g}(Y, W)\theta(X)\theta(Z)\}, \end{aligned}$$

where  $\alpha$  and  $\beta$  are smooth functions, and  $\theta$  is a 1-form defined by

$$(1.2) \quad \theta(X) = \widetilde{g}(X, \zeta),$$

and  $\zeta$  is a unit vector field on  $\widetilde{M}$ , which called the *curvature vector field* of  $\widetilde{M}$ . It is well known that if the curvature tensor  $\widetilde{R}$  is of the form (1.1), then  $\widetilde{M}$  is conformally flat. If  $\beta = 0$ , then  $\widetilde{M}$  is a space of constant curvature  $\alpha$ .

Recently Jin and Jin-Lee studied lightlike hypersurface  $M$  [8] and lightlike submanifolds  $M$  [9] in a semi-Riemannian manifold  $\widetilde{M}$  of quasi-constant curvature subject to the conditions; (1)  $\zeta$  is tangent to  $M$ , (2) the screen distribution  $S(TM)$  is totally geodesic in  $M$  and (3) the co-screen distribution  $S(TM^\perp)$  is conformal Killing distribution. Each of this papers proved several characterization theorems for their lightlike submanifolds.

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The geometry of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds produce models of different types of horizons. Which is also studied in the theory of electromagnetism [2]. Thus, large number of applications but limited information available, motivated us to do research on this subject matter. As for any semi-Riemannian manifold there is a natural existence of lightlike subspaces, Duggal and Bejancu [2] published their work on the general theory of lightlike submanifolds to fill a gap in the study of submanifolds. Since then there has been very active study on lightlike geometry of submanifolds (see up-to date results in two books [4, 5]). The class of lightlike submanifolds of codimension 2 is composed of two classes by virtue of the rank of its radical distribution, named by half lightlike and coisotropic submanifolds [3]. Half lightlike submanifold is a special case of general  $r$ -lightlike submanifold such that  $r = 1$  and its geometry is more general form than that of coisotropic submanifold or lightlike hypersurface. Much of the works on half lightlike submanifolds will be immediately generalized in a formal way to general  $r$ -lightlike submanifolds of arbitrary codimension  $n$  and arbitrary rank  $r$ . Moreover the geometry of half lightlike submanifolds is simple more than that of the general  $r$ -lightlike submanifold. For this reason, in this paper we study the geometry of a semi-Riemannian manifold  $\widetilde{M}$  of quasi-constant curvature with a half lightlike submanifold  $M$ .

The objective of this paper is to study the curvature of semi-Riemannian manifold  $\widetilde{M}$  of quasi-constant curvature admits extended screen homothetic and statical half lightlike submanifolds  $M$  such that  $\zeta$  is tangent to  $M$ . Two natural conditions to impose on this study are that its homothetic factor be either non-zero constant or zero, the latter is equivalent to the screen distribution to be totally geodesic in  $M$ . We prove two characterization theorems for such a semi-Riemannian manifold  $\widetilde{M}$  of quasi-constant curvature:

**Theorem 1.1.** *Let  $\widetilde{M}$  be a semi-Riemannian manifold of quasi-constant curvature admits a solenoidal half lightlike submanifold  $M$  such that  $\dim M > 2$ ,  $\zeta$  is tangent to  $M$  and  $S(TM)$  is totally geodesic in  $M$ . Then the functions  $\alpha$  and  $\beta$ , defined by (1.1), vanish identically and  $\widetilde{M}$  is a flat manifold.*

**Theorem 1.2.** *Let  $\widetilde{M}$  be a semi-Riemannian manifold of quasi-constant curvature admits a screen homothetic and statical half lightlike submanifold  $M$  such that  $\zeta$  is tangent to  $M$  and  $\dim \widetilde{M} > 2$ . Then the functions  $\alpha$  and  $\beta$ , defined by (1.1), vanish identically and  $\widetilde{M}$  is a flat manifold.*

## 2. Half lightlike submanifold

It is well-known [3] that the radical distribution  $Rad(TM) = TM \cap TM^\perp$  of half lightlike submanifold  $(M, g)$  of a semi-Riemannian manifold  $(\widetilde{M}, \widetilde{g})$  of codimension 2 is a vector subbundle of the tangent bundle  $TM$  and the normal bundle  $TM^\perp$ , of rank 1. Therefore there exist complementary non-degenerate

distributions  $S(TM)$  and  $S(TM^\perp)$  of  $Rad(TM)$  in  $TM$  and  $TM^\perp$  respectively, which called the *screen distribution* and *co-screen distribution* on  $M$ , such that

$$(2.1) \quad TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^\perp = Rad(TM) \oplus_{orth} S(TM^\perp),$$

where  $\oplus_{orth}$  denotes the orthogonal direct sum. We denote such a half lightlike submanifold by  $M = (M, g, S(TM), S(TM^\perp))$ . Denote by  $F(M)$  the algebra of smooth functions on  $M$  and by  $\Gamma(E)$  the  $F(M)$  module of smooth sections of any vector bundle  $E$  over  $M$ . Consider the orthogonal complementary distribution  $S(TM)^\perp$  to  $S(TM)$  in  $\widetilde{TM}$ . Certainly  $TM^\perp$  is a subbundle of  $S(TM)^\perp$ . As  $S(TM^\perp)$  is a non-degenerate subbundle of  $S(TM)^\perp$ , the orthogonal complementary distribution  $S(TM^\perp)^\perp$  of  $S(TM^\perp)$  in  $S(TM)^\perp$  is also a non-degenerate distribution such that

$$S(TM)^\perp = S(TM^\perp) \oplus_{orth} S(TM^\perp)^\perp.$$

Clearly  $Rad(TM)$  is a vector subbundle of  $S(TM^\perp)^\perp$ . Choose  $L \in \Gamma(S(TM^\perp)^\perp)$  as a unit vector field with  $\tilde{g}(L, L) = \epsilon = \pm 1$ . For any null section  $\xi$  of  $Rad(TM)$ , there exists a uniquely defined null vector field  $N \in \Gamma(S(TM^\perp)^\perp)$  satisfying

$$\tilde{g}(\xi, N) = 1, \quad \tilde{g}(N, N) = \tilde{g}(N, X) = \tilde{g}(N, L) = 0, \quad \forall X \in \Gamma(S(TM)).$$

Denote by  $ltr(TM)$  the subbundle of  $S(TM^\perp)^\perp$  locally spanned by  $N$ . Then we show that  $S(TM^\perp)^\perp = Rad(TM) \oplus ltr(TM)$ . Let  $tr(TM) = S(TM^\perp) \oplus_{orth} ltr(TM)$ . We call  $N$ ,  $ltr(TM)$  and  $tr(TM)$  the *lightlike transversal vector field*, *lightlike transversal vector bundle* and *transversal vector bundle* of  $M$  with respect to the screen distribution  $S(TM)$  respectively. Then the tangent bundle  $\widetilde{TM}$  of  $\widetilde{M}$  is decomposed as follow :

$$(2.2) \quad \begin{aligned} \widetilde{TM} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp). \end{aligned}$$

Let  $\tilde{\nabla}$  be the Levi-Civita connection of  $\widetilde{M}$  and  $P$  the projection morphism of  $TM$  on  $S(TM)$  with respect to the decomposition (2.1). Then the local Gauss and Weingarten formulas of  $M$  and  $S(TM)$  are given by

$$(2.3) \quad \tilde{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L,$$

$$(2.4) \quad \tilde{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L,$$

$$(2.5) \quad \tilde{\nabla}_X L = -A_L X + \phi(X)N;$$

$$(2.6) \quad \nabla_X PY = \nabla_X^* PY + C(X, PY)\xi,$$

$$(2.7) \quad \nabla_X \xi = -A_\xi^* X - \tau(X)\xi, \quad \forall X, Y \in \Gamma(TM),$$

where  $\nabla$  and  $\nabla^*$  are induced connections on  $TM$  and  $S(TM)$  respectively,  $B$  and  $D$  are called the *local second fundamental forms* of  $M$ ,  $C$  is called the *local second fundamental form* on  $S(TM)$ .  $A_N$ ,  $A_\xi^*$  and  $A_L$  are linear operators on  $TM$  and  $\tau$ ,  $\rho$  and  $\phi$  are 1-forms on  $TM$ . Since  $\tilde{\nabla}$  is torsion-free,  $\nabla$  is

also torsion-free, and  $B$  and  $D$  are symmetric. The above three local second fundamental forms of  $M$  and  $S(TM)$  are related to their shape operators by

$$(2.8) \quad B(X, Y) = g(A_\xi^* X, Y), \quad \tilde{g}(A_\xi^* X, N) = 0,$$

$$(2.9) \quad C(X, PY) = g(A_N X, PY), \quad \tilde{g}(A_N X, N) = 0,$$

$$(2.10) \quad \epsilon D(X, Y) = g(A_L X, Y) - \phi(X)\eta(Y), \quad \tilde{g}(A_L X, N) = \epsilon\rho(X),$$

where  $\eta$  is a 1-form on  $TM$  such that

$$\eta(X) = \tilde{g}(X, N), \quad \forall X \in \Gamma(TM).$$

By (2.8) and (2.9), we show that  $A_\xi^*$  and  $A_N$  are  $S(TM)$ -valued shape operators related to  $B$  and  $C$  respectively and  $A_\xi^*$  is self-adjoint on  $TM$ .

**Definition.** A half lightlike submanifold  $M$  of a semi-Riemannian manifold  $(\tilde{M}, \tilde{g})$  is said to be *statical* if  $\tilde{\nabla}_X L \in \Gamma(S(TM))$  for any  $X \in \Gamma(TM)$ .

From (2.5) and the second equation of (2.10), we show that this definition is equivalent to the following two conditions:  $\phi = 0$  ( $M$  is *irrotational*) [10] and  $\rho = 0$  ( $M$  is *solenoidal*). By  $M$  is *statical* we shall mean not only  $M$  is *irrotational* but also  $M$  is *solenoidal*.

We need the following four Codazzi equations (for a full set of the Gauss-Codazzi equations see [3, 6, 7] for  $M$  and  $S(TM)$ ). Denote by  $\tilde{R}$  and  $R$  the curvature tensors of the Levi-Civita connection  $\tilde{\nabla}$  on  $\tilde{M}$  and the induced connection  $\nabla$  on  $M$  respectively.

$$(2.11) \quad \begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, \xi) &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &\quad + B(Y, Z)\tau(X) - B(X, Z)\tau(Y) \\ &\quad + D(Y, Z)\phi(X) - D(X, Z)\phi(Y), \end{aligned}$$

$$(2.12) \quad \begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, N) &= \tilde{g}(R(X, Y)Z, N) \\ &\quad + \epsilon\{D(X, Z)\rho(Y) - D(Y, Z)\rho(X)\}, \end{aligned}$$

$$(2.13) \quad \begin{aligned} \tilde{g}(\tilde{R}(X, Y)\xi, N) &= g(A_\xi^* X, A_N Y) - g(A_\xi^* Y, A_N X) \\ &\quad - 2d\tau(X, Y) + \rho(X)\phi(Y) - \rho(Y)\phi(X), \end{aligned}$$

$$(2.14) \quad \begin{aligned} \tilde{g}(R(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ &\quad + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X) \end{aligned}$$

for all  $X, Y, Z, W \in \Gamma(TM)$ . The *Ricci curvature tensor* of  $\tilde{M}$  is defined by

$$\widetilde{Ric}(X, Y) = trace\{Z \rightarrow \tilde{R}(Z, X)Y\}$$

for any  $X, Y \in \Gamma(T\tilde{M})$ . Let  $\dim \tilde{M} = m + 3$ . Locally,  $\widetilde{Ric}$  is given by

$$(2.15) \quad \widetilde{Ric}(X, Y) = \sum_{i=1}^{m+3} \epsilon_i \tilde{g}(\tilde{R}(E_i, X)Y, E_i),$$

where  $\{E_1, \dots, E_{m+3}\}$  is an orthonormal frame field of  $T\tilde{M}$  and  $\epsilon_i (= \pm 1)$  denotes the causal character of respective vector field  $E_i$ .

**Definition.** A half lightlike submanifold  $M$  of a semi-Riemannian manifold  $(\widetilde{M}, \widetilde{g})$  is called *extended screen homothetic* if the shape operators  $A_N$  and  $A_\xi^*$  of  $M$  and  $S(TM)$  respectively are related by  $A_N = \varphi A_\xi^*$ , or equivalently, the second fundamental forms  $B$  and  $C$  of  $M$  and  $S(TM)$  respectively satisfy

$$(2.16) \quad C(X, PY) = \varphi B(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

where  $\varphi$  is a constant on a coordinate neighborhood  $\mathcal{U}$  in  $M$ . If  $\varphi \neq 0$  on  $\mathcal{U}$ , then we say that  $M$  is *screen homothetic*. In particular, if  $\varphi = 0$ , i.e.,  $C = 0$  on  $\mathcal{U}$ , then we say that  $S(TM)$  is *totally geodesic* in  $M$  [7].

In this paper, by  $M$  is *extended screen homothetic* we shall mean not only  $M$  is *screen homothetic* but also  $S(TM)$  is *totally geodesic* in  $M$ .

### 3. Proof of theorems

Assume that the curvature vector field  $\zeta$  of  $\widetilde{M}$  is a unit spacelike vector field of  $M$ . In this case, if  $\zeta$  belongs to  $Rad(TM)$ , then we show that  $\zeta = e\xi$ , where  $e = \theta(N) \neq 0$ . From this fact, we have  $1 = \widetilde{g}(\zeta, \zeta) = e^2g(\xi, \xi) = 0$ . It is a contradiction. This enables one to choose a screen distribution  $S(TM)$  which contains  $\zeta$ . This implies that if  $\zeta$  is tangent to  $M$ , then it belongs to  $S(TM)$  which we assume in this paper. Consider a quasi-orthonormal frame field  $\{\xi, W_a, N, L\}$  on  $\widetilde{M}$  such that  $Rad(TM) = Span\{\xi\}$ ,  $S(TM) = Span\{W_a\}$  and  $tr(TM) = Span\{N, L\}$ . By using (2.15), we get

$$(3.1) \quad \begin{aligned} \widetilde{Ric}(X, Y) = & \sum_{a=1}^m \epsilon_a \widetilde{g}(\widetilde{R}(W_a, X)Y, W_a) + \widetilde{g}(\widetilde{R}(\xi, X)Y, N) \\ & + \epsilon \widetilde{g}(\widetilde{R}(L, X)Y, L) + \widetilde{g}(\widetilde{R}(N, X)Y, \xi), \quad \forall X, Y \in \Gamma(TM). \end{aligned}$$

Using (1.1), (2.15) and the facts  $\theta(\xi) = \theta(N) = \theta(L) = 0$ , we have

$$(3.2) \quad \widetilde{Ric}(X, Y) = \{(m + 2)\alpha + \beta\}g(X, Y) + (m + 1)\beta\theta(X)\theta(Y),$$

$$(3.3) \quad \widetilde{g}(\widetilde{R}(\xi, Y)X, N) = \alpha g(X, Y) + \beta\theta(X)\theta(Y),$$

$$(3.4) \quad \epsilon \widetilde{g}(\widetilde{R}(L, Y)X, L) = \alpha g(X, Y) + \beta\theta(X)\theta(Y), \quad \forall X, Y \in \Gamma(TM).$$

#### 3.1. Proof of Theorem 1.1

Replacing  $Z$  by  $\xi$  and  $W$  by  $N$  to (1.1) and using the fact  $\theta(N) = 0$ , we get

$$\widetilde{g}(\widetilde{R}(X, Y)\xi, N) = 0, \quad \forall X, Y \in \Gamma(TM).$$

Substituting  $C = 0$  into (2.14), we get  $\widetilde{g}(R(X, Y)PZ, N) = 0$ . Using this result, (2.12) and the facts  $\rho = 0$  and  $\widetilde{g}(\widetilde{R}(X, Y)\xi, N) = 0$ , we have

$$\widetilde{g}(\widetilde{R}(X, Y)Z, N) = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

Replacing  $X$  by  $\xi$  and  $Z$  by  $X$  to this and then, comparing with (3.3), we have

$$(3.5) \quad \alpha g(X, Y) + \beta\theta(X)\theta(Y) = 0, \quad \forall X, Y \in \Gamma(TM).$$

Taking  $X = Y = \zeta$ , this implies  $\alpha = -\beta$ . Substituting (3.5) into (1.1), we get

$$(3.6) \quad \widetilde{g}(\widetilde{R}(X, Y)Z, W) = -\alpha\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}$$

for all  $X, Y, Z, W \in \Gamma(TM)$ . Substituting (3.2)~(3.4) and (3.6) into (3.1) and using the quasi-orthonormal frame field  $E = \{\xi, W_a, N, L\}$ , we have

$$\widetilde{Ric}(X, Y) = -(m-1)\alpha g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Substituting (3.5) into (3.2) and using the fact  $\alpha = -\beta$ , we have

$$\widetilde{Ric}(X, Y) = 0, \quad \forall X, Y \in \Gamma(TM).$$

From the last two equation, we get  $\alpha = \beta = 0$  as  $m > 1$  and  $\widetilde{M}$  is flat.

### 3.2. Proof of Theorem 1.2

Replacing  $W$  by  $N$  to (1.1) and using the fact  $\theta(N) = 0$ , we get

$$(3.7) \quad \begin{aligned} \widetilde{g}(\widetilde{R}(X, Y)Z, N) &= \alpha\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\} \\ &\quad + \beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z) \end{aligned}$$

for all  $X, Y, Z \in \Gamma(TM)$ . Replacing  $Z$  by  $\xi$  to (3.7) and using the fact  $\theta(\xi) = 0$ , we have  $\widetilde{g}(\widetilde{R}(X, Y)\xi, N) = 0$ . Comparing this and (2.13) and using the facts  $A_N = \varphi A_\xi^*$  and  $\rho = 0$ , we show that  $d\tau = 0$ . Thus, by the cohomology theory there exists a smooth function  $l$  such that  $\tau = dl$ . Thus we get  $\tau(X) = X(l)$ . If we take  $\tilde{\xi} = \gamma\xi$ , then we have  $\tau(X) = \tilde{\tau}(X) + X(\ln \gamma)$ . Setting  $\gamma = \exp(l)$  in this equation, we get  $\tilde{\tau}(X) = 0$ . Although  $S(TM)$  is not unique but it is canonically isomorphic to the factor vector bundle  $S(TM)^\# = TM/Rad(TM)$  [10]. Therefore all  $S(TM)$  are mutually isomorphic. For this reason, we deal with only half lightlike submanifolds  $M$  equipped with  $\tau = 0$ . Replacing  $W$  by  $\xi$  to (1.1) and using (2.11) and the facts  $\tau = 0$  and  $\phi = 0$ , we get

$$(3.8) \quad (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

Substituting (2.16) into (2.14) and using (3.8), we get  $\widetilde{g}(\widetilde{R}(X, Y)PZ, N) = 0$ . From this result, (2.12) and the facts  $\rho = 0$  and  $\widetilde{g}(\widetilde{R}(X, Y)\xi, N) = 0$ , we have

$$\widetilde{g}(\widetilde{R}(X, Y)Z, N) = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

Replacing  $X$  by  $\xi$  and  $Z$  by  $X$  to this and then, comparing with (3.3), we have

$$(3.9) \quad \alpha g(X, Y) + \beta \theta(X)\theta(Y) = 0, \quad \forall X, Y \in \Gamma(TM).$$

Taking  $X = Y = \zeta$ , this implies  $\alpha = -\beta$ . Substituting (3.9) into (1.1), we get

$$(3.10) \quad \widetilde{g}(\widetilde{R}(X, Y)Z, W) = -\alpha\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}$$

for all  $X, Y, Z, W \in \Gamma(TM)$ . Substituting (3.2)~(3.5) and (3.10) into (3.1) and using the quasi-orthonormal frame field  $E = \{\xi, W_a, N, L\}$ , we have

$$\widetilde{Ric}(X, Y) = -(m-1)\alpha g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Substituting (3.9) into (3.2) and using the fact  $\alpha = -\beta$ , we have

$$\widetilde{Ric}(X, Y) = 0, \quad \forall X, Y \in \Gamma(TM).$$

From the last two equations, we have  $\alpha = \beta = 0$  as  $m > 1$  and  $\widetilde{M}$  is flat.

**Example.** Consider a surface  $M$  in  $R_2^4$  given by the equation

$$x^3 = \frac{1}{\sqrt{2}}(x^1 + x^2); \quad x^4 = \frac{1}{2} \ln(1 + (x^1 - x^2)^2).$$

Then  $TM = Span\{U_1, U_2\}$  and  $TM^\perp = Span\{\xi, L\}$  where we set

$$\begin{aligned} U_1 &= \sqrt{2}(1 + (x^1 - x^2)^2) \frac{\partial}{\partial x^1} + (1 + (x^1 - x^2)^2) \frac{\partial}{\partial x^3} + \sqrt{2}(x^1 - x^2) \frac{\partial}{\partial x^4}, \\ U_2 &= \sqrt{2}(1 + (x^1 - x^2)^2) \frac{\partial}{\partial x^2} + (1 + (x^1 - x^2)^2) \frac{\partial}{\partial x^3} - \sqrt{2}(x^1 - x^2) \frac{\partial}{\partial x^4}, \\ \xi &= \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \sqrt{2} \frac{\partial}{\partial x^3}, \\ L &= 2(x^2 - x^1) \frac{\partial}{\partial x^2} + \sqrt{2}(x^2 - x^1) \frac{\partial}{\partial x^3} + (1 + (x^1 - x^2)^2) \frac{\partial}{\partial x^4}. \end{aligned}$$

By direct calculations, we check that  $Rad(TM)$  is a distribution on  $M$  of rank 1 spanned by  $\xi$ . Hence  $M$  is a half-lightlike submanifold of  $R_2^4$ . Choose  $S(TM)$  and  $S(TM^\perp)$  spanned by  $U_2$  and  $L$  which are timelike and spacelike respectively. We obtain the lightlike transversal vector bundle

$$ltr(TM) = Span \left\{ N = -\frac{1}{2} \frac{\partial}{\partial x^1} + \frac{1}{2} \frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^3} \right\},$$

and the transversal vector bundle  $tr(TM) = Span\{N, L\}$ .

Denote by  $\widetilde{\nabla}$  the Levi-Civita connection on  $R_2^4$  and by straightforward calculations obtain

$$\begin{aligned} \widetilde{\nabla}_{U_2} U_2 &= 2(1 + (x^1 - x^2)^2) \left\{ 2(x^2 - x^1) \frac{\partial}{\partial x^2} + \sqrt{2}(x^2 - x^1) \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4} \right\}, \\ \widetilde{\nabla}_\xi U_2 &= 0; \quad \widetilde{\nabla}_X \xi = \widetilde{\nabla}_X N = 0; \quad \forall X \in \Gamma(TM). \end{aligned}$$

Then taking into account of Gauss and Weingarten formulae infer

$$B = 0, \quad A_\xi = 0, \quad A_N = 0, \quad \nabla_X \xi = 0, \quad \tau(X) = 0, \quad \rho(X) = 0,$$

$$D(X, \xi) = 0, \quad D(U_2, U_2) = 2,$$

$$\nabla_X U_2 = \frac{2\sqrt{2}(x^2 - x^1)^3}{1 + (x^1 - x^2)^2} X^2 U_2$$

for any tangent vector field  $X = X^1 \xi + X^2 U_2$  to  $M$ . Thus  $S(TM)$  is totally geodesic in  $M$  ( $M$  is trivial screen homothetic) and solenoidal half-lightlike submanifold of  $R_2^4$ .

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DEPARTMENT OF MATHEMATICS  
DONGGUK UNIVERSITY  
KYONGJU 780-714, KOREA  
*E-mail address:* jindh@dongguk.ac.kr