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# CHOW STABILITY OF CANONICAL GENUS 4 CURVES

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Abstract. In this paper, we give sufficient conditions on a canonical genus 4 curve for it to be Chow (semi)stable.

## 1. Introduction

A Deligne-Mumford stable curve is a complete connected curve $C$ having ample dualising sheaf $\omega_C$ and admitting only nodes as singularities. An $n$-canonical curve $C \subset \mathbb{P}^N$ is a Deligne-Mumford stable curve of arithmetic genus $g$ embedded by the complete linear system $|\omega_C^{\otimes n}|$ where $N = (2n-1)(g-1)-1$ if $n \geq 2$, and $N = g-1$ if $n = 1$.

Let $\mathrm{Chow}_{g,n}$ be the closure of the locus of the Chow forms of $n$-canonical curves of arithmetic genus $g$ in the Chow variety of algebraic cycles of dimension 1 and degree $2g-2$ in $\mathbb{P}^N$. The natural action of $\mathrm{SL}_{N+1}$ on $\mathbb{P}^N$ induces an action on $\mathrm{Chow}_{g,n}$. Denote the corresponding GIT (Geometric Invariant Theory) quotient space by $\mathrm{Chow}_{g,n}//\mathrm{SL_{N+1}}$. To understand this quotient space as a parameter space of curves with some geometric properties, we need to find Chow stability conditions.

Mumford showed that, for $n \geq 5$ and $g \geq 2$, the Chow stable curves are precisely Deligne-Mumford stable curves and there is no strictly Chow semistable curve (cf. [14]). This implies that the quotient space is precisely the moduli space of Deligne-Mumford stable curves $\overline{M}_4$.

The cases when $n = 3$ and $g \geq 3$ were concerned by Schubert in [16]. He proved that a 3-canonical curve of genus $g \geq 3$ is Chow stable if and only if it is pseudo-stable and also showed that there is no strictly Chow semistable curve, and thus the quotient space is the moduli space of pseudo-stable curves $\overline{M}_g^{ps}$. A pseudo-stable curve is a complete connected curve $C$ satisfying the following properties.

- $\omega_C$ is ample,
- it admits at worst nodes and ordinary cusps as singularities, and

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- it has no elliptic components meeting the rest at one point.

Hyeon and Lee proved that, when $n = 3$ and $g = 2$, the pseudo-stable curves are indeed Chow semistable and completely classified the strictly Chow semistable points in [10]. They also concerned the case $n = 2$ and $g = 3$. Hassett and Hyeon studied for the case when $n = 2$ and $g \geq 4$ in [8] and the cases when $n = 4$ and general $g$ were studied by Hyeon and Morrison in [12].

The purpose of this paper is to study the cases when $n = 1$ and $g = 4$. More precisely, we want to give sufficient conditions on a canonical genus 4 curve for it to be Chow stable or semistable. To do this, we use the Hilbert-Mumford criterion (cf. Theorem 2.2). Our main results are presented in Section 3.2. We show that any irreducible curve in $\mathrm{Chow}_{4,1}$ with mild singularities is Chow stable (cf. Theorem 3.8). For reducible curves, we prove that a general curve in $\mathrm{Chow}_{4,1}$ with two irreducible components is Chow stable except when it is a union of two elliptic curves meeting at three points (cf. Theorems 3.10 and 3.11).

After appearing the preliminary version of this paper, Casalaina-Martin, Jensen, and Laza (cf. [2], Theorem 3.1) classified Chow stable and semistable points in $\mathrm{Chow}_{4,1}$ by using the GIT analysis for cubic threefolds. Our results are partial but we make a direct computation of the stability conditions on $\mathrm{Chow}_{4,1}$.

Throughout this paper, we use the following notations and conventions.

- We work over an algebraically closed field $k$ of characteristic zero.
- A curve is a connected, complete scheme of pure dimension 1.
- For a curve $C$, the genus $g(C)$ of $C$ is its arithmetic genus and we write $\omega_C$ for its dualising sheaf.
- We say that a point $p \in C$ is a singular point of type $A_n$ if
$$\hat{\mathcal{O}}_{C,p} \simeq k[[x,y]]/(y^2 - x^{n+1}).$$
In particular, a node (resp. ordinary cusp) is a singular point of type $A_1$ (resp. $A_2$).
- For a polynomial $P(m)$ of degree $n$ in $m$, we denote by n.l.c.$P(m)$ for the coefficient of $\frac{1}{n!}m^n$ in $P(m)$.

## 2. Chow stability and canonical embedding

In this section, we review some basic facts for Chow stability.

### 2.1. Chow stability

A weighted flag $F$ of $\mathbb{P}^n$ consists of a choice of coordinates $X_0, \ldots, X_n$ of $\mathbb{P}^n$ and a sequence of integers $r_0 \geq \cdots \geq r_n = 0$.

Let $F$ be a weighted flag of $\mathbb{P}^n$ as above and $X$ be a variety in $\mathbb{P}^n$ of dimension $r$. Let $\alpha : \tilde{X} \to X$ be a proper birational morphism. Let us define an ideal sheaf $\mathcal{I}(X)$ of $\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}$ by

$$\mathcal{I}(X) \cdot [\alpha^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\mathbb{A}^1}] = \text{ the subsheaf generated by } t^{r_i}X_i,\ i = 1, \ldots, n.$$

It is well known that $\chi(\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}(m)/\mathcal{I}(X)^m\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}(m))$ is a polynomial of degree $r+1$ for $m \gg 0$ (cf. [14], Proposition 2.1). Define

$$e_F(X) := \text{ n.l.c. } \chi(\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}(m)/\mathcal{I}(X)^m\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}(m)).$$

Lemma 5.6 in [14] shows that $e_F(X)$ does not depend on $\alpha$.

For a Chow cycle $X = \sum a_i Y_i$ where $Y_i$ are subvarieties of $\mathbb{P}^n$ of dimension $r$ and $a_i$ are nonnegative integer, define

$$e_F(X) := \sum a_i e_F(Y_i).$$

**Definition 2.1.** The natural action of $SL_{n+1}$ on $\mathbb{P}^n$ induces an action on the Chow variety of $\mathbb{P}^n$. We say that a Chow cycle $X$ in $\mathbb{P}^n$ is Chow stable (resp. semistable, unstable) if its Chow from is GIT stable (resp. semistable, unstable) under the action of $SL_{n+1}$ on the Chow variety of $\mathbb{P}^n$.

The following theorem is the Hilbert-Mumford criterion which is very useful to determine GIT stability.

**Theorem 2.2** ([15], Theorem 2.1)**.** *Let $X$ be a Chow cycle of dimension $r$ in $\mathbb{P}^n$. Then $X$ is Chow semistable* (*resp. Chow stable*) *if and only if*

$$e_F(X) - \frac{r+1}{n+1}\deg X \sum r_i \le 0 \quad (\textit{resp.} < 0)$$

*for any weighted flag $F$ of $\mathbb{P}^n$.*

### 2.2. Criterions for Chow stability

We now review some methods for determining Chow stability. For more detail, we refer to [14, 15, 16].

Let $L_i \subset \mathbb{P}^n$ be the linear subspace defined by $X_i = \cdots = X_n = 0$ and let $P_{L_i} : \mathbb{P}^n - L_i \to \mathbb{P}^{n-i}$ be the natural projection along $L_i$.

**Definition 2.3.** Let $C \subset \mathbb{P}^n$ be an irreducible reduced curve in $\mathbb{P}^n$ with $C \not\subset L_i$. Let $\alpha_{L_i} : \tilde{C} \to \mathbb{P}^{n-i}$ be the morphism extending the composition of $P_{L_i}$ and the normalization $\alpha : \tilde{C} \to C$. Define

$$\deg P_{L_i}(C) := \begin{cases} (\deg \alpha_{L_i})(\deg \alpha_{L_i}(\tilde{C})) & \text{if } \alpha_{L_i}(\tilde{C}) \text{ is a curve} \\ 0 & \text{otherwise} \end{cases}$$

and

$$e_i = e_i^F(C) := \deg C - \deg P_{L_i}(C).$$

For a Chow cycle $C = \sum a_j C_j$ where $C_j$ is a 1 dimensional subvariety of $\mathbb{P}^n$ and $a_i$ are nonnegative integer, assume that $C_j \not\subset L_i$ for all $j$. Define

$$e_i = e_i^F(C) := \sum a_j e_i^F(C_j).$$

From the definition, $e_0 = 0$ and $e_n = \deg C$ if $C \not\subset L_n$.

**Proposition 2.4** ([14], Corollary 4.11)**.** *Let $C \subset \mathbb{P}^n$ be a curve such that each irreducible component of $C$ does not contained in $L_n$. Then, for any sequence $0 = s_0 < \cdots < s_l = n$, it is satisfied that*

$$e_F(C) \leq \sum_{i=0}^{l-1}(r_{s_i} - r_{s_{i+1}})(e_{s_i} + e_{s_{i+1}}).$$

Let $C$ be an irreducible reduced curve in $\mathbb{P}^n$ and let $\alpha : \tilde{C} \to C$ be the normalization of $C$. Pick a point $p$ in $\tilde{C}$ and let $s$ and $t$ be generators of the maximal ideals of $\mathcal{O}_{\tilde{C},p}$ and $\mathcal{O}_{\mathbb{A}^1,0}$, respectively. For the natural valuation $v_p$ on $\mathcal{O}_{\tilde{C},p}$, set $\mathrm{ord}_p X_i := v_p(\alpha^* X_i)$. Recall that $\mathcal{I}(C)$ be the ideal sheaf of $\mathcal{O}_{\tilde{C}\times\mathbb{A}^1}$ defined by

$$\mathcal{I}(C) \cdot [\alpha^*\mathcal{O}_C(1) \otimes \mathcal{O}_{\mathbb{A}^1}] = \text{ the subsheaf generated by } t^{r_i}\alpha^* X_i,\ i = 1, \ldots, n.$$

For each $p \in \tilde{C}$, $\mathcal{I}(C)_{p\times\{0\}} \subset \mathcal{O}_{\tilde{C}\times\mathbb{A}^1,p\times\{0\}}$ is generated by

$$t^{r_0}s^{\mathrm{ord}_p X_0}, t^{r_1}s^{\mathrm{ord}_p X_1}, \ldots, t^{r_n}s^{\mathrm{ord}_p X_n},$$

where $s^{\mathrm{ord}_p X_i} = 0$ if $\mathrm{ord}_p X_i = \infty$. Let us use the notation

$$\mathcal{I}(C)_{p\times\{0\}} = (t^{r_0}s^{\mathrm{ord}_p X_0}, t^{r_1}s^{\mathrm{ord}_p X_1}, \ldots, t^{r_n}s^{\mathrm{ord}_p X_n}).$$

**Definition 2.5.** In the situation above, suppose that there is an $i$ with $r_i = 0$ and $C \not\subset (X_i = 0)$. For each point $p$ in $\tilde{C}$, we define

$$e_F(\tilde{C})_p := \text{ n.l.c.} \dim_k(\mathcal{O}_{\tilde{C}\times\mathbb{A}^1,p\times\{0\}}/\mathcal{I}(C)^m_{p\times\{0\}}).$$

*Remark* 2.6. In the setting of Definition 2.5, the quotient sheaf $\mathcal{O}_{\tilde{C}\times\mathbb{A}^1}/\mathcal{I}(C)$ is supported at the points over $C \cap L_n$ because $r_n = 0$. Therefore

$$\begin{aligned} e_F(C) &= \text{ n.l.c.}\chi(\mathcal{O}_{\tilde{C}\times\mathbb{A}^1}(m)/\mathcal{I}(C)^m\mathcal{O}_{\tilde{C}\times\mathbb{A}^1}(m)) \\ &= \sum_{\alpha(p)\in L_n} \dim_k(\mathcal{O}_{\tilde{C}\times\mathbb{A}^1,p\times\{0\}}/\mathcal{I}(C)^m_{p\times\{0\}}) \\ &= \sum_{\alpha(p)\in L_n} e_F(\tilde{C})_p. \end{aligned}$$

**Lemma 2.7** ([16], Lemma 1.4)**.** *In the situation of Definition 2.5, set $v_i := \mathrm{ord}_p X_i$. If $v_i + r_i \geq a$ for all $i = 0, \ldots, n$, then $e_F(\tilde{C})_p \geq a^2$.*

### 2.3. Canonical curves

**Definition 2.8.** We say that a curve $C$ is honestly hyperelliptic if there is a morphism $C \to \mathbb{P}^1$ of degree 2, and is honestly non-hyperelliptic if it is not honestly hyperelliptic.

A Gorenstein curve is a curve $C$ with $\omega_C \cong \mathcal{O}_C(K_C)$ for a Cartier divisor $K_C$. A generically Gorenstein curve is a curve $C$ such that $\omega_C$ is locally isomorphic to $\mathcal{O}_C$ outside a finite set.

**Theorem 2.9** ([1], Theorem 3.6). *Let $C$ be a numerically 3-connected Gorenstein curve. That is, for any generically Gorenstein strict subcurve $D \subset C$,*

$$\deg \mathcal{O}_D(K_C) - \deg \omega_D \geq 3.$$

*Then either $C$ is honestly hyperelliptic or $K_C$ is very ample.*

If $C$ is a numerically 3-connected curve admitting nodal singularities only, then Theorem 2.9 implies that any irreducible component of $C$ has at least three intersection points with the union of the other components.

**Definition 2.10.** A canonical curve is a numerically 3-connected honestly non-hyperelliptic Gorenstein genus $g$ curve $C \subset \mathbb{P}^{g-1}$ whose embedding is given by $|\omega_C|$.

We remark that any canonical curve $C \subset \mathbb{P}^{g-1}$ is a nondegenerate curve of degree $2g-2$.

## 3. Canonical curves of genus four

From now on, $F$ is a weighted flag of $\mathbb{P}^3$ associated with coordinates $X_0, \ldots, X_3$ and weights $r_0 \geq \cdots \geq r_3 = 0$, and $L_i$ is the linear subspace of $\mathbb{P}^3$ defined by $X_i = \cdots = X_3 = 0$.

Note that any canonical genus 4 curve in $\mathbb{P}^3$ has degree 6. Thus applying Theorem 2.2 we get that a canonical genus 4 curve $C \subset \mathbb{P}^3$ is Chow stable (resp. semistable) if and only if

$$e_F(C) < (\text{resp.} \leq) 3 \sum r_i$$

for any weighted flag $F$.

### 3.1. Upper bounds of $e_F(C)$

In this subsection, we gather some preliminary results which will be used to give upper bounds of $e_F(C)$ in the next subsection.

**Lemma 3.1.** *Let $C \subset \mathbb{P}^3$ be a curve of degree $d$, and let $e_i$ be the same as that in Definition 2.3. Assume that each irreducible component of $C$ does not contained in $L_n$. Then*

$$e_F(C) \leq \min\{dr_0, e_1 r_0 + dr_1, e_2 r_0 + dr_2, e_1 r_0 + e_2 r_1 + (d - e_1) r_2\}.$$

*Proof.* The lemma immediately comes by applying Proposition 2.4 to the sequences $0 < 3$, $0 < 1 < 3$, $0 < 2 < 3$ and $0 < 1 < 2 < 3$. □

**Lemma 3.2.** *Let $R := k[s,t]$ and $I$ an ideal of $R$.*

(1) *If $I = (t^a, s^b)$ for integers $a, b \geq 1$, then* $\text{n.l.c.dim}_k R/I^m = ab$.
(2) *If $I = (t^a, t^p s^q, s^b)$ for integers $a, b, p, q \geq 1$, then*

$$\text{n.l.c.dim}_k R/I^m \leq aq + bp.$$

*Proof.* If $I = (t^a, s^b)$, then $I^m$ is generated by

$$\{t^{an_1+r_1}s^{bn_2+r_2} \mid n_1 + n_2 \geq m, 0 \leq r_1 < a, 0 \leq r_2 < b\}.$$

Thus the following set of monomials

$$\{s^{bi+k}t^j \mid 0 \leq i \leq m-1, 0 \leq j \leq a(m-i)-1, 0 \leq k \leq b-1\}$$

forms a basis of $R/I^m$. Therefore

$$\dim_k R/I^m = \sum_{i=0}^{m-1} a(m-i)b = ab(m^2+m)/2,$$

which implies (1). Similarly, (2) can be proved by describing the set of the monomials spanning $R/I^m$. □

**Lemma 3.3.** *Let $C \subset \mathbb{P}^3$ be a curve of degree $d$, and assume that each irreducible component of $C$ does not lie in the hyperplane $L_3$. Then*

$$e_F(C) \leq (\sum_{\alpha(p)=L_1} \mathrm{ord}_p X_3)r_0 + (\sum_{\alpha(p)\in L_2-L_1} \mathrm{ord}_p X_3)r_1 + (\sum_{\alpha(p)\notin L_2} \mathrm{ord}_p X_3)r_2.$$

*Proof.* We may assume that $C$ is irreducible and reduced. Let $\alpha : \tilde{C} \to C$ be the normalization of $C$. Take a point $p$ in $\tilde{C}$ and set $v_i = \mathrm{ord}_p X_i$. Then

$$\mathcal{I}(C)_{p\times\{0\}} = (t^{r_0}s^{v_0}, t^{r_1}s^{v_1}, t^{r_2}s^{v_2}, s^{v_3}).$$

From this, it is induced that

$$\mathcal{I}(C)_{p\times\{0\}} > \begin{cases} (t^{r_0}, s^{v_3}), & \text{for all } p \\ (t^{r_1}, s^{v_3}), & \text{if } \alpha(p) \neq L_1 \\ (t^{r_2}, s^{v_3}), & \text{if } \alpha(p) \notin L_2. \end{cases}$$

Applying Lemma 3.1 to these inclusions, we obtain that

$$e_F(\tilde{C})_p \leq \begin{cases} r_0 v_3, & \text{for all } p \\ r_1 v_3, & \text{if } \alpha(p) \neq L_1 \\ r_2 v_3, & \text{if } \alpha(p) \notin L_2. \end{cases}$$

Using the equality $e_F(C) = \sum_{p\in\tilde{C}} e_F(\tilde{C})_p$, the desired inequality can be verified. □

**Lemma 3.4.** *Let $C \subset \mathbb{P}^3$ be a reduced irreducible curve of degree $d$ and assume that $C \subset L_3$ and $C \neq L_2$. Then*

$$e_F(C) \leq (\sum_{\alpha(p)=L_1} \mathrm{ord}_p X_3)r_0 + (\sum_{\alpha(p)\neq L_1} \mathrm{ord}_p X_i)r_1 + dr_2.$$

*Proof.* Let $\alpha : \tilde{C} \to C$ be the normalization of $C$. Let $F'$ be the weighted flag of $L_3 \cong \mathbb{P}^2$ associated with the coordinates $X'_0 := X_0|_{L_3}$, $X'_1 := X_1|_{L_3}$,

$X_2' := X_2|_{L_3}$ and the weights $r_0' = r_0 - r_2 \geq r_1' = r_1 - r_2 \geq r_2' = 0$. From the proof of Theorem 2.9 in [14] it is induced that

$$e_F(C) = e_{F'}(C) + 2dr_2.$$

Take a point $p \in \tilde{C} \cap (X_2' = 0)$ and set $v_i := \mathrm{ord}_p \alpha^* X_i$. Then

$$e_{F'}(\tilde{C})_p \leq \begin{cases} r_0' v_2, & \text{for all } p \\ r_1' v_2, & \text{if } \alpha(p) \neq L_1. \end{cases}$$

The first inequality is given by applying from Lemma 3.2 to the inclusion

$$\mathcal{I}(C)_{p\times\{0\}} = (t^{r_0'} s^{v_0}, t^{r_1'} s^{v_1}, s^{v_2}) > (t^{r_0'}, s^{v_2}).$$

If $\alpha(p) \neq L_1$, then $v_1 = 0$, and hence we get the next inclusion

$$\mathcal{I}(C)_{p\times\{0\}} > (t^{r_1'}, s^{v_2})$$

which implies the second inequality by Lemma 3.2. From the equality $e_{F'}(C) = \sum_{p\in\tilde{C}} e_{F'}(\tilde{C})_p$, we get the lemma. □

**Lemma 3.5.** *If $C \subset \mathbb{P}^3$ is equal to $L_2$, then $e_F(C) = r_0 + r_1$.*

*Proof.* The coordinate ring of $C$ is $R = k[X_0, X_1]$. Let $I = (X_0 t^{r_0}, X_1 t^{r_1})$. Applying Lemma 1.3 in [16], we get that

$$e_F(C) = \text{n.l.c.}\dim_k (R[t]/I^m)_m.$$

Since $I^m$ is generated by

$$\{t^{r_0 i + r_1 j} X_0^i X_1^j | i + j = m\},$$

we get that

$$\begin{aligned} \dim_k (R[t]/I^m)_m &= \sum_{i+j=m} r_0 i + r_1 j = \sum_{i=0}^{m} r_0 i + r_1(m-i) \\ &= \sum_{i=0}^{m} i(r_0 - r_1) + m r_1 \\ &= \frac{m(m+1)}{2}(r_0 - r_1) + m(m+1) r_1 \\ &= \frac{r_0 + r_1}{2}(m^2 + m), \end{aligned}$$

and thus the required equality is obtained. □

### 3.2. Main results

Next proposition says that Chow stable curves admit at worst double points.

**Proposition 3.6.** *Let $C \subset \mathbb{P}^3$ be a curve of degree 6. If $C$ admits a singular point of multiplicity $\geq 3$, then it is not Chow stable. Furthermore, if $C$ has a point of multiplicity $\geq 4$, then it is not Chow semistable.*

*Proof.* Let $p$ be a point of $C$ with multiplicity bigger than or equal to 3. Take coordinates $X_0,\ldots,X_3$ so that $X_1$, $X_2$, and $X_3$ vanish at $p$, and let $r_0 = 1, r_1 = r_2 = r_3 = 0$. For the associated weighted flag $F$, it follows that

$$\mathcal{I}_{p\times\{0\}}(C) = (t, m_p)\mathcal{O}_{C\times\mathbb{A}^1,p\times\{0\}}$$

which is the maximal ideal of $\mathcal{O}_{C\times\mathbb{A}^1,p\times\{0\}}$ where $m_p$ is the maximal ideal of $\mathcal{O}_{C,p}$. Hence

$$e_F(C) = e_F(C)_p = \mathrm{mult}_{p\times\{0\}}(C\times\mathbb{A}^1) = \mathrm{mult}_p C \geq 3 = 3\sum r_i.$$

Furthermore, the last inequality is strict if $\mathrm{mult}_p C \geq 4$. □

The next proposition will be used in the proof of the following theorems.

**Proposition 3.7.** *Let $C \subset \mathbb{P}^3$ be an honestly non-hyperelliptic curve of degree 6 in the sense of Definition 2.8, and assume that each irreducible component of $C$ does not contained in $L_n$. Suppose that $e_1 \leq 2$ and $e_2 \leq 4$ where $e_i$ be the same as that in Definition 2.3. Then $C$ is Chow stable with respect to $F$.*

*Proof.* From Lemma 3.1, it follows that

$$e_F(C) \leq \min\{6r_0, 2r_0 + 6r_1, 4r_0 + 6r_2, 2r_0 + 4r_1 + 4r_2\}.$$

If the right hand side in the above inequality is greater than or equal to $3\sum r_i$ simultaneously, then it should be satisfied that

$$6r_0 = 2r_0 + 6r_1 = 4r_0 + 6r_2 = 2r_0 + 4r_1 + 4r_2 = 3\sum r_i$$

which implies that $e_F(C) \leq 3\sum r_i$, and the equality $e_F(C) = 3\sum r_i$ holds only when $r_0 = 3r$, $r_1 = 2r$, $r_2 = r$ for some $r \in \mathbb{Z}_{>0}$, and $e_2 = 2$, $e_4 = 4$.

We now assume that $r_0 = 3r$, $r_1 = 2r$ and $r_2 = r$ for some $r \in \mathbb{Z}_{>0}$, and $e_1 = 2$ and $e_4 = 4$. If $C$ meets $L_3$ at a point not equal to $L_1$, then

$$e_F(C) \leq 5r_0 + r_1 = 17r < 3\sum r_i$$

by Lemma 3.3. On the other hand, if $C$ intersects $L_3$ only at $L_1$, then the restricted projection morphism $P_{L_2}|_{C\cap(\mathbb{P}^3-L_2)}$ extends to a morphism $C \to \mathbb{P}^1$ of degree 2 because $e_2 = 4$ and the assumption that $C \cap L_3$ consists of only one point $L_1$. This gives a contradiction because $C$ is honestly non-hyperelliptic. □

Our next result shows that any irreducible canonical curve admitting only mild singularities is Chow stable.

**Theorem 3.8.** *Let $C \subset \mathbb{P}^3$ be an irreducible canonical curve of genus 4 admitting at worst $A_n$, $n \leq 4$, singularities. Then $C$ is Chow stable.*

*Proof.* From the assumptions, it is induced that $C$ admits at most double points, and is nondegenerate. Thus it follows that $e_1 \leq 2$ and $e_2 \leq 5$. Via Proposition 3.7, it is enough to show that $e_2 \neq 5$. Suppose not. The composition of the partial normalization morphism $\tilde{C} \to C$ at the points in $C \cap L_2$

and the restricted projection morphism $P_{L_2}|_{C\cap(\mathbb{P}^3-L_2)}$ induces an isomorphism $\tilde{C} \to \mathbb{P}^1$. This shows that $C$ has exactly two double points $P$ and $Q$ of type $A_3$ or $A_4$, and $L_2$ meets $C$ at $P$, $Q$ and another point. Let us denote by $H$ the plane determined by $L_2$ and the tangent line of $C$ at $P$. Then the number of intersection points of $H$ and $C$ is greater than or equal to 7 with multiplicity, a contradiction. □

The next theorem deals with double twisted curves in $\mathbb{P}^3$ which are the canonical images of smooth hyperelliptic curves of genus 4.

**Theorem 3.9.** *Let $C \subset \mathbb{P}^3$ be a double curve supported on a twisted cubic curve. Then $C$ is Chow semistable but not stable. Moreover, all such curves are identified in* $\mathrm{Chow}_{4,1}//\mathrm{SL}_4$.

*Proof.* Let $C = 2C_1$ where $C_1$ is the twisted cubic curve in $\mathbb{P}^3$. Then $e_1 \leq 1$ and $e_2 \leq 2$ for $C_1$, and thus $e_F(C) = 2e_F(C_1)$ is less than or equal to $4r_0 + 6r_2$ and $2r_0 + 6r_1$ by Lemma 3.1. On the other hand, the two values $4r_0 + 6r_2$ and $2r_0 + 6r_1$ cannot be bigger than $3\sum r_i$ simultaneously, which implies that $C$ is Chow semistable.

Take a point $p$ in $C_1$, and choose coordinates $X_0, \dots, X_3$ so that $X_1, X_2, X_3$ vanish at $p$, $X_2, X_3$ vanish to order $\geq 2$ at $p$, and $X_3$ vanishes to order $\geq 3$ at $p$. Set $r_0 = 3, r_1 = 2, r_2 = 1, r_3 = 0$. For the corresponding weighted flag $F$, it is obtained that

$$e_F(C) = 2e_F(C_1) \geq 2e_F(C_1)_p \geq 2 \cdot 9 = 3\sum r_i$$

by Lemma 2.7, and thus $C$ is not Chow stable.

The last statement comes from the fact that any two twisted cubic curves are projective equivalent. □

Let $\delta_{i,j} \subset \mathrm{Chow}_{4,1}$ be the closure of the locus parametrizing canonical curves consisting of two smooth components meeting at nodes and having genus $i$ and $j$ respectively. Let $C$ be a curve in $\delta_{i,j}$ with two smooth irreducible components $C_1$ and $C_2$ meeting at $r$ nodes. Then $r \geq 3$ by the remark after Theorem 2.9. Moreover

$$g(C) = g(C_1) + g(C_2) + r - 1 = i + j + r - 1 = 4.$$

Thus the only nontrivial cases are $\delta_{1,1}$, $\delta_{2,0}$, $\delta_{1,0}$ and $\delta_{0,0}$.

Throughout Theorems 3.10 and 3.11, we will show that a general curve in each $\delta_{i,j}$ is Chow stable except when it belongs to a class in $\delta_{1,1}$.

**Theorem 3.10.** *If $C \subset \mathbb{P}^3$ is a general curve in $\delta_{1,1}$, then it is Chow semistable but not Chow stable. Furthermore, all Chow semistable curves in $\delta_{1,1}$ are identified in* $\mathrm{Chow}_{4,1}//\mathrm{SL}_4$.

*Proof.* Without loss of generality, we may assume that $C$ is a union of two smooth elliptic curves $C_1$ and $C_2$ meeting at three nodes denoted by $p_1$, $p_2$ and $p_3$. Note that each $C_i$ is contained in a hyperplane denoted by $H_i$, and has degree 3.

If $L_2$ is not contained in any $H_i$, then $e_1 \leq 2$ and $e_2 \leq 4$ which implies that $e_F(C) \leq 3\sum r_i$ by Proposition 3.7, and thus we may assume that $L_2 \subset H_2$.

If $H_2$ is not equal to $L_3$, then

$$e_F(C) = e_F(C_1) + e_F(C_2) \leq \begin{cases} (r_0+2r_1)+(r_0+2r_1), & \text{if } L_2 = H_1 \cap H_2 \\ (r_0+2r_2)+(r_0+3r_1), & \text{if } L_2 \neq H_1 \cap H_2 \end{cases}$$

which implies that $e_F(C) \leq 3\sum r_i$.

Now assume that $H_2 = L_3$. Then it is easy to check that

$$e_F(C) = e_F(C_1) + e_F(C_2) \leq \begin{cases} (r_0+2r_1)+(r_0+2r_1+3r_2), & \text{if } L_2 = H_1 \cap H_2 \\ (r_0+2r_2)+(r_0+2r_1+3r_2), & \text{if } L_2 \neq H_1 \cap H_2 \end{cases}$$

which yields that $e_F(C) \leq 3\sum r_i$. Finally we showed that $C$ is Chow semistable.

Choose coordinates $X_0,\ldots, X_3$ so that $H_1$ and $H_2$ are hyperplanes defined by $X_2 = 0$ and $X_3 = 0$ respectively. Set $r_0 = r_1 = r$ and $r_2 = 0$. Then for each $i$ it follows that

$$e_F(C_i) = e_F(C_i)_{p_1} + e_F(C_i)_{p_2} + e_F(C_i)_{p_3} = r + r + r = 3r$$

and thus

$$e_F(C) = e_F(C_1) + e_F(C_2) = 6r = 3\sum r_i.$$

This shows that $C$ is not Chow stable.

Now it remains to show the last statement of the theorem. Choose coordinates $X_0,\ldots,X_3$ of $\mathbb{P}^3$ so that $C$ is defined by

$$X_0X_2^2 + X_0X_3^2 - X_1(X_1 - aX_0)(X_1 - bX_0) = 0 \ \text{ and } X_2X_3 = 0,$$

where 1, $a$ and $b$ are distinct where $X_0,\ldots,X_3$ is a homogeneous coordinates on $\mathbb{P}^3$. Note that general curve satisfying the assumptions in the proposition can be defined in this way if we choose suitable coordinates.

Consider the one parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{GL}_4$ defined by

$$\lambda(t)X_0 = tX_0,\ \lambda(t)X_1 = tX_1,\ \lambda(t)X_2 = X_2,\ \text{and } \lambda(t)X_3 = X_3.$$

Let $\bar{C}$ be the limit of $C$ as $t \to \infty$ under the action $\lambda$. Applying the computation in [9], it follows that $\bar{C}$ is given by

$$X_1(X_1 - aX_0)(X_1 - bX_0) = 0 \ \text{ and } X_2X_3 = 0.$$

We note that $\bar{C}$ is a union of $\bar{C}_1$ and $\bar{C}_2$ satisfying

(a) each $\bar{C}_i$ is contained in $H_i$,
(b) $\bar{C}_1 = L_{1,1} \cup L_{1,2} \cup L_{1,3}$ and $\bar{C}_2 = L_{2,1} \cup L_{2,2} \cup L_{2,3}$ where each $L_{i,j}$ is a line,
(c) $L_{i,1}$, $L_{i,2}$ and $L_{i,3}$ intersect at one point $q_i$ for each $i = 1, 2$, and
(d) $L_{1,j}$ and $L_{2,j}$ meet at a point $p_j$.

From Section 11.3 in [4], it is induced that $\bar{C}$ is Chow semistable. Note that any two curves satisfying (a)∼(d) are projectively equivalent which yields the last statement in the theorem. □

**Theorem 3.11.** *If $C \subset \mathbb{P}^3$ is a general curve in $\delta_{2,0}$, $\delta_{1,0}$, or $\delta_{0,0}$, then it is Chow stable.*

*Proof.* Without loss of generality, we may assume that $C$ is a canonical curve consisting of two smooth components $C_1$ and $C_2$ meeting at nodes. It is easy to check that $e_1 \leq 2$ and $e_2 \leq 4$ for any weighted flag $F$. Therefore by Proposition 3.7, we can also assume that $C_2$ is contained in $L_3$.

If $C$ belongs to a class in $\delta_{0,0}$, then $C_1$ and $C_2$ are twisted cubic curves by Fig. 18 in p. 354 [5], and thus they are nondegenerate, a contradiction.

Assume that $C$ belongs to a class in $\delta_{1,0}$. From Fig. 18 in p. 354 [5], we obtain that $\deg C_1 = 4$ and $\deg C_2 = 2$. We note that the intersection $C_1 \cap C_2$ consists of four distinct nodes of $C$ and $C_2 \subset L_3$. Therefore the points in $C_1 \cap L_3$ are exactly the same as that in $C_1 \cap C_2$. Hence in $C_1 \cap L_3$, there exist at least two points not lying on $L_2$, and at least three points not equal to $L_1$, which implies that $e_F(C_1) \leq r_0 + r_1 + 2r_2$ by Lemma 3.3. Applying Lemma 3.4, it is induced that $e_F(C_2) \leq 2r_0 + 2r_2$. Therefore

$$e_F(C) = e_F(C_1) + e_F(C_2) \leq 3r_0 + r_1 + 4r_2 \leq 3\sum r_i.$$

In the last inequality, the equality holds if and only if $r_1 = r_2 = 0$. In the case when $r_1 = r_2 = 0$, it is induced that

$$e_F(C) = e_F(C_1) + e_F(C_2) = e_F(C_1)_p + e_F(C_2)_p \leq 2r < 3\sum r_i,$$

where $p$ is the point on which $X_1$, $X_2$ and $X_3$ vanish.

The cases when $C$ belongs to a class in $\delta_{2,0}$ can be proved by similar arguments. $\square$

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