APPROXIMATING FIXED POINTS OF NONEXPANSIVE TYPE MAPPINGS IN BANACH SPACES WITHOUT UNIFORM CONVEXITY

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Abstract. Approximate fixed point property problem for Mann iteration sequence of a nonexpansive mapping has been resolved on a Banach space independent of uniform (strict) convexity by Ishikawa [*Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), 65–71]. In this paper, we solve this problem for a class of mappings wider than the class of asymptotically nonexpansive mappings on an arbitrary normed space. Our results generalize and extend several known results.

1. Introduction

Let C be a nonempty subset of a normed space X and $T : C \to C$ be a mapping. Then T is said to be

(1) *Lipschitzian* if for each $n \in \mathbb{N}$, there exists a positive real number k_n such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $x, y \in C$,

(2) *uniformly k-Lipschitzian* if T is Lipschitzian with sequence $\{k_n\}$ such that $k_n = k$ for all $n \in \mathbb{N}$,

(3) *asymptotically nonexpansive* ([8]) if T is Lipschitzian with sequence $\{k_n\}$ such that $k_n \ge 1$ for all $n \in \mathbb{N}$ with $\lim_{n \to \infty} k_n = 1$.

Clearly, every nonexpansive mapping T (i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$) is asymptotically nonexpansive with sequence $\{1\}$ and every asymptotically nonexpansive mapping is uniformly k-Lipschitzian with $k = \sup_{n \in \mathbb{N}} k_n$.

Iterative methods for approximating fixed points of nonexpansive mappings have been studied by many authors using the following two iteration processes:

(a) *Mann iteration process* ([18]): Let C be a convex subset of a linear space X and $T : C \to C$ be a mapping. Then the Mann iteration sequence $\{x_n\}$ is given by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying appropriate conditions.

We denote $(1 - \alpha_n)x_n + \alpha_n T x_n$ by $M(x_n, \alpha_n, T)$.

(b) *Ishikawa iteration process* ([9]): With X, C and T as in (a), the Ishikawa iteration sequence $\{x_n\}$ is given by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying appropriate conditions.

Let C be a nonempty convex subset of a normed space X, $T : C \to C$ be a mapping with $F(T) := \{x \in C : x = Tx\} \neq \emptyset$ and $\{x_n\}$ a sequence in C. Following Sahu [23], we say that $\{x_n\}$ has

(D1) *limit existence property* (for short, LE property) for T if $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$,

(D2) *approximate fixed point property* (for short, AF point property) for T if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$,

(D3) *LEAF point property* if $\{x_n\}$ has both LE property and AF point property.

The important feature of a sequence with AF point property for a mapping T is that any weak subsequential limit of the sequence is a fixed point of T under the assumption that $I - T$ is demiclosed at zero (see Lemma 2.6).

Let $T : C \to C$ be a nonexpansive mapping and $T_\alpha = (1 - \alpha)I + \alpha T$, $\alpha \in (0, 1)$, where I denotes the identity mapping. In 1955, Krasnosel'skiǐ [15] proved that if X is uniformly convex, then for each $x \in C$, $\{T_{\frac{1}{2}}^n x\}$ has the AF point property for T. Schaefer [26] proved that if X is uniformly convex, then for each $x \in C$, $\{T_\alpha^n x\}$ has the AF point property for T. Edelstein [6] observed that for Scheafer's result strict convexity of X suffices. The removal of strict convexity remained an open problem for many years. In 1976, this problem was solved by Ishikawa [10] as follows:

Theorem 1.1. *Let C be a nonempty subset of a normed space X and $T : C \to X$ be a nonexpansive mapping. For arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = M(x_n, \alpha_n, T), \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence satisfying

(i) $0 \leq \alpha_n \leq 1$ *for all* $n \in \mathbb{N}$, $\limsup_{n\to\infty} \alpha_n < 1$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{x_n\}$ *is contained in C.*

If $\{x_n\}$ is bounded, then $\{x_n\}$ has the AF point property.

It is obvious that for each $x \in C$, $\{T^n x\}$ has the *AF* point property for contraction mappings. Asymptotic regularity assumption of arbitrary nonlinear mapping T at $x \in C$, i.e., $\lim_{n \to \infty} \|T^n x - T^{n+1} x\| = 0$ provides the facility that $\{T^n x\}$ has the *AF* point property for T. Using this fact, Bose [3] initiated study of approximation of fixed points of asymptotically nonexpansive mappings in uniformly convex Banach space satisfying the Opial condition. In [28], Schu proved that the Mann iteration sequence enjoys the *AF* point property for asymptotically nonexpansive mappings on a uniformly convex Banach space. More precisely, he proved the following:

Theorem 1.2. *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the condition $\varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \in \mathbb{N}$ and for some $\varepsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by*

$$(1.1) \qquad x_{n+1} = M(x_n, \alpha_n, T^n), \quad n \in \mathbb{N}$$

has the AF point property for the mapping T.

Subsequently, Chang [4], Khan, Domlo and Fukhar-ud-din [12], Liu and Kang [17], Osilike and Aniagbosore [20], Rhoades [22], Tan and Xu [29] extended Schu's result, using variants of Mann and Ishikawa sequences in the setting of uniformly convex Banach spaces (see also [7, 11, 13, 14, 25]).

It is remarked that convexity (uniform convexity) condition is needed even on a metric space to establish approximate fixed point property for Ishikawa iterates of nonexpansive type mappings (see [11, 13]).

It is still an open problem whether or not the sequence $\{x_n\}$ defined by (1.1) has the *AF* point property for the class of asymptotically nonexpansive mappings in normed spaces without the uniform convexity. This brings us to the following question.

Question 1.3. Is it possible to extend Theorem 1.1 for a class of mappings more general than asymptotically nonexpansive mappings ?

In this paper, we deal with the problem of approximating fixed points of nearly asymptotically nonexpansive mappings in Banach spaces without uniform (strict) convexity. In particular, if $\{T_n\}$ is a sequence of self-mappings defined on a nonempty convex subset C of a normed space X satisfying

$$\|T_n x - T_n y\| \leq L_n(\|x - y\| + \rho_n)$$

for all $x, y \in C$ and $n \in \mathbb{N}$, and $\{x_n\}$ is a sequence in C defined by

$$x_{n+1} = M(x_n, \alpha_n, T_n), \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{L_n\}$ and $\{\rho_n\}$ satisfying suitable conditions, then we prove that $x_n - T_n x_n \to 0$ as $n \to \infty$. This provides an affirmative answer to Question 1.3. Moreover, it is shown that the sequence $\{x_n\}$ defined by (1.1) converges weakly to fixed points of uniformly continuous nearly asymptotically nonexpansive mappings on a Banach space without uniform convexity. Some strong convergence theorems for approximation of fixed points of nearly nonexpansive type mappings in Banach spaces are also proved. Our results improve several known convergence theorems in the setting of an arbitrary Banach space.

2. Preliminaries

Sahu [24] introduced the notion of nearly Lipschitzian mappings: Let C be a nonempty subset of a Banach space X and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : C \to C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$(2.1) \qquad \|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n)$$

for all $x, y \in C$. The infimum of constants k_n for which (2.1) holds is denoted by $\eta(T^n)$ and called the *nearly Lipschitz constant* of T^n.

A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n)\}$ is said to be
(4) *nearly nonexpansive* if $\eta(T^n) \leq 1$ for all $n \in \mathbb{N}$,
(5) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$,
(6) *nearly uniformly k-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,
(7) *nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Example 2.1. Let $X = \mathbb{R}$, $C = [0, 1]$ and $T : C \to C$ be a mapping defined by

$$Tx = \begin{cases} \frac{1}{2} & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 0 & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Clearly, T is a discontinuous and non-Lipschitzian mapping. However, it is a nearly nonexpansive mapping and hence nearly asymptotically nonexpansive.

Indeed, for a sequence $\{a_n\}$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$, we have

$$\|Tx - Ty\| \leq \|x - y\| + a_1$$

for all $x, y \in C$ and

$$\|T^n x - T^n y\| \leq \|x - y\| + a_n$$

for all $x, y \in C$ and $n \geq 2$ since $T^n x = \frac{1}{2}$ for all $x \in [0, 1]$ and $n \geq 2$.

Sahu [24] developed a nearly contraction mapping principle for the existence and uniqueness of fixed points of demicontinuous mappings, more general than contraction mappings, in Banach spaces. The details of the fixed point theory of nearly Lipschitzian mappings can be found in [1].

A Banach space X is said to satisfy the *Opial condition* ([19]) if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

for all $y \in X$ $(y \neq x)$.

Let X be a Banach space. Then the mapping $J : X \to 2^{X^*}$ defined by

$$J(u) = \{j \in X^* : \langle u, j \rangle = \|u\|^2, \|j\| = \|u\|\}, \quad u \in X$$

is called the *normalized duality mapping*. Suppose that J is single-valued. Then J is said to be *weakly sequentially continuous* if, for each $\{x_n\}$ in X with $x_n \rightharpoonup x$ ($\rightharpoonup$ denotes weak convergence), we have $J(x_n) \rightharpoonup^* J(x)$. It is well known that if X admits a weakly sequentially continuous duality mapping, then X satisfies the Opial condition.

A mapping $T : C \to X$ is said to be *demicompact* if for every bounded sequence $\{x_n\}$ in C such that $\{x_n - Tx_n\}$ converges strongly, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ that converges strongly to some $x^* \in C$.

Throughout this paper, the set of all weak subsequential limits of $\{x_n\}$ will be denoted by $\omega_w(\{x_n\})$. In what follows, we shall make use of the following known lemmas.

Lemma 2.2 ([5]). *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences in a normed space X. If there is a sequence $\{t_n\}$ of real number satisfying*

(i) $0 \leq t_n \leq t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) $a_{n+1} = (1 - t_n)a_n + t_n b_n$ *for all* $n \in \mathbb{N}$,
(iii) $\lim_{n \to \infty} \|a_n\| = a$,
(iv) $\limsup_{n \to \infty} \|b_n\| \leq a$ *and* $\{\sum_{i=1}^{n} t_i b_i\}$ *is bounded,*

then $a = 0$.

Lemma 2.3 ([20]). *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that $\beta_n \geq 1$ and $\delta_{n+1} \leq \beta_n \delta_n + \gamma_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n \to \infty} \delta_n$ exists.*

Lemma 2.4 ([30]). *Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences satisfying*

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n \to \infty} \gamma_n \leq 0$.

Let $\{\lambda_n\}$ be a sequence of nonnegative numbers which satisfies the inequality

$$\lambda_{n+1} \leq (1 - \alpha_n)\lambda_n + \alpha_n \gamma_n, \quad n \in \mathbb{N}.$$

Then $\lim_{n \to \infty} \lambda_n = 0$.

The following lemmas play important role in approximation of fixed points of nonlinear operators.

Lemma 2.5 ([1, Theorem 5.6.3]). *Let X be a Banach space with a weakly continuous duality mapping $J_\varphi : X \to X^*$ with gauge function φ. Let C be a nonempty closed convex subset of X and $T : C \to C$ be a nearly asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero, i.e., for $\{x_n\}$ in C converging weakly to x and $\{(I - T)x_n\}$ converging strongly to 0, we have $(I - T)x = 0$.*

Lemma 2.6 ([2, Lemma 2.10]). *Let X be a reflexive Banach space satisfying the Opial condition, C be a nonempty closed convex subset of X and $T : C \to X$ be a mapping such that*

(i) $F(T) \neq \emptyset$,
(ii) $I - T$ *is demiclosed at zero.*

If a sequence $\{x_n\}$ in C has LEAF point property for T, then $\{x_n\}$ converges weakly to a fixed point of T.

3. Weak convergence theorems

Our first result extends Theorem 1 of Deng [5] and the corresponding result of Ishikawa [10] for a sequence of nearly asymptotically nonexpansive mappings.

Theorem 3.1. *Let C be a nonempty convex subset of a normed space X and for each $n \in \mathbb{N}$, let $T_n : C \to C$ be a mapping satisfying*

$$\|T_n x - T_n y\| \leq L_n(\|x - y\| + \rho_n)$$

for all $x, y \in C$ and $n \in \mathbb{N}$, where $\{(\rho_n, L_n)\}$ is a sequence in $[0, \infty) \times [1, \infty)$ such that $\sum \rho_n < \infty$ and $\sum_{n=1}^\infty (L_n - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. For arbitrary $x_1 \in C$, define a sequence $\{x_n\}$ in C by

$$(3.1) \qquad x_{n+1} = M(x_n, \alpha_n, T_n), \quad n \in \mathbb{N}.$$

Suppose that $\{x_n\}$ is bounded and $\sum_{n=1}^\infty \|T_n x_n - T_{n+1} x_n\| < \infty$. Then we have the following:

(a) $(I - T_n)x_n \to 0$ *as $n \to \infty$,*
(b) $x_n - x_{n+1} \to 0$ *as $n \to \infty$.*

Proof. (a) Set $d_n := x_n - T_n x_n$ and $\varepsilon_n := \|T_n x_n - T_{n+1} x_n\|$ for all $n \in \mathbb{N}$. By (3.1), we have

$$
\begin{aligned}
\|d_{n+1}\| &= \|x_{n+1} - T_{n+1} x_{n+1}\| \\
&\leq (1 - \alpha_n)\|x_n - T_{n+1} x_{n+1}\| + \alpha_n\|T_n x_n - T_{n+1} x_{n+1}\| \\
&\leq (1 - \alpha_n)(\|x_n - x_{n+1}\| + \|x_{n+1} - T_{n+1} x_{n+1}\|) \\
&\quad + \alpha_n(\|T_n x_n - T_{n+1} x_n\| + \|T_{n+1} x_n - T_{n+1} x_{n+1}\|) \\
&\leq (1 - \alpha_n)(\alpha_n\|d_n\| + \|d_{n+1}\|) \\
&\quad + \alpha_n(\varepsilon_n + L_{n+1}(\|x_n - x_{n+1}\| + \rho_n)),
\end{aligned}
$$

which implies that

$$\|d_{n+1}\| \leq (1-\alpha_n)\|d_n\| + \varepsilon_n + L_{n+1}(\alpha_n\|d_n\| + \rho_n)$$
$$\leq L_{n+1}\|d_n\| + \varepsilon_n + K\rho_n,$$

where $K = \sup_{n\in\mathbb{N}} L_n$. Since $\sum_{n=1}^{\infty}(\varepsilon_n + K\rho_n) < \infty$, it follows from Lemma 2.3 that $\lim_{n\to\infty}\|d_n\|$ exists.

Let $\lim_{n\to\infty}\|d_n\| = d$ and set $b_n := \alpha_n^{-1}(T_n x_n - T_{n+1} x_{n+1})$. Then

$$d_{n+1} = (1-\alpha_n)d_n + \alpha_n b_n.$$

Observe that

$$\|b_n\| = \alpha_n^{-1}\|T_n x_n - T_{n+1} x_{n+1}\|$$
$$\leq \alpha_n^{-1}(\|T_n x_n - T_{n+1} x_n\| + \|T_{n+1} x_n - T_{n+1} x_{n+1}\|)$$
$$\leq \alpha_n^{-1}(\varepsilon_n + L_{n+1}(\|x_n - x_{n+1}\| + \rho_n))$$
$$\leq \alpha_n^{-1}(\varepsilon_n + L_{n+1}(\alpha_n\|d_n\| + \rho_n))$$
$$\leq L_{n+1}\|d_n\| + a^{-1}(\varepsilon_n + L_{n+1}\rho_n),$$

which implies that $\limsup_{n\to\infty}\|b_n\| \leq d$. Moreover, for all $n \in \mathbb{N}$, we have

$$\left\|\sum_{i=1}^{n}\alpha_i b_i\right\| = \left\|\sum_{i=1}^{n}(T_i x_i - T_{i+1} x_{i+1})\right\|$$
$$= \|T_1 x_1 - T_{n+1} x_{n+1}\|$$
$$\leq \|T_1 x_1 - x_n\| + \|x_n - T_n x_n\| + \|T_n x_n - T_{n+1} x_n\|$$
$$\quad + \|T_{n+1} x_n - T_{n+1} x_{n+1}\|$$
$$\leq \|T_1 x_1 - x_n\| + \|d_n\| + \varepsilon_n + L_{n+1}(\alpha_n\|d_n\| + \rho_n).$$

Note that $\{\|d_n\|\}$, $\{L_n\}$, $\{\varepsilon_n\}$, $\{\rho_n\}$ are convergent sequences and $\{x_n\}$ is bounded. So $\{\sum_{i=1}^{n}\alpha_i b_i\}$ is bounded. Since $\{\alpha_n\}$ is bounded away from 0 and 1, it follows that $\sum_{n=1}^{\infty}\alpha_n = \infty$. Applying Lemma 2.2, we conclude that $(I - T_n)x_n \to 0$ as $n \to \infty$.

(b) From (3.1), we have

$$(3.2) \qquad \begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n\|x_n - T_n x_n\| \\ &\leq b\|x_n - T_n x_n\|, \quad n \in \mathbb{N}. \end{aligned}$$

By part (a), we have $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$. It follows from (3.2) that $x_{n+1} - x_n \to 0$ as $n \to \infty$. This completes the proof. $\qquad\square$

Remark 3.2. (1) Conditions similar to $\sum_{n=1}^{\infty}\|T_n x_n - T_{n+1} x_n\| < \infty$ have been imposed in the literature. Reich [21] imposed the condition $\sum_{n=0}^{\infty}\alpha_n^2\|Tx_n\|^2 < \infty$, where $\{\alpha_n\}$ is a real sequence in $(0,1)$, satisfying appropriate conditions to prove the convergence of the Mann iteration process to the solution of an operator equation involving a strongly accretive operator T on a uniformly smooth Banach space.

(2) Observe that if each $T_n = T$ is nonexpansive, then this condition is trivially satisfied.

Next result shows that the sequence $\{x_n\}$ generated by modified Mann iteration sequence enjoys the AF point property for the class of nearly asymptotically nonexpansive mappings on an arbitrary normed space.

Theorem 3.3. *Let C be a nonempty convex subset of a normed space X and $T : C \to C$ be a nearly asymptotically nonexpansive mappings with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. For arbitrary $x_1 \in C$, define a sequence $\{x_n\}$ in C by*

$$(3.3) \qquad x_{n+1} = M(x_n, \alpha_n, T^n), \quad n \in \mathbb{N}.$$

Suppose that $\{x_n\}$ is bounded and $\sum_{n=1}^{\infty} \|(I - T)T^n x_n\| < \infty$. Then we have the following:

(a) $\lim_{n \to \infty} \|x_n - T^n x_n\| = 0$,
(b) *If T is uniformly continuous, then the sequence $\{x_n\}$ has the AF point property for T.*

Proof. (a) Set $T_n := T^n$ for all $n \in \mathbb{N}$. Then the result follows from Theorem 3.1(a).

(b) Since T is uniformly continuous and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$, it follows that

$$\|T^{n+1} x_n - T x_n\| \to 0 \quad \text{as } n \to \infty.$$

So,

$$\|x_n - T x_n\| \leq \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + \|T^{n+1} x_n - T x_n\|$$
$$\to 0 \quad \text{as } n \to \infty.$$

This completes the proof. □

Remark 3.4. Theorem 3.3 improves results of Chang [4], Liu and Kang [17], Osilike and Aniagbosore [20], Rhoades [22], Schu [27, 28], Tan and Xu [29] and others in the sense of computability of key term $\lim_{n \to \infty} \|x_n - T x_n\| = 0$ in the following ways.

(1) The requirement of uniform convexity is dropped.
(2) The assumption $F(T) \neq \emptyset$ is not required.
(3) The mapping T may not be Lipschitzian.

Corollary 3.5. *Let C be a nonempty convex subset of a normed space X and $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $0 < a \leq \alpha_n \leq b < 1$. For arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (3.3). Suppose that $\{x_n\}$ is bounded and $\sum_{n=1}^{\infty} \|(I - T)T^n x_n\| < \infty$. Then $(I - T)x_n \to 0$ as $n \to \infty$.*

Proof. Since every asymptotically nonexpansive mapping is uniformly continuous, the result follows from Theorem 3.3(b). □

As a direct consequence of Theorem 3.1(b), we have:

Corollary 3.6. *Let C be a nonempty convex subset of a normed space X and $T : C \to C$ be a nonexpansive mapping. For $\alpha \in (0,1)$, assume that $\{T_\alpha^n x\}$ is bounded for some $x \in C$. Then T_α is asymptotically regular at x, i.e., $\lim_{n\to\infty} \|(I - T_\alpha)T_\alpha^n x\| = 0$.*

In case of an asymptotically nonexpansive mapping T, T_α is not necessarily asymptotically regular. However, $T_{n,\alpha} := (1-\alpha)I + \alpha T^n$ satisfies a certain asymptotic regularity type condition which gives partial answer to the question of Lim and Xu ([16, Remark 1]) as follows:

Corollary 3.7. *Let C be a nonempty convex subset of normed space X and $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. For an arbitrary $x_1 \in C$ and $\alpha \in (0,1)$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = T_{n,\alpha}x_n, \quad n \in \mathbb{N}.$$

Suppose that $\{x_n\}$ is bounded and $\sum_{n=1}^{\infty} \|(I-T)T^n x_n\| < \infty$. Then

$$\lim_{n\to\infty} \|T_{n,\alpha} \cdots T_{2,\alpha}T_{1,\alpha}x_1 - T_{n+1,\alpha} \cdots T_{2,\alpha}T_{1,\alpha}x_1\| = 0.$$

The following result extends [5, Theorem 2] from the class of nonexpansive mappings to the class of nearly asymptotically nonexpansive mappings.

Theorem 3.8. *Let X be a Banach space satisfying the Opial condition, C be a nonempty weakly compact convex subset of X and $T : C \to C$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ such that*

   (i) *$\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$,*
   (ii) *$I - T$ is demiclosed at zero.*

Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $0 < a \le \alpha_n \le b < 1$ for all $n \in \mathbb{N}$. For arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (3.3) and assume that $\sum_{n=1}^{\infty} \|(I-T)T^n x_n\| < \infty$. Then $\{x_n\}$ converges weakly to an element of $F(T)$.

Proof. By Theorem 3.3(b), $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. As $\{x_n\}$ is bounded, so there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$. By demiclosedness of $I - T$ at zero, we have $v \in F(T)$. From (3.3), we have

$$
\begin{aligned}
\|x_{n+1} - v\| &\le (1-\alpha_n)\|x_n - v\| + \alpha_n \|T^n x_n - v\| \\
&\le (1-\alpha_n)\|x_n - v\| + \alpha_n(\|x_n - v\| + a_n) \\
&\le \|x_n - v\| + a_n, \quad n \in \mathbb{N}.
\end{aligned}
$$

(3.4)

By Lemma 2.3, we obtain that $\lim_{n\to\infty} \|x_n - v\|$ exists. Suppose not, that is, $\{x_n\}$ does not converge weakly to v. Then $w_\omega(\{x_n\})$ is not singleton. There

exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in C$ with $z \neq v$. It is easy to see that $z \in F(T)$ and $\lim_{n \to \infty} \|x_n - z\|$ exists. Since X has the Opial condition

$$\lim_{n \to \infty} \|x_n - v\| = \lim_{i \to \infty} \|x_{n_i} - v\| < \lim_{i \to \infty} \|x_{n_i} - z\| = \lim_{n \to \infty} \|x_n - z\|,$$
$$\lim_{n \to \infty} \|x_n - z\| = \lim_{j \to \infty} \|x_{n_j} - z\| < \lim_{j \to \infty} \|x_{n_j} - v\| = \lim_{n \to \infty} \|x_n - v\|,$$

a contradiction. Hence the result follows. $\qquad\square$

An asymptotically nonexpansive mapping is a nearly asymptotically nonexpansive mapping, so we have:

Corollary 3.9. *Let X be a Banach space satisfying the Opial condition, C be a nonempty weakly compact convex subset of X and $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that*

(i) $\sum_{n=1}^{\infty}(k_n - 1) < \infty,$
(ii) $I - T$ *is demiclosed at zero.*

If $\{\alpha_n\}$, $\{x_n\}$ and $\sum_{n=1}^{\infty} \|(I - T)T^n x_n\| < \infty$ are same as in Theorem 3.8, then $\{x_n\}$ converges weakly to an element of $F(T)$.

Remark 3.10. If X is a Banach space having weakly continuous duality mapping, then, by Lemma 2.5, the assumption "$I - T$ is demiclosed at zero" in Theorem 3.8 and Corollary 3.9 can be dropped.

4. Strong convergence theorems

In this section, we establish strong convergence theorems for nearly nonexpansive type mappings in an arbitrary Banach space.

Theorem 4.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{a_n, \eta(T^n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $0 < a \le \alpha_n \le b < 1$ for all $n \in \mathbb{N}$. Assume that the sequence $\{x_n\}$ in C defined by (3.3) is bounded and $\sum_{n=1}^{\infty} \|(I-T)T^n x_n\| < \infty$. Suppose that T^m is demicompact for some $m \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to an element of $F(T)$.*

Proof. Since T is uniformly continuous and $x_n - Tx_n \to 0$ as $n \to \infty$ by Theorem 3.3(b), therefore, we have $\|T^i x_n - T^{i+1} x_n\| \to 0$ as $n \to \infty$ for all $i = 0, 1, 2, \ldots$. It follows that

$$\|x_n - T^m x_n\| \le \sum_{i=0}^{m-1} \|T^i x_n - T^{i+1} x_n\| \to 0 \quad \text{as } n \to \infty.$$

By the demicompactness of T^m, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $v \in C$ such that

$$\lim_{i \to \infty} x_{n_i} = \lim_{i \to \infty} T^m x_{n_i} = v.$$

As $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ by Theorem 3.3(b), so it follows from continuity of T that $v \in F(T)$. Now $\lim_{n\to\infty} \|x_n - v\|$ exists by (3.4) and Lemma 2.3 for $v \in F(T)$, so we conclude that $\{x_n\}$ converges strongly to v. $\square$

We now turn our attention to approximate fixed points of nearly nonexpansive type mappings.

Theorem 4.2. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a nearly uniformly k-contraction mapping with sequence $\{a_n\}$ such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$. Then, for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ in C defined by (3.3) converges strongly to the unique fixed point of T.*

Proof. Let $p, q \in F(T)$. Then

$$\|p - q\| = \|T^n p - T^n q\| \leq k\|p - q\| + a_n, \quad n \in \mathbb{N}.$$

It follows that $(1 - k)\|p - q\| \leq a_n \to 0$ as $n \to \infty$. Thus, T has a unique fixed point. Suppose that $F(T) = \{v\}$ for some $v \in C$. Then from (3.3), we have

$$\begin{aligned}
\|x_{n+1} - v\| &\leq (1 - \alpha_n)\|x_n - v\| + \alpha_n\|T^n x_n - v\| \\
&\leq (1 - \alpha_n)\|x_n - v\| + \alpha_n(k\|x_n - v\| + a_n) \\
&\leq (1 - (1 - k)\alpha_n)\|x_n - v\| + a_n, \quad n \in \mathbb{N}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$, we conclude from Lemma 2.4 that $\{x_n\}$ converges strongly to the unique fixed point v of T. $\square$

Recently, the first author [23] introduced the normal S-iteration process as follows:
Let C be a convex subset of a linear space X and $T : C \to C$ a mapping. Then the normal S-iteration process is defined by

$$x_{n+1} = T[(1 - \alpha_n)x_n + \alpha_n Tx_n], \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying appropriate conditions.

We note that the normal S-iteration process is useful for approximating fixed points of nonexpansive mappings and at the same time, the rate of convergence of the normal S-iteration process is faster than Picard iteration process (and hence Mann iteration process) for contraction mappings (for details see Sahu [23]).

We now apply modified normal S-iteration process for approximating fixed points of nearly uniformly k-contraction mappings.

Theorem 4.3. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a nearly uniformly k-contraction mapping with sequence $\{a_n\}$ such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$*

and $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$. *Then, for arbitray $x_1 \in C$, the sequence $\{x_n\}$ in C defined by*

$$(4.1) \qquad x_{n+1} = T^n[(1 - \alpha_n)x_n + \alpha_n T^n x_n], \quad n \in \mathbb{N}$$

converges strongly to the unique fixed point of T.

Proof. As in the proof Theorem 4.2, we can show that T has a unique fixed point. Suppose that $F(T) = \{v\}$ for some $v \in C$. From (4.1), we have

$$\begin{aligned}
\|x_{n+1} - v\| &= \|T^n[(1 - \alpha_n)x_n + \alpha_n T^n x_n] - T^n v\| \\
&\le k\|(1 - \alpha_n)x_n + \alpha_n T^n x_n - v\| + a_n \\
&\le k[(1 - \alpha_n)\|x_n - v\| + \alpha_n\|T^n x_n - v\|] + a_n \\
&\le k[(1 - \alpha_n)\|x_n - v\| + \alpha_n(k\|x_n - v\| + a_n)] + a_n \\
&\le k(1 - (1-k)\alpha_n)\|x_n - v\| + 2a_n, \quad n \in \mathbb{N}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 0$, we conclude from Lemma 2.4 that $\{x_n\}$ converges strongly to the unique fixed point v of T. $\square$

Theorem 4.3 is an improvement of Theorem 3.7 in [2] as it is independent of $\sum_{n=1}^{\infty} a_n < \infty$ and v is unique.

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