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SELF-RECIPROCAL POLYNOMIALS WITH RELATED MAXIMAL ZEROS

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Abstract. For each real number $n > 6$, we prove that there is a sequence $\{p_k(n,z)\}_{k=1}^{\infty}$ of fourth degree self-reciprocal polynomials such that the zeros of $p_k(n,z)$ are all simple and real, and every $p_{k+1}(n,z)$ has the largest (in modulus) zero $\alpha\beta$ where α and β are the first and the second largest (in modulus) zeros of $p_k(n,z)$, respectively. One such sequence is given by $p_k(n,z)$ so that

$$p_k(n,z) = z^4 - q_{k-1}(n)\, z^3 + (q_k(n)+2)\, z^2 - q_{k-1}(n)\, z + 1,$$

where $q_0(n) = 1$ and other $q_k(n)$'s are polynomials in n defined by the severely nonlinear recurrence

$$4q_{2m-1}(n) = q_{2m-2}^2(n) - (4n+1)\prod_{j=0}^{m-2} q_{2j}^2(n),$$

$$4q_{2m}(n) = q_{2m-1}^2(n) - (n-2)(n-6)\prod_{j=0}^{m-2} q_{2j+1}^2(n)$$

for $m \geq 1$, with the usual empty product conventions, i.e., $\prod_{j=0}^{-1} b_j = 1$.

1. Introduction and statements of results

There are infinitely many sequences of monic integral polynomials

$$p_1(z),\ p_2(z),\ p_3(z),\ \ldots$$

whose largest (in modulus) zero of $p_{k+1}(z)$ is $\alpha\beta$ where α and β are the first two largest (in modulus) zeros of $p_k(z)$. An example is taken when $p_1(z)$ is the minimal polynomial of a Salem number in which case we can take $p_k(z) = p_1(z)$ for all $k \geq 1$ because a Salem number is a real algebraic integer > 1 all of whose conjugates lie inside or on the unit circle, and at least one of these conjugates has modulus exactly 1. It is known that there are infinitely many Salem numbers. It does not seem obvious how to find such a sequence of

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distinct polynomials each of which has the same degree. We exhibit here an explicit such sequence consisting of monic self-reciprocal polynomials of degree 4 with integer coefficients. For reference about self-reciprocal polynomial, see Chapter 7 of [1].

Consider the self-reciprocal polynomial

$$z^4 - z^3 - kz^2 - z + 1.$$

One may check that, for $0 \le k \le 4$, it has at least two zeros on the unit circle. But, for $k > 4$, no zeros lie on the unit circle. As a generalization of this, we define, for a real number $n > 6$,

$$p_1(n, z) = z^4 - z^3 - (n-2)z^2 - z + 1.$$

In this paper we show the following.

Theorem 1. *For each real number $n > 6$, there is a sequence $\{p_k(n, z)\}_{k=1}^{\infty}$ of fourth degree self-reciprocal polynomials such that the zeros of $p_k(n, z)$ are all simple and real, and every $p_{k+1}(n, z)$ has the largest (in modulus) zero $\alpha\beta$ where α and β are the first and the second largest (in modulus) zeros of $p_k(n, z)$, respectively. One such sequence is given by $p_k(n, z)$ so that*

$$p_k(n, z) = z^4 - q_{k-1}(n)\, z^3 + (q_k(n) + 2)\, z^2 - q_{k-1}(n)\, z + 1,$$

where $q_0(n) = 1$ and other $q_k(n)$'s are polynomials in n defined by the severely nonlinear recurrence

$$(1)\qquad \begin{aligned} 4q_{2m-1}(n) &= q_{2m-2}^2(n) - (4n+1) \prod_{j=0}^{m-2} q_{2j}^2(n), \\ 4q_{2m}(n) &= q_{2m-1}^2(n) - (n-2)(n-6) \prod_{j=0}^{m-2} q_{2j+1}^2(n) \end{aligned}$$

for $m \ge 1$, with the usual empty product conventions, i.e., $\prod_{j=0}^{-1} b_j = 1$.

2. Proof of Theorem 1

We define

$$u_k(n) = \begin{cases} 4n+1 & \text{for } k \text{ odd}, \\ (n-2)(n-6) & \text{for } k \text{ even}. \end{cases}$$

Then one may write (1) as

$$(2)\qquad 4q_k(n) = q_{k-1}^2(n) - u_k(n) \prod_{j=1}^{\left[\frac{k-1}{2}\right]} q_{k-2j-1}^2(n)$$

for $k \ge 1$. Note that $u_k(n) > 0$ for $n > 6$, and $u_k(n) = u_{k+2}(n)$ for any integer $k \ge 1$. For notational convenience, write $q_k,\ u_k$ instead of $q_k(n),\ u_k(n)$ when n is irrelevant to the context. The values of q_k for $0 \le k \le 5$ are as follows.

$$q_0 = 1,$$

$$\begin{aligned} q_1 &= -n, \\ q_2 &= 2n - 3, \\ q_3 &= n^2 - 4n + 2, \\ q_4 &= 2n^2 - 4n + 1, \\ q_5 &= n^4 - 8n^3 + 16n^2 - 8n - 2. \end{aligned}$$

We begin by establishing some properties of the q_k polynomials.

Lemma 2. *For any integer $k \geq 1$,*

$$q_{k+1}^2 - 4q_{k+2} = q_{k-1}^2(q_{k-1}^2 - 4q_k). \tag{3}$$

Proof. From the equation (2) with $k+2$, we have

$$\begin{aligned} q_{k+1}^2 - 4q_{k+2} &= u_{k+2} \prod_{j=1}^{\left[\frac{k+1}{2}\right]} q_{k-2j+1}^2 \\ &= \left(u_k \prod_{j=1}^{\left[\frac{k-1}{2}\right]} q_{k-2j-1}^2 \right) q_{k-1}^2 = (q_{k-1}^2 - 4q_k)\; q_{k-1}^2. \end{aligned}$$

□

Lemma 3. *For any integer $k \geq 1$, we have*

(i) $q_{k+1} = q_{k-1}^2 - 2q_k - 4$,

(ii) $4q_k < q_{k-1}^2 < 2q_{k+1} + 8$ *for* $n > 6$.

Proof. To prove (i), we use an induction on k. From

$$q_0^2 - 2q_1 - 4 = 1 + 2n - 4 = 2n - 3 = q_2,$$

we observe that (i) holds for $k = 1$. Assume $q_{k+1} = q_{k-1}^2 - 2q_k - 4$. Then by (3),

$$\begin{aligned} 4q_{k+2} &= q_{k+1}^2 - q_{k-1}^2(q_{k-1}^2 - 4q_k) \\ &= (q_{k-1}^2 - 2q_k - 4)^2 - q_{k-1}^4 + 4q_{k-1}^2 q_k \\ &= 4q_k^2 - 8q_{k-1}^2 + 16q_k + 16 = 4q_k^2 - 8(q_{k-1}^2 - 2q_k - 4) - 16 \\ &= 4q_k^2 - 8q_{k+1} - 16 = 4(q_k^2 - 2q_{k+1} - 4). \end{aligned}$$

For (ii), it follows from equation (2) that

$$q_{k-1}^2 - 4q_k = u_k \prod_{j=1}^{\left[\frac{k-1}{2}\right]} q_{k-2j-1}^2 > 0, \tag{4}$$

and this implies the first inequality in (ii). Now applying (i) and (4), we obtain

$$\begin{aligned} 2q_{k+1} + 8 - q_{k-1}^2 &= 2(q_{k-1}^2 - 2q_k - 4) + 8 - q_{k-1}^2 \\ &= q_{k-1}^2 - 4q_k > 0, \end{aligned}$$

which proves the second inequality in (ii). □

Lemma 4. *If $n > 6$, then for any integer $k \geq 2$, we have*

$$9 < q_k < q_{k+1}.$$

Proof. First, we show that $9 < q_k$ by using induction on k. Obviously, $q_2 = 2n - 3 > 9$ since $n > 6$. Let us assume $9 < q_k$. Then by Lemma 3(ii), we have the inequality $2q_{k+1} + 8 > 4q_k > 36$ or $q_{k+1} > 14 > 9$ which completes the induction. And again Lemma 3(ii) says that $4q_k < 2q_{k+1} + 8 < 4q_{k+1}$ or $q_k < q_{k+1}$. □

Lemma 5. *For $n > 6$, we have*

$$(4q_k - q_{k-1}^2)^2((q_k + 4)^2 - (2q_{k-1})^2) \neq 0.$$

Proof. By Lemma 3(ii), $4q_k \neq q_{k-1}^2$ is obvious. Next, we show $(q_k + 4)^2 - (2q_{k-1})^2 > 0$. For $k = 1$, $(q_1 + 4)^2 - (2q_0)^2 = (n - 4)^2 - 4 > 0$ since $n > 6$. And for $k = 2$, again we see that $(q_2 + 4)^2 - (2q_1)^2 = (2n + 1)^2 - 4n^2 > 0$. For $k \geq 3$, since $q_k + 4 > 2q_{k-1} > 0$ by Lemma 3(ii) and Lemma 4, we have $(q_k + 4)^2 - (2q_{k-1})^2 > 0$. □

We are now ready to prove Theorem 1.

Proof. The resultant of $p_k(n, z)$ and $p'_k(n, z)$ in z is

$$\begin{aligned} &(4q_k - q_{k-1}^2)^2(q_k + 2q_{k-1} + 4)(q_k - 2q_{k-1} + 4) \\ &= (4q_k - q_{k-1}^2)^2((q_k + 4)^2 - (2q_{k-1})^2) \end{aligned}$$

which is nonzero by Lemma 5. So all zeros of $p_k(n, z)$ are simple. Let

$$w = z + \frac{1}{z}.$$

Then

$$\begin{aligned} p_k(n, z) &= z^2 \left(\left(z^2 + \frac{1}{z^2} \right) - q_{k-1} \left(z + \frac{1}{z} \right) + (q_k + 2) \right) \\ &= z^2(w^2 - q_{k-1}w + q_k), \end{aligned}$$

and so the zeros of $p_k(n, z)$ satisfy

$$z^2 - wz + 1 = 0,$$

where

$$w = w_{k,0} = \frac{1}{2} \left(q_{k-1} + \sqrt{q_{k-1}^2 - 4q_k} \right)$$

or

$$w = w_{k,1} = \frac{1}{2} \left(q_{k-1} - \sqrt{q_{k-1}^2 - 4q_k} \right).$$

We note that all zeros of $p_k(n, z)$ are real because the discriminant of the quadratic equation $z^2 - wz + 1 = 0$ is

$$w^2 - 4 = \frac{1}{2}q_{k-1}^2 - q_k - 4 \pm \frac{1}{2}q_{k-1}\sqrt{q_{k-1}^2 - 4q_k}$$

$$= \frac{1}{2}\left((q_{k-1}^2 - 2q_k - 8) \pm q_{k-1}\sqrt{q_{k-1}^2 - 4q_k}\right)$$
$$> \frac{1}{2}\left((q_{k-1}^2 - 2q_k - 8) - \left|q_{k-1}\sqrt{q_{k-1}^2 - 4q_k}\right|\right)$$

and by Lemmas 3 and 4,

$$q_{k-1}^2 - 2q_k - 8 = q_{k+1} - 4 > 0 \quad \text{and}$$
$$(q_{k-1}^2 - 2q_k - 8)^2 - \left|q_{k-1}\sqrt{q_{k-1}^2 - 4q_k}\right|^2$$
$$= 4q_k^2 - 16(q_{k-1}^2 - 2q_k - 4) = 4(q_k^2 - 4q_{k+1}) > 0.$$

We now prove that

$$w_{1,0} > 0,\ \ w_{1,1} < 0,\ \ w_{2,0} < 0,\ \ w_{2,1} < 0, \quad \text{and}$$
$$w_{k,0} > 0,\ \ w_{k,1} > 0 \quad \text{for}\ \ k \geq 3,$$

and that the product of the first two largest (in modulus) zeros of $p_k(n, z)$ is

$$\frac{1}{4}\left(w_{1,0} + \sqrt{w_{1,0}^2 - 4}\right)\left(w_{1,1} - \sqrt{w_{1,1}^2 - 4}\right),$$
$$\frac{1}{4}\left(w_{2,0} - \sqrt{w_{2,0}^2 - 4}\right)\left(w_{2,1} - \sqrt{w_{2,1}^2 - 4}\right),$$
$$\frac{1}{4}\left(w_{k,0} + \sqrt{w_{k,0}^2 - 4}\right)\left(w_{k,1} + \sqrt{w_{k,1}^2 - 4}\right), \quad k \geq 3,$$

respectively. By direct calculation, we have

$$w_{1,0} > 0 > w_{1,1} > -w_{1,0} \quad \text{and} \quad w_{2,1} < w_{2,0} < 0.$$

For $k \geq 3$, the fact $w_{k,0} > w_{k,1} > 0$ follows immediately from

$$q_{k-1} > \sqrt{q_{k-1}^2 - 4q_k} > 0.$$

Thus, for $k \geq 3$, the first two largest zeros of $p_k(n, z)$ are the larger positive zero of

$$z^2 - w_{k,0}z + 1 = 0$$

and the larger positive zero of

$$z^2 - w_{k,1}z + 1 = 0.$$

This is because the product of the two zeros of each equation equals 1 and all of the four zeros are positive (since $w_{k,0} > w_{k,1} > 0$). Hence the product the first two largest zeros of $p_k(n, z)$ is

$$\frac{1}{4}\left(w_{k,0} + \sqrt{w_{k,0}^2 - 4}\right)\left(w_{k,1} + \sqrt{w_{k,1}^2 - 4}\right).$$

So, since $w_{k,0} > w_{k,1}$, it is enough to show that

$$\frac{1}{2}\left(w_{k+1,0} + \sqrt{w_{k+1,0}^2 - 4}\right) = \frac{1}{4}\left(w_{k,0} + \sqrt{w_{k,0}^2 - 4}\right)\left(w_{k,1} + \sqrt{w_{k,1}^2 - 4}\right).$$

On the other hand, for $k=1,2$, we need to show that

$$\frac{1}{2}\left(w_{2,1}-\sqrt{w_{2,1}^2-4}\right)=\frac{1}{4}\left(w_{1,0}+\sqrt{w_{1,0}^2-4}\right)\left(w_{1,1}-\sqrt{w_{1,1}^2-4}\right),$$
$$\frac{1}{2}\left(w_{3,0}+\sqrt{w_{3,0}^2-4}\right)=\frac{1}{4}\left(w_{2,0}-\sqrt{w_{2,0}^2-4}\right)\left(w_{2,1}-\sqrt{w_{2,1}^2-4}\right)$$

or

$$2\left(w_{2,1}-\sqrt{w_{2,1}^2-4}\right)=\left(w_{1,0}+\sqrt{w_{1,0}^2-4}\right)\left(w_{1,1}-\sqrt{w_{1,1}^2-4}\right),$$
$$2\left(w_{3,0}+\sqrt{w_{3,0}^2-4}\right)=\left(w_{2,0}-\sqrt{w_{2,0}^2-4}\right)\left(w_{2,1}-\sqrt{w_{2,1}^2-4}\right).$$

Note that

$$w_{1,0}=\frac{1+\sqrt{4n+1}}{2},\qquad w_{1,1}=\frac{1-\sqrt{4n+1}}{2},$$
$$w_{2,0}=\frac{-n+\sqrt{n^2-8n+12}}{2},\quad w_{2,1}=\frac{-n-\sqrt{n^2-8n+12}}{2},$$
$$w_{3,0}=\frac{2n-3+\sqrt{4n+1}}{2},\qquad w_{3,1}=\frac{2n-3-\sqrt{4n+1}}{2}.$$

So, with a little calculation, we have

$$\begin{aligned}
&\left(w_{1,0}+\sqrt{w_{1,0}^2-4}\right)\left(w_{1,1}-\sqrt{w_{1,1}^2-4}\right)\\
&=w_{1,0}\cdot w_{1,1}-\sqrt{(w_{1,0}^2-4)(w_{1,1}^2-4)}+w_{1,1}\sqrt{w_{1,0}^2-4}-w_{1,0}\sqrt{w_{1,1}^2-4}\\
&=\ -n-\sqrt{n^2-8n+12}\\
&\quad-\left(\sqrt{n^2-4n-2+2\sqrt{4n+1}}+\sqrt{n^2-4n-2-2\sqrt{4n+1}}\right)\\
&=2w_{2,1}-\sqrt{2(n^2-4n-2)+2\sqrt{(n^2-4n-2)^2-\left(2\sqrt{4n+1}\right)^2}}\\
&=2w_{2,1}-\sqrt{2(n^2-4n-2)+2n\sqrt{n^2-8n+12}}\\
&=2\left(w_{2,1}-\sqrt{w_{2,1}^2-4}\right).
\end{aligned}$$

The fact that $\sqrt{A+B}+\sqrt{A-B}=\sqrt{2A+2\sqrt{A^2-B^2}}$ is used in the third equality. And similarly we have

$$\begin{aligned}
&\left(w_{2,0}-\sqrt{w_{2,0}^2-4}\right)\left(w_{2,1}-\sqrt{w_{2,1}^2-4}\right)\\
&=w_{2,0}\cdot w_{2,1}+\sqrt{(w_{2,0}^2-4)(w_{2,1}^2-4)}-w_{2,1}\sqrt{w_{2,0}^2-4}-w_{2,0}\sqrt{w_{2,1}^2-4}\\
&=2n-3+\sqrt{4n+1}+\sqrt{2n^2-4n-3-2n\sqrt{n^2-8n+12}}\\
&\quad+\sqrt{2n^2-4n-3+2n\sqrt{n^2-8n+12}}
\end{aligned}$$

$$= 2w_{3,0} + \sqrt{2(2n^2-4n-3) + 2\sqrt{(2n^2-4n-3)^2 - \left(2n\sqrt{n^2-8n+12}\right)^2}}$$
$$= 2w_{3,0} + \sqrt{2(2n^2-4n-3) + 2(2n-3)\sqrt{4n+1}}$$
$$= 2\left(w_{3,0} + \sqrt{w_{3,0}^2 - 4}\right).$$

Finally, we prove

$$\frac{1}{2}\left(w_{k+1,0} + \sqrt{w_{k+1,0}^2 - 4}\right) = \frac{1}{4}\left(w_{k,0} + \sqrt{w_{k,0}^2 - 4}\right)\left(w_{k,1} + \sqrt{w_{k,1}^2 - 4}\right)$$

or

$$\begin{aligned} &8\left(w_{k+1,0} + \sqrt{w_{k+1,0}^2 - 4}\right) \\ &= 4\left(w_{k,0} + \sqrt{w_{k,0}^2 - 4}\right)\left(w_{k,1} + \sqrt{w_{k,1}^2 - 4}\right), \end{aligned} \tag{5}$$

where

$$w_{k,0} = \frac{1}{2}\left(q_{k-1} + \sqrt{q_{k-1}^2 - 4q_k}\right), \quad w_{k,1} = \frac{1}{2}\left(q_{k-1} - \sqrt{q_{k-1}^2 - 4q_k}\right).$$

Direct calculation gives

$$\begin{aligned} &4(w_{k,0}^2 - 4) \\ &= 2q_{k-1}^2 - 4q_k - 16 + 2q_{k-1}\sqrt{q_{k-1}^2 - 4q_k} \\ &= 2q_{k+1} - 8 + 2q_{k-1}\sqrt{q_{k-1}^2 - 4q_k} \\ &= 2q_{k+1} - 8 + 2\sqrt{q_{k+1}^2 - 4q_{k+2}}, \end{aligned}$$

where the second and the third equalities follow from Lemma 3(i) and Lemma 2, respectively. Hence we have

$$\begin{aligned} &2\left(w_{k,0} + \sqrt{w_{k,0}^2 - 4}\right) \\ &= q_{k-1} + \sqrt{q_{k-1}^2 - 4q_k} + \sqrt{2q_{k+1} - 8 + 2\sqrt{q_{k+1}^2 - 4q_{k+2}}}, \\ &2\left(w_{k+1,0} + \sqrt{w_{k+1,0}^2 - 4}\right) \\ &= q_k + \sqrt{q_k^2 - 4q_{k+1}} + \sqrt{2q_{k+2} - 8 + 2\sqrt{q_{k+2}^2 - 4q_{k+3}}}. \end{aligned}$$

And the similar calculation gives

$$2\left(w_{k,1} + \sqrt{w_{k,1}^2 - 4}\right) = q_{k-1} - \sqrt{q_{k-1}^2 - 4q_k} + \sqrt{2q_{k+1} - 8 - 2\sqrt{q_{k+1}^2 - 4q_{k+2}}}.$$

Let $a = q_{k-1},\ b = q_{k-1}^2 - 4q_k,\ c = 2q_{k+1} - 8,\ d = q_{k+1}^2 - 4q_{k+2}$. Then

$$4\left(w_{k,0} + \sqrt{w_{k,0}^2 - 4}\right)\left(w_{k,1} + \sqrt{w_{k,1}^2 - 4}\right)$$

$$= \left(a+\sqrt{b}+\sqrt{c+2\sqrt{d}}\right)\left(a-\sqrt{b}+\sqrt{c-2\sqrt{d}}\right)$$
$$= a^2-b+\sqrt{c^2-4d}+\left(a+\sqrt{b}\right)\sqrt{c-2\sqrt{d}}+\left(a-\sqrt{b}\right)\sqrt{c+2\sqrt{d}}.$$

Since $a^2-b=4q_k$ and

$$\begin{aligned}\sqrt{c^2-4d} &= \sqrt{(2q_{k+1}-8)^2-4(q_{k+1}^2-4q_{k+2})}\\ &= \sqrt{-32q_{k+1}+64+16q_{k+2}}\\ &= \sqrt{-32q_{k+1}+64+16(q_k^2-2q_{k+1}-4)}\\ &= 4\sqrt{q_k^2-4q_{k+1}}\,,\end{aligned}\tag{6}$$

in order to prove (5), it is enough to show that

$$4\sqrt{2q_{k+2}-8+2\sqrt{q_{k+2}^2-4q_{k+3}}} = \left(a+\sqrt{b}\right)\sqrt{c-2\sqrt{d}}+\left(a-\sqrt{b}\right)\sqrt{c+2\sqrt{d}}$$

or, by squaring both sides,

$$\begin{aligned}&16\left(2q_{k+2}-8+2\sqrt{q_{k+2}^2-4q_{k+3}}\right)\\ &=\left(\left(a+\sqrt{b}\right)\sqrt{c-2\sqrt{d}}+\left(a-\sqrt{b}\right)\sqrt{c+2\sqrt{d}}\right)^2.\end{aligned}\tag{7}$$

Applying (6) and

$$a^2b\,d = q_{k-1}^2(q_{k-1}^2-4q_k)(q_{k+1}^2-4q_{k+2}) = (q_{k+1}^2-4q_{k+2})^2 \quad \text{and}$$
$$a^2+b = 2q_{k-1}^2-4q_k = 2(q_{k-1}^2-2q_k) = 2(q_{k+1}+4),$$

in the following calculation, we see that the expansion of right side of (7) is equal to

$$\begin{aligned}&2(a^2+b)c-8\sqrt{a^2bd}+2(a^2-b)\sqrt{c^2-4d}\\ &= 2\cdot 2(q_{k+1}+4)(2q_{k+1}-8)-8(q_{k+1}^2-4q_{k+2})+2\cdot 4q_k\cdot 4\sqrt{q_k^2-4q_{k+1}}\\ &= 16\left(2q_{k+2}-8+2\sqrt{q_k^2(q_k^2-4q_{k+1})}\right) = 16\left(2q_{k+2}-8+2\sqrt{q_{k+2}^2-4q_{k+3}}\right)\end{aligned}$$

of which last expression is exactly the left side of (7). This completes the proof. □

Remarks. Our proof has the advantage of directness, but it would be desirable to have a less computational proof.

If we replace n by z and consider the $q_k(z)$ as polynomials in a complex variable, their zero distribution is of interest. For $k \geq 5$, k odd, the number of zeros in the interval $(2,6)$ seems to follow the Jacobsthal $2x \pm 1$ sequence $\{1,3,5,11,21,43,85,\ldots\}$. Machine computation also suggests the following.

For k large and odd, the zeros are real, no zeros exceeds 6, and the zero "density" increases as z goes from 2 to 6. In small neighborhoods of 2, there are more zeros less than 2 than greater than 2. Every zero exceeds $-1/4$. Some similar conjectures may be made for k even, except that here the negative zeros seem to be unbounded.

For n large and k even, the $p_k(n, x)$ seem to be irreducible, but, for n large and k odd, the $p_k(n, x)$ seem to have remarkable factorizations precisely when $n = t(t + 1)$. For example,

$$\begin{aligned}
p_1(n, z) &= (z^2 + tz + 1)(z^2 - (t + 1)z + 1),\\
p_3(n, z) &= (z^2 - (t^2 - 2)z + 1)(z^2 - (t^2 + 2t - 1)z + 1),\\
p_5(n, z) &= (z^2 - (t^4 - 4t^2 + 2)z + 1)(z^2 - (t^4 + 4t^3 + 2t^2 - 4t - 1)z + 1),\\
p_7(n, z) &= (z^2 - (t^8 - 8t^6 + 20t^4 - 16t^2 + 2)z + 1)\\
&\quad (z^2 - (t^8 + 8t^7 + 20t^6 + 8t^5 - 30t^4 - 24t^3 + 12t^2 + 8t - 1)z + 1),
\end{aligned}$$

and so on.

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