

ON THE GALERKIN-WAVELET METHOD FOR HIGHER ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The Galerkin method has been developed mainly for 2nd order differential equations. To get numerical solutions, there are some choices of Riesz bases for the approximation subspace $V_j \subset L^2$. In this paper we shall propose a uniform approach to find suitable Riesz bases for higher order differential equations. Especially for the beam equation (4-th order equation), we also report numerical results.

1. Introduction

1.1. The Galerkin method

The Galerkin method is a numerical method for converting a continuous operator problem to a discrete problem (see, [10, 13, 15], etc.). With respect to the bilinear form, the approximation error between the solution u of the differential equation and the solution u_j of the Galerkin equation is orthogonal to the approximation subspace $V_j \subset L^2$, where j is a fixed dilation parameter. For instance, let us consider the $2m$ -th order differential equation $(-1)^m \frac{d^{2m}}{dx^{2m}} u + u = f$ and its Galerkin equation $a(u_j, \varphi_{j,k}) = \langle f, \varphi_{j,k} \rangle_{L^2}$, where

$$a(g, \phi) = \langle g, \phi \rangle_{L^2} + \left\langle \frac{d^m}{dx^m} g, \frac{d^m}{dx^m} \phi \right\rangle_{L^2}.$$

Then the error $e_j := u - u_j$ satisfies by integration by parts

$$a(e_j, \varphi_{j,k}) = \left\langle \left((-1)^m \frac{d^{2m}}{dx^{2m}} + 1 \right) u, \varphi_{j,k} \right\rangle_{L^2} - a(u_j, \varphi_{j,k}) = 0$$

for all $\varphi_{j,k}$ ($k \in \mathbb{Z}$), where k is the translation parameter to span the whole V_j . The subspace V_j on which u is projected, characterizes the visual shape of u_j . However, it is not always concerned with the sharpness of the approximation, because the residual error e_j is often estimated by the base which is not uniquely

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determined for the subspace V_j . So, the choices of the subspace and its base are up to us.

Generally, the compactly-supported bases are preferable to get numerical solutions of differential equations. In particular, the Galerkin method with piecewise polynomials called Finite element method (FEM). These compactly-supported bases enable us to localize a few elements in the whole interval, but are not orthogonal in translation. The Galerkin-wavelet method can be dealt with the (sufficiently smooth) Daubechies scaling function (or wavelet) which is not only orthogonal but also compactly-supported (see [1, 5]). To get a better approximation, there are two ways: by increasing the degree of the scaling function (or wavelet) and the level j . In general, the stiffness matrix with orthogonal bases are full. To consider higher order differential equations, orthogonal bases require many computations of stiffness matrix. In this paper, our aim is to find suitable (non orthogonal) Riesz bases for higher order differential equations in the sense that stiffness matrices are more sparse.

1.2. Uniform approach to find suitable bases

According to differential equations, we expect certain smoothness (at least Lipschitz continuity) for the subspace. Let us put the B-splines of orders 1 and 2 as follows:

$$N_1(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases} \quad N_2(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 2-x & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

$N_1(x)$ is called the Haar scaling function. $\{N_2(x-k) : k \in \mathbb{Z}\}$ which is a Riesz basis for the space V_0 of piecewise linear continuous functions on the intervals $[k, k+1]$ for all $k \in \mathbb{Z}$, is used in the standard FEM. We remark that the Franklin scaling function and the Strömberg scaling function can be also orthogonal bases for V_0 (see [7, 11, 14]). The Lipschitz continuity of functions in the subspace comes from the property of these bases. Therefore, our task is to determine a base scaling function rather than a subspace.

From the point of view of the study of differential equations, the coefficient of the highest order derivative has much more influence on the behavior of the solution. After the translation of the continuous problem into the discrete one, if the matrix corresponding to the principal part becomes simpler, the approximate solution will be more stable as an appropriate numerical treatment. In this section we shall give a uniform approach to find suitable bases such that the matrix corresponding to the principal part has just a form of 3-point formula or more generally, $(2m+1)$ -point formula.

At first, for the simplicity, let us consider the 2nd order equation $-\frac{d^2}{dx^2}u + u = f$ and V_0 , i.e., $j = 0$. We are concerned with the following matrix coming from the principal part:

$$a_{k,\ell} := - \left\langle \frac{d}{dx} \varphi_{0,k}, \frac{d}{dx} \varphi_{0,\ell} \right\rangle_{L^2} \left(= \left\langle \frac{d^2}{dx^2} \varphi_{0,k}, \varphi_{0,\ell} \right\rangle_{L^2} \text{ if } \varphi \in C^2 \right).$$

Since $\varphi_{0,k}(x) = \varphi(x - k)$, by Parseval's theorem we see that

$$\left\langle \frac{d}{dx}\varphi_{0,k}, \frac{d}{dx}\varphi_{0,\ell} \right\rangle_{L^2} = \frac{1}{2\pi} \langle i\xi e^{-ik\xi} \hat{\varphi}, i\xi e^{-i\ell\xi} \hat{\varphi} \rangle_{L^2} = \mathcal{F}^{-1} \left[|\xi \hat{\varphi}(\xi)|^2 \right] (\ell - k).$$

On the other hand, in order to get 3-point formula for 2nd order derivative, we need the tridiagonal matrix

$$(1.1) \quad \{a_{k,\ell}\}_{1 \leq k, \ell \leq N} = \begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & -2 \end{pmatrix},$$

where N depends on the interval in which $-\frac{d^2}{dx^2}u + u = f$ is considered. Thus, φ must satisfy the condition

$$(1.2) \quad \mathcal{F}^{-1} \left[|\xi \hat{\varphi}(\xi)|^2 \right] (\ell - k) = \begin{cases} 2 & \text{if } k = \ell, \\ -1 & \text{if } k = \ell \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

It would not be easy to find φ from (1.2). Therefore, we shall try to change the condition (1.2).

Further computations yield

$$\begin{aligned} \mathcal{F}^{-1} \left[|\xi \hat{\varphi}(\xi)|^2 \right] (\ell - k) &= \frac{1}{2\pi} \sum_{q \in \mathbb{Z}} \int_{2q\pi}^{2(q+1)\pi} e^{i(\ell-k)\xi} |\xi \hat{\varphi}(\xi)|^2 d\xi \\ &= \frac{1}{2\pi} \sum_{q \in \mathbb{Z}} \int_0^{2\pi} e^{i(\ell-k)\xi} |(\xi + 2q\pi) \hat{\varphi}(\xi + 2q\pi)|^2 d\xi \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{i(\ell-k)\xi} \sum_{q \in \mathbb{Z}} |(\xi + 2q\pi) \hat{\varphi}(\xi + 2q\pi)|^2 d\xi. \end{aligned}$$

Hence, we find that (1.2) is equivalent to

$$\sum_{q \in \mathbb{Z}} |(\xi + 2q\pi) \hat{\varphi}(\xi + 2q\pi)|^2 \equiv -e^{i\xi} + 2e^{i0\xi} - e^{-i\xi} \left(= 4 \sin^2 \frac{\xi}{2} \right)$$

for almost everywhere $\xi \in \mathbb{R}$. Denoting the sinc function by $\text{sinc } \xi = \frac{\sin \xi}{\xi}$, we see that the Haar scaling function $N_1(x)$ satisfies $\hat{N}_1(\xi) = e^{-i\xi/2} \text{sinc } \frac{\xi}{2}$. We shall define $\Phi(x)$ by

$$(1.3) \quad \hat{\Phi}(\xi) = \frac{\hat{\varphi}(\xi)}{\hat{N}_1(\xi)}.$$

Then we also get

$$\sum_{q \in \mathbb{Z}} |\hat{\Phi}(\xi + 2q\pi)|^2 = \sum_{q \in \mathbb{Z}} \frac{|(\xi + 2q\pi)\hat{\varphi}(\xi + 2q\pi)|^2}{4 \sin^2 \frac{\xi + 2q\pi}{2}} = \frac{\sum_{q \in \mathbb{Z}} |(\xi + 2q\pi)\hat{\varphi}(\xi + 2q\pi)|^2}{-e^{i\xi} + 2e^{i0\xi} - e^{-i\xi}},$$

here we used

$$4 \sin^2 \frac{\xi + 2q\pi}{2} = 4 \sin^2 \frac{\xi}{2} = -e^{i\xi} + 2e^{i0\xi} - e^{-i\xi}.$$

This means that

$$(1.4) \quad \sum_{q \in \mathbb{Z}} |\hat{\Phi}(\xi + 2q\pi)|^2 \equiv 1 \quad \text{a.e. } \xi \in \mathbb{R}.$$

So, the condition (1.2) has been reduced to the conditions (1.3) and (1.4). Now we can easily find φ from (1.3) and (1.4), because the identity (1.4) is well-known as the orthonormal condition. The definition (1.3) yields

$$(1.5) \quad \varphi(x) = \mathcal{F}^{-1} \left[\hat{N}_1(\xi)\hat{\Phi}(\xi) \right] (x) = N_1 * \Phi(x) \left(= \int_{x-1}^x \Phi(y)dy \right).$$

The new function φ is the *elevation* of Φ with N_1 . Therefore N_1 is also called the *elevator* (see [16, 17]). More generally, let us represent the elevator by \mathcal{E} and define

$$(1.6) \quad \varphi(x) = \mathcal{E} * \Phi(x).$$

Remark 1.1. The most typical example is the case when the elevator $\mathcal{E}(x)$ is $N_1(x)$ and $\Phi(x)$ is the Haar scaling function, i.e., $\mathcal{E}(x) = \Phi(x) = N_1(x)$. By (1.5) we obtain

$$\varphi(x) = N_1 * N_1(x) = N_2(x).$$

This case just coincides with the standard FEM. Choosing other scaling functions for $\Phi(x)$, we can obtain various types of bases.

1.3. Definition of elevator

We shall derive some properties for the case when the elevator $\mathcal{E}(x)$ is $N_1(x)$. By Taylor expansion we see that for $v \in C^4$

$$(1.7) \quad \sum_{\nu=-1}^1 a_{k,k+\nu} v(x + \nu h) = h^2 \frac{d^2}{dx^2} v(x) + O(h^4) \quad \text{for all } k \in \mathbb{Z}.$$

Moreover, we assume that

$$(1.8) \quad \hat{\Phi}(0) = 1,$$

which allows scaling functions, but excludes wavelet functions. Hence, by (1.5) it follows that

$$\sum_{k \in \mathbb{Z}} \varphi_{j,k}(x) = \sum_{k \in \mathbb{Z}} \varphi(2^j x - k) = \sum_{k \in \mathbb{Z}} \int_{2^j x - k - 1}^{2^j x - k} \Phi(y) dy$$

$$(1.9) \quad = \int_{-\infty}^{\infty} \Phi(y)dy = \hat{\Phi}(0) = 1.$$

This is just the partition of unity. Let us put $h = 2^{-j}$ and

$$w_j(x) = \sum_{\ell \in \mathbb{Z}} w_{j,\ell} \varphi_{j,\ell}(x).$$

If w_j is sufficiently smooth and φ has compact support (or decays sufficiently fast), by (1.9) we have for $k \in \mathbb{Z}$

$$(1.10) \quad w_j(kh) = \sum_{\ell \in \mathbb{Z}} w_{j,\ell} \varphi_{j,\ell}(kh) \sim w_{j,k} \sum_{\ell \in \mathbb{Z}} \varphi_{j,\ell}(kh) = w_{j,k}.$$

Indeed, it holds that $w_j(kh) = w_{j,k}$ in the standard FEM.

Meanwhile we also get the following identity:

$$\sum_{q \in \mathbb{Z}} |\hat{\varphi}(\xi + 2q\pi)|^2 = \sum_{\nu \in \mathbb{Z}} c_{k,k+\nu} e^{i\nu\xi},$$

where $c_{k,\ell} := \langle \varphi_{0,k}, \varphi_{0,\ell} \rangle_{L^2}$. In particular, taking $\xi = 0$, by (1.3) and (1.8) we find that for all $k \in \mathbb{Z}$

$$\sum_{\nu \in \mathbb{Z}} c_{k,k+\nu} = \sum_{q \in \mathbb{Z}} |\hat{\varphi}(2q\pi)|^2 = \sum_{q \in \mathbb{Z}} |\hat{N}_1(2q\pi) \hat{\Phi}(2q\pi)|^2 = |\hat{\Phi}(0)|^2 = 1,$$

here we used $\hat{\Phi}(2q\pi) = 0$ if $q \neq 0$, since $\sum_{q \neq 0} |\hat{\Phi}(2q\pi)|^2 - |\hat{\Phi}(0)|^2 = 0$ by (1.8) and (1.4) with $\xi = 0$. Noting that $c_{k,k+\nu} = c_{k,k-\nu}$, by Taylor expansion we see that for $v \in C_0^2$

$$(1.11) \quad \sum_{\nu \in \mathbb{Z}} c_{k,k+\nu} v(x + \nu h) = v(x) + O(h^2) \quad \text{for all } k \in \mathbb{Z}.$$

In our construction, to get the approximate solution $u_j(x) = \sum_{\ell \in \mathbb{Z}} u_{j,\ell} \varphi_{j,\ell}(x)$ in the interval $(0, 1)$ for the equation $-\frac{d^2}{dx^2}u + u = f$, we solve the following system corresponding to the Galerkin equation:

$$\begin{aligned} & \left[-\{a_{k,\ell} h^{-2}\}_{1 \leq k, \ell \leq N} + \{c_{k,\ell} h\}_{1 \leq k, \ell \leq N} \right]^t \{u_{j,\ell}\}_{1 \leq \ell \leq N} \\ & = \{c_{k,\ell}\}_{1 \leq k, \ell \leq N}^t \{f_{j,\ell}\}_{1 \leq \ell \leq N}. \end{aligned}$$

By (1.10) this can be regarded as

$$\begin{aligned} & \left[-\{a_{k,\ell} h^{-2}\}_{1 \leq k, \ell \leq N} + \{c_{k,\ell} h\}_{1 \leq k, \ell \leq N} \right]^t \{u_j(\ell h)\}_{1 \leq \ell \leq N} \\ & = \{c_{k,\ell}\}_{1 \leq k, \ell \leq N}^t \{f_j(\ell h)\}_{1 \leq \ell \leq N}. \end{aligned}$$

Paying attention to each row, by (1.7) and (1.11) we find that

$$-\sum_{1 \leq \ell \leq 2^j} a_{k,\ell} h^{-2} u_j(\ell h) = -\sum_{\nu} a_{k,k+\nu} h^{-2} u_j(kh + \nu h) = -\frac{d^2}{dx^2} u_j(kh) + O(h^2),$$

$$\begin{aligned} \sum_{1 \leq \ell \leq 2^j} c_{k,\ell} u_j(\ell h) &= \sum_{\nu} c_{k,k+\nu} u_j(kh + \nu h) = u_j(kh) + O(h^2), \\ \sum_{1 \leq \ell \leq 2^j} c_{k,\ell} f_j(\ell h) &= \sum_{\nu} c_{k,k+\nu} f_j(kh + \nu h) = f_j(kh) + O(h^2). \end{aligned}$$

These give the numerical difference equation of the original differential equation $-\frac{d^2}{dx^2}u + u = f$ at the point $x = kh$. The accuracy of (1.10) depends on the case of application. We remark that (1.7) and (1.11) play an important role to guarantee the accuracy.

From the above observations for $\mathcal{E} = N_1$, we shall propose the following conditions to characterize qualitative elevators for the Galerkin method:

Definition 1.1. Let Φ be a scaling function such that $\hat{\Phi}(0) = 1$ and $\hat{\Phi}(\xi) \neq 0$ for $-\pi \leq \xi \leq \pi$. Put $c_{k,\ell} := \langle \varphi_{0,k}, \varphi_{0,\ell} \rangle_{L^2}$ and $a_{k,\ell} := -\langle \frac{d}{dx}\varphi_{0,k}, \frac{d}{dx}\varphi_{0,\ell} \rangle_{L^2}$ for $\varphi(x) = \mathcal{E} * \Phi(x)$. The elevator \mathcal{E} for the Galerkin method is a function satisfying

- (i) $\hat{\mathcal{E}}(\xi) \neq 0$ for $-\pi \leq \xi \leq \pi$, in particular, $\hat{\mathcal{E}}(0) = 1$.
- (ii) It holds that for $v \in C_0^4$

$$\begin{aligned} \sum_{\nu \in \mathbb{Z}} c_{k,k+\nu} v(x + \nu h) &= v(x) + O(h^2), \\ \sum_{\nu \in \mathbb{Z}} a_{k,k+\nu} v(x + \nu h) &= h^2 \frac{d^2}{dx^2} v(x) + O(h^4). \end{aligned}$$

- (iii) There exists a 2π -periodic function $m_{\mathcal{E}}(\xi)$ such that $\hat{\mathcal{E}}(2\xi) = m_{\mathcal{E}}(\xi)\hat{\mathcal{E}}(\xi)$.

It is known that the exact frame is equivalent to the Riesz basis. The condition for the Riesz basis is given by

$$(1.12) \quad A \leq \sum_{q \in \mathbb{Z}} |\hat{\varphi}(\xi + 2q\pi)|^2 \leq B$$

for $0 < A \leq B < \infty$ (see [2]). If $\hat{\Phi}(\xi) \neq 0$ for $-\pi \leq \xi \leq \pi$, by (i) we note that

$$\begin{aligned} \sum_{q \in \mathbb{Z}} |\hat{\varphi}(\xi + 2q\pi)|^2 &= \sum_{q \in \mathbb{Z}} |\hat{\mathcal{E}}(\xi + 2q\pi)\hat{\Phi}(\xi + 2q\pi)|^2 \\ &\geq |\hat{\mathcal{E}}(\xi - 2n\pi)\hat{\Phi}(\xi - 2n\pi)|^2 \\ &\geq \exists A > 0 \end{aligned}$$

for $2n\pi - \pi \leq \xi \leq 2n\pi + \pi$ ($n \in \mathbb{Z}$), that is, $\xi \in \mathbb{R}$. Rewriting $\varphi(x) = N_1 * \Phi^\sharp(x)$ with $\hat{\Phi}^\sharp(\xi) = \frac{\hat{\mathcal{E}}(\xi)\hat{\Phi}(\xi)}{N_1(\xi)}$, from (i) we can expect that the properties corresponding to (1.9), (1.10) and (1.11) still hold, since $\hat{\Phi}^\sharp(0) = 1$. In fact, we may omit $\sum_{\nu \in \mathbb{Z}} c_{k,k+\nu} v(x + \nu h) = v(x) + O(h^2)$ in (ii).

Replacing the definition $\varphi_{j,k}(x) = \varphi(2^j x - k)$ by $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$, we could also get wavelet expansions. Thanks to the condition (iii) we obtain a semi-orthogonal wavelet $\hat{\psi}(\xi) = e^{i\xi/2} \overline{m(\xi/2 + \pi)} \hat{\varphi}(\xi/2)$, where $m(\xi) =$

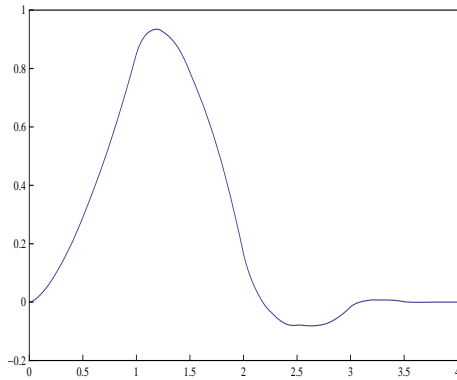
$m_\varphi(\xi) \sum_{q \in \mathbb{Z}} |\hat{\varphi}(\xi + 2q\pi)|^2 = m_\mathcal{E}(\xi)m_\Phi(\xi) \sum_{\nu \in \mathbb{Z}} c_{k,k+\nu} e^{i\nu\xi}$ (2π -periodic). A biorthogonal wavelet for the elevated φ can be also considered (see [6]). By (1.7) we have already found the elevator \mathcal{E} which allows the finite sum of $a_{k,k+\nu}v(x + \nu h)$ in (ii) and improves the standard FEM with $\varphi = N_2$ in §4.

2. Riesz basis of Daubechies type

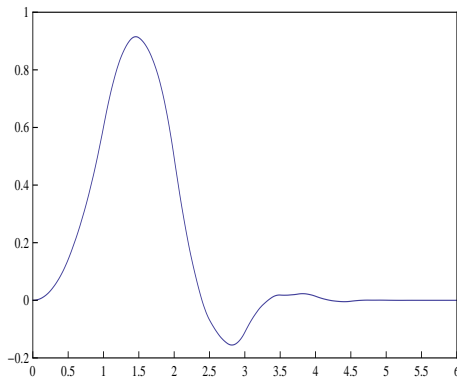
2.1. 3-point formula for 2nd order derivative

To get compactly-supported and also more smooth base than N_2 , we may choose the Daubechies scaling function of order p for $\Phi \equiv \Phi_p^D$ satisfying (1.8). Then by (1.5) we have

$$(2.1) \quad \varphi_p^D(x) = N_1 * \Phi_p^D(x).$$



(a) φ_2^D (supp $\varphi_2^D \subset [0, 4]$)



(b) φ_3^D (supp $\varphi_3^D \subset [0, 6]$)

FIGURE 1. Graphs of φ_2^D and φ_3^D .

The basis $\{\varphi_p^D(x - k) : k \in \mathbb{Z}\}$ had been derived by [16] and [17]. Their approach is motivated from the observation that the integration of the Haar wavelet becomes N_2 . Therefore, the pseudoframe was firstly considered by the integration of the Daubechies wavelet, and secondly it was arranged for the efficiency of the computation and arrived at φ_p^D (see also [12]).

In order to solve numerically the equation $-\frac{d^2}{dx^2}u + u = f$ with some base $\{\varphi(x - k) : k \in \mathbb{Z}\}$, we need to know the matrices $\{c_{k,\ell}\}_{1 \leq k, \ell \leq N}$ and $\{a_{k,\ell}\}_{1 \leq k, \ell \leq N}$. If one considers the orthogonal Daubechies scaling function, it holds that the matrix $\{c_{k,\ell}\}_{1 \leq k, \ell \leq N} = I$. On the other hand, the matrix $\{a_{k,\ell}\}_{1 \leq k, \ell \leq N}$ for the Daubechies scaling function is well studied in [1]. For all the bases constructed by the approach in §1.2, the matrix $\{a_{k,\ell}\}_{1 \leq k, \ell \leq N}$ is just (1.1).

Remark 2.1. It would be preferable that bases are at least C^1 or Lipschitz continuous as N_2 in order that the weak form $-\langle \frac{d}{dx}\varphi_{0,k}, \frac{d}{dx}\varphi_{0,\ell} \rangle_{L^2}$ of $\langle \frac{d^2}{dx^2}\varphi_{0,k}, \varphi_{0,\ell} \rangle_{L^2}$ has a meaning. Especially for $p = 2$, the Daubechies scaling function $\Phi_2^D \in C^{0.55}$ fails to satisfy the differentiability, but gives $\varphi_2^D \in C^{1.55}$.

We shall also compute the exact value of $c_{k,\ell}$ for $\varphi_2^D(x)$. Putting $\hat{\phi}(\xi) = |\hat{\Phi}_2^D(\xi)|^2$, by Parseval's theorem we have

$$c_{k,\ell} = \langle \varphi_2^D(x - k), \varphi_2^D(x - \ell) \rangle_{L^2} = \phi(\ell - k).$$

By (2.1) it holds that

$$\hat{\phi}(\xi) = \text{sinc}^2 \frac{\xi}{2} \hat{\Phi}_2^D(\xi)^2 = \prod_{j=1}^{\infty} \cos^2 \left(\frac{\xi}{2^{j+1}} \right) \left| m_2^D \left(\frac{\xi}{2^j} \right) \right|^2 \equiv \prod_{j=1}^{\infty} \tilde{m} \left(\frac{\xi}{2^j} \right),$$

where m_2^D is the Daubechies low pass filter. Since $m_2^D(\xi) = \sum_{k=0}^3 \eta_k e^{-ik\xi}$ with $\{\eta_0, \eta_1, \eta_2, \eta_3\} = \left\{ \frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8} \right\}$ and $\cos^2 \xi = \frac{e^{2i\xi} + 2 + e^{-2i\xi}}{4}$, we find that $\tilde{m}(\xi) = \sum_{k=-4}^4 \mu_k e^{-ik\xi}$ and its coefficients are given by

$$\{\mu_0, \mu_{\pm 1}, \mu_{\pm 2}, \mu_{\pm 3}, \mu_{\pm 4}\} = \left\{ \frac{25}{64}, \frac{17}{64}, \frac{1}{16}, -\frac{1}{64}, -\frac{1}{128} \right\}.$$

The function ϕ satisfies

$$\phi(x) = 2 \sum_{k=-4}^4 \mu_k \phi(2x - k)$$

and $\text{supp } \phi \subset [-4, 4]$ since $\text{supp } \varphi_2^D \subset [0, 4]$ and $\phi(x) = \int_{\mathbb{R}} \varphi_2^D(t+x) \overline{\varphi_2^D(t)} dt$. Hence we have

$$M^t \{\phi(k)\}_{-3 \leq k \leq 3} = \mathbf{0},$$

where

$$M = \begin{pmatrix} 1 - 2\mu_{-3} & -2\mu_{-4} & 0 & 0 & 0 & 0 & 0 \\ -2\mu_{-1} & 1 - 2\mu_{-2} & -2\mu_{-3} & -2\mu_{-4} & 0 & 0 & 0 \\ -2\mu_1 & -2\mu_0 & 1 - 2\mu_{-1} & -2\mu_{-2} & -2\mu_{-3} & -2\mu_{-4} & 0 \\ -2\mu_3 & -2\mu_2 & -2\mu_1 & 1 - 2\mu_0 & -2\mu_{-1} & -2\mu_{-2} & -2\mu_{-3} \\ 0 & -2\mu_4 & -2\mu_3 & -2\mu_2 & 1 - 2\mu_1 & -2\mu_0 & -2\mu_{-1} \\ 0 & 0 & 0 & -2\mu_4 & -2\mu_3 & 1 - 2\mu_2 & -2\mu_{-1} \\ 0 & 0 & 0 & 0 & 0 & -2\mu_4 & 1 - 2\mu_3 \end{pmatrix}.$$

We also remark that $\sum_{k=-3}^3 \phi(k) = \sum_{k \in \mathbb{Z}} \phi(k) = \int_{\mathbb{R}} \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{ik\xi} \hat{\phi}(\xi) d\xi = \hat{\phi}(0) = 1$. Deriving the eigenvector with 0 eigenvalue such that $\sum_{k=-3}^3 \phi(k) = 1$, we find that

$$\{\phi(0), \phi(\pm 1), \phi(\pm 2), \phi(\pm 3)\} = \left\{ \frac{131}{180}, \frac{37}{240}, -\frac{11}{600}, \frac{1}{3600} \right\}.$$

Thus we obtain

$$c_{k,\ell} \left(= \phi(\ell - k) \right) = \begin{cases} 131/180 & \text{if } k = \ell, \\ 37/240 & \text{if } k = \ell \pm 1, \\ -11/600 & \text{if } k = \ell \pm 2, \\ 1/3600 & \text{if } k = \ell \pm 3, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we get the following theorem:

Theorem 2.1. For $\varphi_2^D(x)$ defined by (2.1) with $\Phi = \Phi_2^D$ we have (2.2)

$$c_{k,\ell} = \begin{cases} 131/180 & \text{if } k = \ell, \\ 37/240 & \text{if } k = \ell \pm 1, \\ -11/600 & \text{if } k = \ell \pm 2, \\ 1/3600 & \text{if } k = \ell \pm 3, \\ 0 & \text{otherwise,} \end{cases} \text{ and } a_{k,\ell} = \begin{cases} -2 & \text{if } k = \ell, \\ 1 & \text{if } k = \ell \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, it holds that for $v \in C_0^4$

$$\sum_{\nu \in \mathbb{Z}} c_{k,k+\nu} v(x + \nu h) = v(x) + O(h^2),$$

$$\sum_{\nu=-1}^1 a_{k,k+\nu} v(x + \nu h) = h^2 \frac{d^2}{dx^2} v(x) + O(h^4).$$

2.2. 5-point formula for 2nd order derivative

With small changes of the approach in §1.2 we can also consider the 5-point formula for 2nd order derivative. For this purpose, we replace (1.2) by

$$\mathcal{F}^{-1} \left[|\xi \hat{\varphi}(\xi)|^2 \right] (\ell - k) = \begin{cases} \frac{5}{2} & \text{if } k = \ell, \\ -\frac{4}{3} & \text{if } k = \ell \pm 1, \\ \frac{1}{12} & \text{if } k = \ell \pm 2, \\ 0 & \text{otherwise,} \end{cases}$$

which is equivalent to

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} |(\xi + 2q\pi) \hat{\varphi}(\xi + 2q\pi)|^2 \\ & \equiv \frac{1}{12} e^{2i\xi} - \frac{4}{3} e^{i\xi} + \frac{5}{2} e^{i0\xi} - \frac{4}{3} e^{-i\xi} + \frac{1}{12} e^{-2i\xi} \left(= \frac{4}{3} \sin^2 \frac{\xi}{2} \left(\sin^2 \frac{\xi}{2} + 3 \right) \right) \end{aligned}$$

for almost everywhere $\xi \in \mathbb{R}$. We shall define $\Phi(x)$ by

$$(2.3) \quad \hat{\Phi}(\xi) = \frac{\hat{\varphi}(\xi)}{(\gamma^- + \gamma^+ e^{-i\xi}) \hat{N}_1(\xi)},$$

where $\gamma^\pm = \frac{1}{2} \pm \frac{1}{\sqrt{3}}$. Then we also get

$$\begin{aligned} \sum_{q \in \mathbb{Z}} |\hat{\Phi}(\xi + 2q\pi)|^2 &= \sum_{q \in \mathbb{Z}} \frac{|(\xi + 2q\pi) \hat{\varphi}(\xi + 2q\pi)|^2}{4 \sin^2 \frac{\xi + 2q\pi}{2} |\gamma^- + \gamma^+ e^{-i(\xi + 2q\pi)}|^2} \\ &= \frac{\sum_{p \in \mathbb{Z}} |(\xi + 2q\pi) \hat{\varphi}(\xi + 2q\pi)|^2}{\frac{1}{12} e^{2i\xi} - \frac{4}{3} e^{i\xi} + \frac{5}{2} e^{i0\xi} - \frac{4}{3} e^{-i\xi} + \frac{1}{12} e^{-2i\xi}}, \end{aligned}$$

here we used

$$\begin{aligned} & 4 \sin^2 \frac{\xi + 2q\pi}{2} |\gamma^- + \gamma^+ e^{-i(\xi + 2q\pi)}|^2 \\ &= 4 \sin^2 \frac{\xi}{2} |\gamma^- + \gamma^+ e^{-i\xi}|^2 = \frac{4}{3} \sin^2 \frac{\xi}{2} \left(\sin^2 \frac{\xi}{2} + 3 \right) \\ &= \frac{1}{12} e^{2i\xi} - \frac{4}{3} e^{i\xi} + \frac{5}{2} e^{i0\xi} - \frac{4}{3} e^{-i\xi} + \frac{1}{12} e^{-2i\xi}. \end{aligned}$$

Hence, the identity (1.4) still holds. Thus, the definition (2.3) yields

$$(2.4) \quad \tilde{\varphi}(x) = \{ \gamma^- N_1(\cdot) + \gamma^+ N_1(\cdot - 1) \} * \Phi(x).$$

Let us put $\tilde{c}_{k,\ell} := \langle \tilde{\varphi}_{0,k}, \tilde{\varphi}_{0,\ell} \rangle_{L^2}$ and $\tilde{a}_{k,\ell} := -\langle \frac{d}{dx} \tilde{\varphi}_{0,k}, \frac{d}{dx} \tilde{\varphi}_{0,\ell} \rangle_{L^2}$. Similarly as §1.2, by (2.4) we also find that for all $\ell \in \mathbb{Z}$

$$\begin{aligned} \sum_{\nu \in \mathbb{Z}} \tilde{c}_{k,k+\nu} &= \sum_{q \in \mathbb{Z}} | \{ \gamma^- \hat{N}_1(2q\pi) + \gamma^+ e^{-2iq\pi} \hat{N}_1(2q\pi) \} \hat{\Phi}(2q\pi) |^2 \\ &= |(\gamma^- + \gamma^+) \hat{\Phi}(0)|^2 = 1. \end{aligned}$$

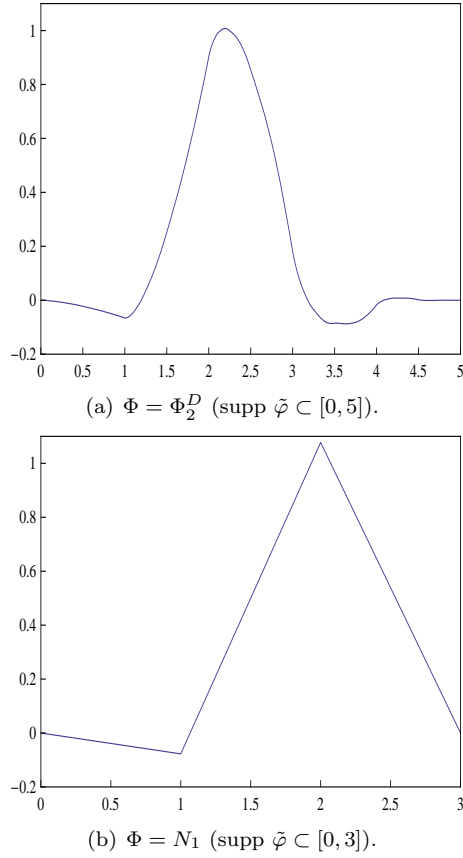


FIGURE 2. Graphs of $\tilde{\varphi} = \{\gamma^- N_1(x) + \gamma^+ N_1(x - 1)\} * \Phi(x)$.

We remark that $E(x) = \gamma^+ N_1(x - 1) + \gamma^- N_1(x)$ satisfies (i) in Definition 1.1, since $|\hat{E}(\xi)| = |\gamma^- + \gamma^+ e^{i\xi}| \left| \text{sinc} \frac{\xi}{2} \right| = \sqrt{\frac{1}{3} \sin^2 \frac{\xi}{2}} \left| \text{sinc} \frac{\xi}{2} \right|$.

It remains to compute the precise value of $\tilde{c}_{k,\ell}$ for $\tilde{\varphi}(x)$. Put $\varphi = N_1 * \Phi$ and $c_{k,\ell} = \langle \varphi_{0,k}, \varphi_{0,\ell} \rangle_{L^2}$ as in §2.1. Since $\tilde{\varphi}(x) = \gamma^- \varphi(x) + \gamma^+ \varphi(x - 1)$, we get

$$\begin{aligned} \tilde{c}_{k,\ell} &= \langle \gamma^- \varphi_{0,k} + \gamma^+ \varphi_{0,k+1}, \gamma^- \varphi_{0,\ell} + \gamma^+ \varphi_{0,\ell+1} \rangle_{L^2} \\ &= (\gamma^{+2} + \gamma^{-2}) \langle \varphi_{0,k}, \varphi_{0,\ell} \rangle_{L^2} + \gamma^+ \gamma^- (\langle \varphi_{0,k}, \varphi_{0,\ell+1} \rangle_{L^2} + \langle \varphi_{0,k+1}, \varphi_{0,\ell} \rangle_{L^2}) \\ &= \frac{7}{6} c_{k,\ell} - \frac{1}{12} (c_{k,\ell+1} + c_{k+1,\ell}). \end{aligned}$$

In the case of $\Phi = \Phi_2^D$, each $c_{k,\ell}$ is already given by (2.2). In the case of $\Phi = N_1$, since $\varphi = N_2$ we can easily see that

$$c_{k,\ell} = \begin{cases} 2/3 & \text{if } k = \ell, \\ 1/6 & \text{if } k = \ell \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we get the following theorem corresponding to Theorem 2.1:

Theorem 2.2. For $\tilde{\varphi}_2^D(x)$ defined by (2.4) with $\Phi = \Phi_2^D$ (resp. N_1) we have

$$\tilde{c}_{k,\ell} = \begin{cases} 3557/4320 & \text{if } k = \ell, \\ 163/1350 & \text{if } k = \ell \pm 1, \\ -37/1080 & \text{if } k = \ell \pm 2, \\ 1/540 & \text{if } k = \ell \pm 3, \\ -1/43200 & \text{if } k = \ell \pm 4, \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(\text{resp. } \tilde{c}_{k,\ell} = \begin{cases} 3/4 & \text{if } k = \ell, \\ 5/36 & \text{if } k = \ell \pm 1, \\ -1/72 & \text{if } k = \ell \pm 2, \\ 0 & \text{otherwise,} \end{cases} \right)$$

and

$$(2.5) \quad \tilde{a}_{k,\ell} = \begin{cases} -5/2 & \text{if } k = \ell, \\ 4/3 & \text{if } k = \ell \pm 1, \\ -1/12 & \text{if } k = \ell \pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, it holds that for $v \in C_0^6$

$$\sum_{\nu \in \mathbb{Z}} \tilde{c}_{k,k+\nu} v(x + \nu h) = v(x) + O(h^2),$$

$$\sum_{\nu=-2}^2 \tilde{a}_{k,k+\nu} v(x + \nu h) = h^2 \frac{d^2}{dx^2} v(x) + O(h^6).$$

3. Riesz basis for higher order derivatives

3.1. (2m + 1)-point formula for 2m-th order derivative

Let $N_m(x)$ be the B-spline of order $m(\geq 1)$ which can be defined by (see [3])

$$\hat{N}_m(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi} \right)^m \equiv \{\hat{N}_1(\xi)\}^m.$$

We shall consider the following instead of (1.3):

$$(3.1) \quad \hat{\Phi}(\xi) = \frac{\hat{\varphi}^{(2m)}(\xi)}{\hat{N}_m(\xi)}.$$

Combining (3.1) with (1.4) we obtain $\varphi^{(2m)}(x)$ with the elevator $\mathcal{E}(x) = N_m(x)$. In this section, we shall derive a further property of the elevator N_m for applications of differential equations.

To get the $(2m + 1)$ -point formula for $2m$ -th order derivative, we must find $\alpha_\sigma^{(r)}$ such that for any $u \in C^{2m}$

$$\lim_{h \rightarrow 0} \frac{\sum_{\sigma=0}^{2m} \alpha_\sigma^{(2m)} u(x + (m - \sigma)h)}{h^{2m}} = \left(\frac{d}{dx}\right)^{2m} u(x).$$

So, we expect that $\alpha_\sigma^{(2m)}$'s satisfy

$$(3.2) \quad \sum_{\sigma=0}^{2m} \alpha_\sigma^{(2m)} u(x + (m - \sigma)h) = h^{2m} \left(\frac{d}{dx}\right)^{2m} u(x) + O(h^{2m+2}).$$

In case when $m = 1$ and $r = 2$, we see that $\alpha_0^{(2)} = 1$, $\alpha_1^{(2)} = -2$ and $\alpha_2^{(2)} = 1$ satisfy $\alpha_0^{(2)}u(x+h) + \alpha_1^{(2)}u(x) + \alpha_2^{(2)}u(x-h) = h^2 \frac{d^2}{dx^2}u(x) + O(h^4)$. This is just the case of 3-point formula for 2nd order derivative (see (2.2)). Using Taylor expansion, we see that cancellations occur during the odd order derivatives.

Remark 3.1. $\alpha_\sigma^{(2m)}$'s uniquely determined later from (3.2) are the best constants in sense that the remainder order is highest for the $(2m + 1)$ -point formula for $2m$ -th order derivative.

Using Taylor expansion for $u(x + (m - \sigma)h)$ in (3.2) and equating the coefficients of equal order derivatives on both sides of (3.2), we have

$$\begin{aligned} & \begin{pmatrix} \frac{1}{0!} & \frac{1}{0!} & \cdots & \frac{1}{0!} \\ \frac{(m-0)h}{1!} & \frac{(m-1)h}{1!} & \cdots & \frac{(m-2m)h}{1!} \\ \vdots & \vdots & & \vdots \\ \frac{\{(m-0)h\}^{2m}}{(2m)!} & \frac{\{(m-1)h\}^{2m}}{(2m)!} & \cdots & \frac{\{(m-2m)h\}^{2m}}{(2m)!} \end{pmatrix} \begin{pmatrix} \alpha_0^{(0)} & \alpha_0^{(1)} & \cdots & \alpha_0^{(2m)} \\ \alpha_1^{(0)} & \alpha_1^{(1)} & \cdots & \alpha_1^{(2m)} \\ \vdots & \vdots & & \vdots \\ \alpha_{2m}^{(0)} & \alpha_{2m}^{(1)} & \cdots & \alpha_{2m}^{(2m)} \end{pmatrix} \\ = & \begin{pmatrix} h^0 & 0 & \cdots & 0 \\ 0 & h^1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & h^{2m} \end{pmatrix}. \end{aligned}$$

Let us put $\lambda_\sigma = m - \sigma$ and $K = \{\alpha_\sigma^{(r)}\}_{0 \leq \sigma, r \leq 2m}$. Then we have

$$\begin{pmatrix} \frac{h^0}{0!} & 0 & \cdots & 0 \\ 0 & \frac{h^1}{1!} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{h^{2m}}{m!} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{2m} \\ \vdots & \vdots & & \vdots \\ \lambda_0^{2m} & \lambda_1^{2m} & \cdots & \lambda_{2m}^{2m} \end{pmatrix} K = \begin{pmatrix} h^0 & 0 & \cdots & 0 \\ 0 & h^1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & h^{2m} \end{pmatrix}.$$

Hence, we can solve

$$K = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{2m} & \lambda_1^{2m} & \cdots & \lambda_{2m}^{2m} \end{pmatrix}^{-1} \begin{pmatrix} 0! & 0 & \cdots & 0 \\ 0 & 1! & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (2m)! \end{pmatrix}.$$

Our concern in this section is $(2m + 1)$ -point formula for the highest even $2m$ -th order derivatives. Therefore, we need to know the $(2m + 1)$ -th and $2m$ -th columns of the inverse matrix of the Vandermonde-type and ones of K . From the formula of the inverse matrix of the Vandermonde-type in [4] we obtain

$$\alpha_\sigma^{(2m)} = \frac{(2m)!}{\prod_{\ell=0, \ell \neq \sigma}^{2m} (\lambda_\ell - \lambda_\sigma)} - \frac{(2m - 1)! \sum_{\ell=0, \ell \neq \sigma}^{2m} \lambda_\ell}{\prod_{\ell=0, \ell \neq \sigma}^{2m} (\lambda_\ell - \lambda_\sigma)}$$

for $\sigma = 0, 1, \dots, 2m$. We note that

$$\sum_{\ell=0, \ell \neq \sigma}^{2m} \lambda_\ell = \sum_{\ell=0}^{2m} \lambda_\ell - \lambda_\sigma = -\lambda_\sigma = \sigma - m$$

and

$$\begin{aligned} \prod_{\ell=0, \ell \neq \sigma}^{2m} (\lambda_\ell - \lambda_\sigma) &= \prod_{\ell=0}^{\sigma-1} (\lambda_\ell - \lambda_\sigma) \prod_{\ell=\sigma+1}^{2m} (\lambda_\ell - \lambda_\sigma) \\ &= (-1)^{2m-(\sigma+1)+1} \prod_{\ell=0}^{\sigma-1} (\sigma - \ell) \prod_{\ell=\sigma+1}^{2m} (\ell - \sigma) \\ &= (-1)^{-\sigma} \sigma! (2m - \sigma). \end{aligned}$$

Hence, with the binomial coefficient ${}_{2m}C_\sigma$ we can write

$$(3.3) \quad \alpha_\sigma^{(2m)} = (-1)^{-\sigma} {}_{2m}C_\sigma (-1)^{-\sigma} {}_{2m}C_\sigma \frac{m - \sigma}{2m}$$

for $\sigma = 0, 1, \dots, 2m$. Here we remark that $\alpha_\sigma^{(2m)} = \alpha_{2m-\sigma}^{(2m)}$ and that $\alpha_\sigma^{(2m-1)} = -\alpha_{2m-\sigma}^{(2m-1)}$ which means $\alpha_m^{(2m-1)} = 0$.

Similarly as §1.2, φ must satisfies the condition

$$\mathcal{F}^{-1} \left[\left| \xi^m \hat{\varphi}^{(2m)}(\xi) \right|^2 \right] (\ell - k) = \begin{cases} (-1)^m \alpha_m^{(2m)} & \text{if } k = \ell, \\ (-1)^m \alpha_{m-1}^{(2m)} & \text{if } k = \ell \pm 1, \\ \vdots & \\ (-1)^m \alpha_{m-m}^{(2m)} & \text{if } k = \ell \pm m, \\ 0 & \text{otherwise,} \end{cases}$$

which is equivalent to

$$\sum_{q \in \mathbb{Z}} \left| (\xi + 2q\pi)^m \hat{\varphi}^{(2m)}(\xi + 2q\pi) \right|^2 \equiv (-1)^m \sum_{\sigma=0}^{2m} \alpha_\sigma^{(2m)} e^{(m-\sigma)i\xi} \left(= 4^m \sin^{2m} \frac{\xi}{2} \right)$$

for almost everywhere $\xi \in \mathbb{R}$, here we used by (3.3)

$$\begin{aligned} (-1)^m \sum_{\sigma=0}^{2m} \alpha_{\sigma}^{(2m)} e^{(m-\sigma)i\xi} &= \sum_{j=0}^{2m} {}_{2m}C_j \{ -e^{i\xi} \}^{m-\sigma} \\ &= (-e^{-i\xi})^m (1 - e^{i\xi})^{2m} = 4^m \sin^{2m} \frac{\xi}{2}. \end{aligned}$$

Hence, the identity (1.4) still holds for $\Phi(x)$. Thus, the definition (3.1) yields

$$(3.4) \quad \varphi^{(2m)}(x) = N_m * \Phi(x).$$

We remark that $\mathcal{E}(x) = N_m(x)$ satisfies (i) in Definition 1.1. As for (ii), we get the following theorem:

Theorem 3.1. *Let us put*

$$a_{k,\ell}^{(2m)} := (-1)^m \left\langle \left(\frac{d}{dx} \right)^m \varphi_{0,k}^{(2m)}, \left(\frac{d}{dx} \right)^m \varphi_{0,\ell}^{(2m)} \right\rangle_{L^2} \left(= \left\langle \left(\frac{d}{dx} \right)^{2m} \varphi_{0,k}^{(2m)}, \varphi_{0,\ell}^{(2m)} \right\rangle_{L^2} \right)$$

and

$$a_{k,\ell}^{(2m,r)} := (-1)^{\lfloor \frac{r}{2} \rfloor} \left\langle \left(\frac{d}{dx} \right)^{\lfloor \frac{r+1}{2} \rfloor} \varphi_{0,k}^{(2m)}, \left(\frac{d}{dx} \right)^{\lfloor \frac{r}{2} \rfloor} \varphi_{0,\ell}^{(2m)} \right\rangle_{L^2} = \left\langle \left(\frac{d}{dx} \right)^r \varphi_{0,k}^{(2m)}, \varphi_{0,\ell}^{(2m)} \right\rangle_{L^2}$$

for $0 \leq r \leq 2m - 1$. Then, for $\varphi^{(2m)}$ defined by (3.4) we have

$$(3.5) \quad a_{k,\ell}^{(2m)} = \begin{cases} (-1)^m {}_{2m}C_m & \text{if } k = \ell, \\ (-1)^{m-1} {}_{2m}C_{m-1} & \text{if } k = \ell \pm 1, \\ \vdots & \\ (-1)^{m-m} {}_{2m}C_{m-m} & \text{if } k = \ell \pm m, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, it holds that for $v \in C_0^{2m+2}$

$$(3.6) \quad \sum_{\nu \in \mathbb{Z}} a_{k,k+\nu}^{(2m,r)} v(x + \nu h) = h^r \frac{d^r}{dx^r} v(x) + O(h^{r+2}) \quad (0 \leq r \leq 2m - 1),$$

$$(3.7) \quad \sum_{\nu=-m}^m a_{k,k+\nu}^{(2m)} v(x + \nu h) = h^{2m} \frac{d^{2m}}{dx^{2m}} v(x) + O(h^{2m+2}).$$

Remark 3.2. For the discretization of the $2m$ -th order derivative we need the information on at least $(2m + 1)$ -points. Therefore, the stiffness matrix $\{a_{k,\ell}^{(2m)}\}$ given by (3.5) is the most sparse (the band width is $2m + 1$).

Proof of Theorem 3.1. (3.5) can be obtained by (3.3). Noting that $a_{k,k+\nu}^{(2m)} = \alpha_{m-\nu}^{(2m)}$, from the construction we immediately get (3.7). It remains to prove (3.6). By Parseval's theorem we see that

$$a_{k,\ell}^{(2m,r)} := \left\langle \frac{d^r}{dx^r} \varphi_{0,k}^{(2m)}, \varphi_{0,\ell}^{(2m)} \right\rangle_{L^2} = \mathcal{F}^{-1} \left[(i\xi)^r |\hat{\varphi}^{(2m)}(\xi)|^2 \right] (\ell - k)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{i(\ell-k)\xi} \sum_{q \in \mathbb{Z}} \{i(\xi + 2q\pi)\}^r |\hat{\varphi}^{(2m)}(\xi + 2q\pi)|^2 d\xi.$$

Hence, we find that

$$\sum_{q \in \mathbb{Z}} \{i(\xi + 2q\pi)\}^r |\hat{\varphi}^{(2m)}(\xi + 2q\pi)|^2 d\xi = \sum_{\nu \in \mathbb{Z}} a_{k,k+\nu}^{(2m,r)} e^{i\nu\xi},$$

here we remark that $a_{k,k+\nu}^{(2m,r)}$ is constant for all $k \in \mathbb{Z}$. Furthermore, we have for $0 \leq r \leq 2m - 1$

$$\begin{aligned} & \lim_{\xi \rightarrow 0} \frac{1}{(i\xi)^r} \sum_{\nu \in \mathbb{Z}} a_{k,k+\nu}^{(2m,r)} e^{i\nu\xi} = \lim_{\xi \rightarrow 0} \sum_{q \in \mathbb{Z}} \frac{\{i(\xi + 2q\pi)\}^r |\hat{\varphi}^{(2m)}(\xi + 2q\pi)|^2}{(i\xi)^r} \\ &= \lim_{\xi \rightarrow 0} \sum_{q \in \mathbb{Z}, q \neq 0} \frac{\{i(\xi + 2q\pi)\}^r |\sin^m(\xi/2)\hat{\Phi}(\xi + 2q\pi)|^2}{|(\xi + 2q\pi)/2|^{2m}(i\xi)^r} + \lim_{\xi \rightarrow 0} \frac{|\sin^m(\xi/2)\hat{\Phi}(\xi)|^2}{|\xi/2|^{2m}} \\ &= \lim_{\xi \rightarrow 0} \sum_{q \in \mathbb{Z}, q \neq 0} \frac{\{i(2q\pi)\}^r |\sin^m \xi \hat{\Phi}(2q\pi)|^2}{|q\pi|^{2m}(i\xi)^r} + |\hat{\Phi}(0)|^2 = 1. \end{aligned}$$

Replacing ξ by $h\xi$ and multiplying the both sides by $(i\xi)^r \hat{u}(\xi)$, we also get the identity

$$\lim_{h \rightarrow 0} \frac{1}{h^r} \sum_{\nu \in \mathbb{Z}} a_{k,k+\nu}^{(2m,r)} e^{i\nu\xi} \hat{u}(\xi) = (i\xi)^r \hat{u}(\xi).$$

By inverse Fourier transform it follows that

$$\lim_{h \rightarrow 0} \frac{\sum_{\nu \in \mathbb{Z}} a_{k,k+\nu}^{(2m,r)} u(x + \nu h)}{h^r} = \frac{d^r}{dx^r} u(x),$$

which means (3.6). □

4. Numerical results

Let us introduce some examples and numerical results in this section.

| Riesz base | Choice of Φ | Elevator \mathcal{E} | Length of support | Regularity in x | Remainder |
|-----------------------|------------------|---------------------------------------|-------------------|-------------------|-----------|
| φ | N_1 | N_1 | 2 | C^1 (Lip) | $O(h^2)$ |
| φ_2^D | Φ_2^D | N_1 | 4 | $C^{1.5}$ | $O(h^2)$ |
| \tilde{N}_2 | N_1 | $\gamma^+ N_1(x-1) + \gamma^- N_1(x)$ | 3 | C^2 | $O(h^4)$ |
| $\tilde{\varphi}_2^D$ | Φ_2^D | $\gamma^+ N_1(x-1) + \gamma^- N_1(x)$ | 5 | $C^{1.5}$ | $O(h^4)$ |
| \tilde{N}_3 | N_1 | N_2 | 3 | C^2 | $O(h^2)$ |
| $\tilde{\varphi}_2^D$ | Φ_2^D | N_2 | 5 | $C^{2.5}$ | $O(h^2)$ |

The boundary valued problem for

$$-\varepsilon^2 \frac{d^2}{dx^2} u^{(\varepsilon)} + u^{(\varepsilon)} = f, \quad 0 < x < 1, \quad u^{(\varepsilon)}(0) = u^{(\varepsilon)}(1) = 0,$$

has a solution represented by

$$(4.1) \quad u^{(\varepsilon)}(x) = -\frac{\sinh(x/\varepsilon)}{\varepsilon \sinh(1/\varepsilon)} \int_0^1 \sinh \frac{y-1}{\varepsilon} f(y) dy + \frac{1}{\varepsilon} \int_0^x \sinh \frac{y-x}{\varepsilon} f(y) dy.$$

For $f(x) = \sin 10\pi x$, by (4.1) the exact solution is $u^{(\varepsilon)}(x) = \frac{\sin 10\pi x}{1+100\varepsilon^2\pi^2}$ and the errors with the Riesz bases N_2 and φ_2^D are given by the following:

(i) The case of $\varepsilon = 1$.

| Mesh size 2^{-j} | $E_j^{N_2}$ | Q^{N_2} | $E_j^{\varphi_2^D}$ | $Q^{\varphi_2^D}$ |
|--------------------|-----------------------|-----------|-----------------------|-------------------|
| $j = 6$ | 1.57×10^{-4} | 2.67 | 2.02×10^{-2} | 6.07 |
| $j = 7$ | 5.67×10^{-5} | 2.77 | 3.58×10^{-3} | 5.64 |
| $j = 8$ | 2.02×10^{-5} | 2.81 | 8.47×10^{-4} | 4.23 |

(ii) The case of $\varepsilon = 10^{-6}$.

| Mesh size 2^{-j} | $E_j^{N_2}$ | Q^{N_2} | $E_j^{\varphi_2^D}$ | $Q^{\varphi_2^D}$ |
|--------------------|-----------------------|-----------|-----------------------|-------------------|
| $j = 6$ | 1.58×10^{-1} | 2.82 | 6.21×10^{-4} | 5.57 |
| $j = 7$ | 5.62×10^{-2} | 2.81 | 1.17×10^{-4} | 5.32 |
| $j = 8$ | 2.00×10^{-2} | 2.82 | 6.21×10^{-4} | 5.57 |

For $f(x) = -\varepsilon^2(9N_1(3x) - 18N_1(3x-1) + N_1(3x-2)) + N_3(3x)$, the exact solution $u(x) = N_3(3x)$ belongs to H^2 , since $\int_{\mathbb{R}} \langle \xi \rangle^4 |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}} \langle \xi \rangle^4 \left(\frac{\sin \xi/6}{\xi/6}\right)^6 d\xi < \infty$. The errors are given by the following:

(iii) The case of $\varepsilon = 1$.

| Mesh size 2^{-j} | $E_j^{N_2}$ | Q^{N_2} | $E_j^{\varphi_2^D}$ | $Q^{\varphi_2^D}$ |
|--------------------|-----------------------|-----------|-----------------------|-------------------|
| $j = 6$ | 1.43×10^{-4} | 2.81 | 6.08×10^{-4} | 4.00 |
| $j = 7$ | 5.20×10^{-5} | 2.75 | 1.52×10^{-4} | 4.00 |
| $j = 8$ | 1.84×10^{-5} | 2.82 | 3.82×10^{-5} | 3.98 |

(iv) The case of $\varepsilon = 10^{-6}$.

| Mesh size 2^{-j} | $E_j^{N_2}$ | Q^{N_2} | $E_j^{\varphi_2^D}$ | $Q^{\varphi_2^D}$ |
|--------------------|-----------------------|-----------|-----------------------|-------------------|
| $j = 6$ | 4.68×10^{-3} | 2.79 | 5.59×10^{-4} | 4.00 |
| $j = 7$ | 1.69×10^{-3} | 2.77 | 1.40×10^{-4} | 3.99 |
| $j = 8$ | 6.00×10^{-4} | 2.82 | 3.51×10^{-5} | 3.99 |

Here E_j^φ is relative L^2 -error between the exact solution $u^{(\varepsilon)}(x)$ and the approximation $\tilde{u}^{(\varepsilon)}(x) = \sum_{\ell=1}^N u_\ell \varphi_{j,\ell}(x)$ on $[0, 1]$ defined by

$$E_j^\varphi = E_j^\varphi(\varepsilon) = \frac{1}{\|u^{(\varepsilon)}\|_{L^2}} \sqrt{\sum_{\ell=0}^{2^j} \left\{ u^{(\varepsilon)}\left(\frac{\ell}{2^j}\right) - \tilde{u}^{(\varepsilon)}\left(\frac{\ell}{2^j}\right) \right\}^2}$$

and the ratio Q^φ is defined by $Q^\varphi = E_{j-1}^\varphi / E_j^\varphi$.

Now, we also consider the beam equation (4th order differential equation)

$$\frac{d^4}{dx^4}u + u = f, \quad 0 < x < 1, \quad u(0) = u(1) = u'(0) = u'(1) = 0,$$

and its bilinear form

$$\left\langle \frac{d^2}{dx^2}u, \frac{d^2}{dx^2}\varphi \right\rangle_{L^2} + \langle u, \varphi \rangle_{L^2} = \langle f, \varphi \rangle_{L^2}.$$

To derive the approximate solution we solve the following equation:

$$\left[\{a_{k,\ell}h^{-3}\}_{1 \leq k, \ell \leq N} + \{c_{k,\ell}h\}_{1 \leq k, \ell \leq N} \right]^t \{u_\ell\}_{1 \leq \ell \leq N} = {}^t \{ \langle f, \varphi_{j,\ell} \rangle_{L^2} \}_{1 \leq \ell \leq N},$$

where $c_{k,\ell} = \langle \varphi_{0,k}, \varphi_{0,\ell} \rangle$. If φ satisfies (3.1), from (3.5) it holds that

$$(4.2) \quad a_{k,\ell} = \begin{cases} 6 & \text{if } k = \ell, \\ -4 & \text{if } k = \ell \pm 1, \\ 1 & \text{if } k = \ell \pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

For $N_3 = N_2 * N_1$ and $\overset{\circ}{\varphi}_2^D = N_2 * \Phi_2^D$, we get the following numerical results:

(v) The case of $f(x) = \sin 10\pi x$ and

$$u(x) = \frac{10\sqrt{2}\pi \left(e^{\frac{1}{\sqrt{2}}} (1 - e^{\sqrt{2}x}) \sin \frac{x-1}{\sqrt{2}} - (e^{\sqrt{2}} - e^{\sqrt{2}x}) \sin \frac{x}{\sqrt{2}} \right) + e^{\frac{x}{\sqrt{2}}} \left(e^{\sqrt{2}} - 1 - 2e^{\frac{1}{\sqrt{2}}} \sin \frac{1}{\sqrt{2}} \right) \sin 10\pi x}{(1 + 10000\pi^4) e^{\frac{x}{\sqrt{2}}} \left(e^{\sqrt{2}} - 1 - 2e^{\frac{1}{\sqrt{2}}} \sin \frac{1}{\sqrt{2}} \right)}.$$

| Mesh size 2^{-j} | $E_j^{N_3}$ | Q^{N_3} | $E_j^{\overset{\circ}{\varphi}_2^D}$ | $\tilde{Q}^{\overset{\circ}{\varphi}_2^D}$ |
|--------------------|-----------------------|-----------|--------------------------------------|--|
| $j = 6$ | 5.02×10^{-2} | 2.86 | 2.44 | 3.03 |
| $j = 7$ | 1.77×10^{-2} | 2.84 | 3.11×10^{-1} | 7.86 |
| $j = 8$ | 6.25×10^{-3} | 2.83 | 3.39×10^{-1} | 0.918 |

(vi) The case of $f(x) = x$ and

$$\begin{aligned} u(x) = & \left\{ \left(\left(e^{\frac{2x+3}{\sqrt{2}}} + e^{\frac{1}{\sqrt{2}}} \right) \left(\sqrt{2}e^{\sqrt{2}} - \sqrt{2} + 1 \right) - 2e^{\frac{1}{\sqrt{2}}} \right) \sin \frac{x-1}{\sqrt{2}} \right. \\ & - e^{\frac{1}{\sqrt{2}}} \left(e^{\sqrt{2}x} (e^{\sqrt{2}} - 2 - \sqrt{2}) - e^{\sqrt{2}} (2 - \sqrt{2}) + 1 \right) \cos \frac{x-1}{\sqrt{2}} \\ & + \sqrt{2} (e^{\sqrt{2}} - 1) (e^{\sqrt{2}x} - e^{\sqrt{2}}) \sin \frac{x}{\sqrt{2}} + \left(e^{\frac{3}{\sqrt{2}}} - e^{\frac{2x+1}{\sqrt{2}}} \right) \sin \frac{x+1}{\sqrt{2}} \\ & + \sqrt{2} e^{\sqrt{2}} \left(e^{\sqrt{2}x} - 1 \right) \left(\cos \frac{x}{\sqrt{2}} - \cos \frac{x-2}{\sqrt{2}} \right) \\ & + \left. \left((\sqrt{2} - 1) e^{\frac{3}{\sqrt{2}}} - (1 + \sqrt{2}) e^{\frac{2x+1}{\sqrt{2}}} \right) \cos \frac{x+1}{\sqrt{2}} \right\} \\ & \times \left\{ 2e^{\frac{x+2}{\sqrt{2}}} \left(-2 + \cos \sqrt{2} + \cosh \sqrt{2} \right) \right\}^{-1} \\ & + x. \end{aligned}$$

| Mesh size 2^{-j} | $E_j^{N_3}$ | Q^{N_3} | $E_j^{\varphi_2^D}$ | $\tilde{Q}^{\varphi_2^D}$ |
|--------------------|-----------------------|-----------|-----------------------|---------------------------|
| $j = 6$ | 9.02×10^{-3} | 2.83 | 1.33 | 1.33 |
| $j = 7$ | 3.19×10^{-3} | 2.83 | 9.71×10^{-1} | 1.37 |
| $j = 8$ | 1.13×10^{-3} | 2.83 | 6.97×10^{-1} | 1.39 |

(vii) The case of $f(x) = N_5(5x) + 625N_1(5x) - 2500N_1(5x-1) + 3750N_1(5x-2) - 2500N_1(5x-3) + 625N_1(5x-4)$ and $u(x) = N_5(5x) \in H^4$.

| Mesh size 2^{-j} | $E_j^{N_3}$ | Q^{N_3} | $E_j^{\varphi_2^D}$ | $\tilde{Q}^{\varphi_2^D}$ |
|--------------------|-----------------------|-----------|-----------------------|---------------------------|
| $j = 6$ | 7.81×10^{-3} | 2.83 | 1.14×10^{-4} | 5.18 |
| $j = 7$ | 2.76×10^{-3} | 2.83 | 1.14×10^{-4} | 5.18 |
| $j = 8$ | 9.75×10^{-4} | 2.83 | 2.20×10^{-4} | 5.17 |

(viii) The case of $f(x) = N_4(4x) - 256\delta(x-1/4) + 384\delta(x-1/2) - 256\delta(x-3/4)$ and $u(x) = N_4(4x) \in H^s$ for $s < 3.5 (< 4)$.

| Mesh size 2^{-j} | $E_j^{N_3}$ | Q^{N_3} | $E_j^{\varphi_2^D}$ | $\tilde{Q}^{\varphi_2^D}$ |
|--------------------|-----------------------|-----------|-----------------------|---------------------------|
| $j = 6$ | 6.15×10^{-3} | 2.85 | 5.98×10^{-2} | 2.87 |
| $j = 7$ | 2.17×10^{-3} | 2.83 | 2.10×10^{-2} | 2.85 |
| $j = 8$ | 7.67×10^{-4} | 2.83 | 7.41×10^{-3} | 2.83 |

Concluding remarks

The method with the elevated Riesz bases converts a continuous operator to a discrete problem by featuring the highest order derivatives. Therefore, we can consider the following advantages:

(1) For $-\varepsilon^2 \frac{d^2}{dx^2} u(\varepsilon) + u(\varepsilon) = f$, if $\varepsilon > 0$ is smaller, $-\varepsilon^2 \frac{d^2}{dx^2}$ gives more perturbation to the solution. The influence of the parameter $\varepsilon > 0$ can be reduced to some extent. Actually, for φ_2^D the relative L^2 -error $E_j^{\varphi_2^D}$ is stable for a smaller $\varepsilon > 0$ (see $E_j^{\varphi_2^D}$ in (i), (ii), (iii), (iv)).

(2) For $(-1)^m \frac{d^{2m}}{dx^{2m}} u + u = f$, if the solution has lower regularity, $(-1)^m \frac{d^{2m}}{dx^{2m}}$ plays a more role in the structure of the equation. When we consider the solution in H^{2m} or of less regularity in H^s ($s < 2m$), for instance $m = 2$, the relative L^2 -errors $E_j^{N_3}$ and $E_j^{\varphi_2^D}$ keep good results even in comparison with $m = 1$ (see (iii), (vii), (viii)).

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