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SEMI-SLANT SUBMERSIONS

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Abstract. We introduce semi-slant submersions from almost Hermitian manifolds onto Riemannian manifolds as a generalization of slant submersions, semi-invariant submersions, anti-invariant submersions, etc. We obtain characterizations, investigate the integrability of distributions and the geometry of foliations, etc. We also find a condition for such submersions to be harmonic. Moreover, we give lots of examples.

1. Introduction

Let F be a C^∞-submersion from a semi-Riemannian manifold (M, g_M) onto a semi-Riemannian manifold (N, g_N). Then according to the conditions on the map $F : (M, g_M) \mapsto (N, g_N)$, we have the following submersions:

Semi-Riemannian submersion and Lorentzian submersion [7], Riemannian submersion ([8], [14]), slant submersion ([5], [17]), almost Hermitian submersion [20], contact-complex submersion [9], quaternionic submersion [10], almost h-slant submersion and h-slant submersion [15], anti-invariant submersion [19], semi-invariant submersion [18], h-semi-invariant submersion [16], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([3], [21]), Kaluza-Klein theory ([2], [11]), Supergravity and superstring theories ([12], [13]), etc. Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F : M \mapsto N$ a C^∞-submersion. The map F is said to be *Riemannian submersion* if the differential F_* preserves the lengths of horizontal vectors [10]. Let (M, g_M, J) and (M_1, g_{M_1}, J_1) be almost Hermitian manifolds. A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *slant submersion* if the angle $\theta(X)$ between JX and the space $\ker(F_*)_p$ is constant for any nonzero $X \in T_pM$ and $p \in M$ [17]. We call $\theta(X)$ a *slant angle.* A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called an *anti-invariant submersion* if $JX \in \Gamma((\ker F_*)^\perp)$ for $X \in \Gamma(\ker F_*)$ [19]. A Riemannian submersion $F : (M, g_M, J) \mapsto (M_1, g_{M_1}, J_1)$ is called an *almost Hermitian submersion* if F is an almost complex map, i.e., $F_* \circ J = J_1 \circ F_*$ [20]. A Riemannian

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submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *semi-invariant submersion* if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ J(\mathcal{D}_1) = \mathcal{D}_1,\ J(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$ [17]. Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F : (M, g_M) \mapsto (N, g_N)$ a smooth map. The second fundamental form of F is given by

$$(\nabla F_*)(X, Y) := \nabla^F_X F_* Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where ∇^F is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N [4]. Recall that F is said to be *harmonic* if $trace(\nabla F_*) = 0$ and F is called a *totally geodesic* map if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [4]. The paper is organized as follows. In Section 2 we give the definition of the semi-slant submersion and obtain some interesting properties on them. In Section 3 we construct some examples for the semi-slant submersion.

2. Semi-slant submersions

Definition 2.1. Let (M, g_M, J) be an almost Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *semi-slant submersion* if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We call the angle θ a *semi-slant angle*.

Remark 2.2. As we know, a semi-slant submersion is the generalized version of a slant submersion. There are some similarities and differences between them. For the condition for such submersions to be harmonic, a semi-slant submersion has much more nice form than a slant submersion. But for the one for such submersions to be totally geodesic, two cases have the same condition. With the tensor ω to be parallel, we obtain some results on the slant submersions. For the semi-slant submersions with totally umbilical fibers, we have some results for the mean curvature vector field.

Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a semi-slant submersion. Then there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

Then for $X \in \Gamma(\ker F_*)$, we have

$$X = PX + QX,$$

where $PX \in \Gamma(\mathcal{D}_1)$ and $QX \in \Gamma(\mathcal{D}_2)$.

For $X \in \Gamma(\ker F_*)$, we get

$$JX = \phi X + \omega X,$$

where $\phi X \in \Gamma(\ker F_*)$ and $\omega X \in \Gamma((\ker F_*)^\perp)$.

For $Z \in \Gamma((\ker F_*)^\perp)$, we obtain

$$JZ = BZ + CZ,$$

where $BZ \in \Gamma(\ker F_*)$ and $CZ \in \Gamma((\ker F_*)^\perp)$.

For $U \in \Gamma(TM)$, we have

$$U = \mathcal{V}U + \mathcal{H}U,$$

where $\mathcal{V}U \in \Gamma(\ker F_*)$ and $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$.

Then

$$(\ker F_*)^\perp = \omega\mathcal{D}_2 \oplus \mu,$$

where μ is the orthogonal complement of $\omega\mathcal{D}_2$ in $(\ker F_*)^\perp$ and is invariant under J. Furthermore,

$$\phi\mathcal{D}_1 = \mathcal{D}_1,\ \omega\mathcal{D}_1 = 0,\ \phi\mathcal{D}_2 \subset \mathcal{D}_2,\ B((\ker F_*)^\perp) = \mathcal{D}_2$$
$$\phi^2 + B\omega = -id,\ C^2 + \omega B = -id,\ \omega\phi + C\omega = 0,\ BC + \phi B = 0.$$

Define the tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \quad \mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

for vector fields E, F on M, where ∇ is the Levi-Civita connection of g_M. Define

$$(\nabla_X\phi)Y := \widehat{\nabla}_X\phi Y - \phi\widehat{\nabla}_X Y$$

and

$$(\nabla_X\omega)Y := \mathcal{H}\nabla_X\omega Y - \omega\widehat{\nabla}_X Y$$

for $X, Y \in \Gamma(\ker F_*)$, where $\widehat{\nabla}_X Y := \mathcal{V}\nabla_X Y$. Then we easily have:

Lemma 2.3. *Let (M, g_M, J) be a Kähler manifold and (N, g_N) a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a semi-slant submersion. Then we get*

(a)
$$\widehat{\nabla}_X\phi Y + \mathcal{T}_X\omega Y = \phi\widehat{\nabla}_X Y + B\mathcal{T}_X Y,$$
$$\mathcal{T}_X\phi Y + \mathcal{H}\nabla_X\omega Y = \omega\widehat{\nabla}_X Y + C\mathcal{T}_X Y$$

for $X, Y \in \Gamma(\ker F_*)$.

(b)
$$\mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW = \phi\mathcal{A}_Z W + B\mathcal{H}\nabla_Z W,$$
$$\mathcal{A}_Z BW + \mathcal{H}\nabla_Z CW = \omega\mathcal{A}_Z W + C\mathcal{H}\nabla_Z W$$

for $Z, W \in \Gamma((\ker F_*)^\perp)$.

(c)
$$\widehat{\nabla}_X BZ + \mathcal{T}_X CZ = \phi\mathcal{T}_X Z + B\mathcal{H}\nabla_X Z,$$
$$\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ = \omega\mathcal{T}_X Z + C\mathcal{H}\nabla_X Z$$

for $X \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$.

Theorem 2.4. *Let* F *be a semi-slant submersion from an almost Hermitian manifold* (M, g_M, J) *onto a Riemannian manifold* (N, g_N). *Then the complex distribution* $\mathcal{D}_1$ *is integrable if and only if we have*

$$\omega(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X) = C(\mathcal{T}_Y X - \mathcal{T}_X Y) \quad \text{for } X, Y \in \Gamma(\mathcal{D}_1).$$

Proof. For $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma((\ker F_*)^\perp)$, since $[X, Y] \in \Gamma(\ker F_*)$, we obtain

$$\begin{aligned} g_M(J[X,Y], Z) &= g_M(J(\nabla_X Y - \nabla_Y X), Z) \\ &= g_M(\phi\widehat{\nabla}_X Y + \omega\widehat{\nabla}_X Y + B\mathcal{T}_X Y + C\mathcal{T}_X Y - \phi\widehat{\nabla}_Y X - \omega\widehat{\nabla}_Y X \\ &\quad - B\mathcal{T}_Y X - C\mathcal{T}_Y X, Z) \\ &= g_M(\omega\widehat{\nabla}_X Y + C\mathcal{T}_X Y - \omega\widehat{\nabla}_Y X - C\mathcal{T}_Y X, Z). \end{aligned}$$

Therefore, we have the result. □

Similarly, we get:

Theorem 2.5. *Let* F *be a semi-slant submersion from an almost Hermitian manifold* (M, g_M, J) *onto a Riemannian manifold* (N, g_N). *Then the slant distribution* $\mathcal{D}_2$ *is integrable if and only if we obtain*

$$P(\phi(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X) + B(\mathcal{T}_X Y - \mathcal{T}_Y X)) = 0 \quad \text{for } X, Y \in \Gamma(\mathcal{D}_2).$$

Lemma 2.6. *Let* (M, g_M, J) *be a Kähler manifold and* (N, g_N) *a Riemannian manifold. Let* $F : (M, g_M, J) \mapsto (N, g_N)$ *be a semi-slant submersion. Then the slant distribution* $\mathcal{D}_2$ *is integrable if and only if we obtain*

$$P(\widehat{\nabla}_X \phi Y - \widehat{\nabla}_Y \phi X + \mathcal{T}_X \omega Y - \mathcal{T}_Y \omega X) = 0 \quad \text{for } X, Y \in \Gamma(\mathcal{D}_2).$$

Proof. For $X, Y \in \Gamma(\mathcal{D}_2)$ and $Z \in \Gamma(\mathcal{D}_1)$, since $[X, Y] \in \Gamma(\ker F_*)$, we have

$$\begin{aligned} g_M(J[X,Y], Z) &= g_M(\nabla_X JY - \nabla_Y JX, Z) \\ &= g_M(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \phi Y + \mathcal{T}_X \omega Y + \mathcal{H}\nabla_X \omega Y - \widehat{\nabla}_Y \phi X - \mathcal{T}_Y \phi X \\ &\quad - \mathcal{T}_Y \omega X - \mathcal{H}\nabla_Y \omega X, Z) \\ &= g_M(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y - \widehat{\nabla}_Y \phi X - \mathcal{T}_Y \omega X, Z). \end{aligned}$$

Therefore, the result follows. □

In a similar way, we have:

Lemma 2.7. *Let* (M, g_M, J) *be a Kähler manifold and* (N, g_N) *a Riemannian manifold. Let* $F : (M, g_M, J) \mapsto (N, g_N)$ *be a semi-slant submersion. Then the complex distribution* $\mathcal{D}_1$ *is integrable if and only if we get*

$$Q(\widehat{\nabla}_X \phi Y - \widehat{\nabla}_Y \phi X) = 0 \text{ and } \mathcal{T}_X \phi Y = \mathcal{T}_Y \phi X \quad \text{for } X, Y \in \Gamma(\mathcal{D}_1).$$

Define an endomorphism $\widehat{F}$ of $\ker F_*$ by

$$\widehat{F} := JP + \phi Q,$$

where $(\nabla_X \widehat{F})Y := \widehat{\nabla}_X \widehat{F} Y - \widehat{F} \widehat{\nabla}_X Y$ for $X, Y \in \Gamma(\ker F_*)$. Then it is not difficult to get.

Lemma 2.8. *Let F be a semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then we have*

$$(\nabla_X \widehat{F})Y = \phi(\widehat{\nabla}_X PY - \widehat{\nabla}_X Y) + B\mathcal{T}_X PY + \widehat{\nabla}_X \phi QY \quad \text{for } X, Y \in \Gamma(\ker F_*).$$

Proposition 2.9. *Let F be a semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then we obtain*

$$\phi^2 X = -\cos^2\theta X \quad \text{for } X \in \Gamma(\mathcal{D}_2),$$

where θ denotes the semi-slant angle of $\mathcal{D}_2$.

Proof. Since

$$\cos\theta = \frac{g_M(JX, \phi X)}{|JX| \cdot |\phi X|} = \frac{-g_M(X, \phi^2 X)}{|X| \cdot |\phi X|} \text{ and } \cos\theta = \frac{|\phi X|}{|JX|},$$

we have

$$\cos^2\theta = -\frac{g_M(X, \phi^2 X)}{|X|^2} \quad \text{for } X \in \Gamma(\mathcal{D}_2).$$

Hence,

$$\phi^2 X = -\cos^2\theta X \quad \text{for } X \in \Gamma(\mathcal{D}_2).$$

□

Remark 2.10. In particular, we easily see that the converse of Proposition 2.9 is also true.

Assume that the semi-slant angle θ is not equal to $\frac{\pi}{2}$ and define an endomorphism $\widehat{J}$ of $\ker F_*$ by

$$\widehat{J} := JP + \frac{1}{\cos\theta}\phi Q.$$

Then,

$$\widehat{J}^2 = -id \quad \text{on } \ker F_*. \tag{1}$$

Remark 2.11. Let F be a semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Assume that $\dim M = 2m$, $\dim N = n$, and $\theta \in [0, \frac{\pi}{2})$. From (1), we have

$$\dim(\ker(F_*)_p) = 2k \text{ and } \dim((\ker(F_*)_p)^\perp) = 2m - 2k \quad \text{for } p \in M,$$

where k is a non-negative integer.

Therefore, n must be even.

Theorem 2.12. *Let F be a semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) with the semi-slant angle $\theta \in [0, \frac{\pi}{2})$. Then N is an even-dimensional manifold.*

Proposition 2.13. *Let F be a semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distribution* $\ker F_*$ *defines a totally geodesic foliation if and only if*

$$\omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) = 0 \quad \textit{for } X, Y \in \Gamma(\ker F_*).$$

Proof. For $X, Y \in \Gamma(\ker F_*)$,

$$\begin{aligned}\nabla_X Y &= -J\nabla_X JY \\ &= -J(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \phi Y + \mathcal{T}_X \omega Y + \mathcal{H}\nabla_X \omega Y) \\ &= -(\phi\widehat{\nabla}_X \phi Y + \omega\widehat{\nabla}_X \phi Y + B\mathcal{T}_X \phi Y + C\mathcal{T}_X \phi Y + \phi\mathcal{T}_X \omega Y + \omega\mathcal{T}_X \omega Y \\ &\quad + B\mathcal{H}\nabla_X \omega Y + C\mathcal{H}\nabla_X \omega Y).\end{aligned}$$

Thus,

$$\nabla_X Y \in \Gamma(\ker F_*) \Leftrightarrow \omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) = 0. \quad \square$$

Similarly, we have:

Proposition 2.14. *Let F be a semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distribution* $(\ker F_*)^\perp$ *defines a totally geodesic foliation if and only if*

$$\phi(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY) + B(\mathcal{A}_X BY + \mathcal{H}\nabla_X CY) = 0 \quad \textit{for } X, Y \in \Gamma((\ker F_*)^\perp).$$

Proposition 2.15. *Let F be a semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distribution $\mathcal{D}_1$ defines a totally geodesic foliation if and only if*

$$Q(\phi\widehat{\nabla}_X \phi Y + B\mathcal{T}_X \phi Y) = 0 \textit{ and } \omega\widehat{\nabla}_X \phi Y + C\mathcal{T}_X \phi Y = 0$$

for $X, Y \in \Gamma(\mathcal{D}_1)$.

Proof. For $X, Y \in \Gamma(\mathcal{D}_1)$, we get

$$\begin{aligned}\nabla_X Y &= -J\nabla_X JY \\ &= -J(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \phi Y) \\ &= -(\phi\widehat{\nabla}_X \phi Y + \omega\widehat{\nabla}_X \phi Y + B\mathcal{T}_X \phi Y + C\mathcal{T}_X \phi Y).\end{aligned}$$

Hence,

$$\nabla_X Y \in \Gamma(\mathcal{D}_1) \Leftrightarrow Q(\phi\widehat{\nabla}_X \phi Y + B\mathcal{T}_X \phi Y) = 0 \text{ and } \omega\widehat{\nabla}_X \phi Y + C\mathcal{T}_X \phi Y = 0. \square$$

In a similar way, we obtain:

Proposition 2.16. *Let F be a semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distribution $\mathcal{D}_2$ defines a totally geodesic foliation if and only if*

$$P(\phi(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + B(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y)) = 0,$$

$$\omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

Theorem 2.17. *Let F be a semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then F is a totally geodesic map if and only if*

$$\omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) = 0,$$
$$\omega(\widehat{\nabla}_X BZ + \mathcal{T}_X CZ) + C(\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ) = 0$$

for $X, Y \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$.

Proof. Since F is a Riemannian submersion, we have

$$(\nabla F_*)(Z_1, Z_2) = 0 \quad \text{for } Z_1, Z_2 \in \Gamma((\ker F_*)^\perp).$$

For $X, Y \in \Gamma(\ker F_*)$, we obtain

$$\begin{aligned}(\nabla F_*)(X, Y) &= -F_*(\nabla_X Y)\\ &= F_*(J\nabla_X(\phi Y + \omega Y))\\ &= F_*(\phi\widehat{\nabla}_X \phi Y + \omega\widehat{\nabla}_X \phi Y + B\mathcal{T}_X \phi Y + C\mathcal{T}_X \phi Y + \phi\mathcal{T}_X \omega Y\\ &\quad + \omega\mathcal{T}_X \omega Y + B\mathcal{H}\nabla_X \omega Y + C\mathcal{H}\nabla_X \omega Y).\end{aligned}$$

Thus,

$$(\nabla F_*)(X, Y) = 0 \Leftrightarrow \omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) = 0.$$

For $X \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$, we get

$$\begin{aligned}(\nabla F_*)(X, Z) &= -F_*(\nabla_X Z)\\ &= F_*(J\nabla_X(BZ + CZ))\\ &= F_*(\phi\widehat{\nabla}_X BZ + \omega\widehat{\nabla}_X BZ + B\mathcal{T}_X BZ + C\mathcal{T}_X BZ + \phi\mathcal{T}_X CZ\\ &\quad + \omega\mathcal{T}_X CZ + B\mathcal{H}\nabla_X CZ + C\mathcal{H}\nabla_X CZ).\end{aligned}$$

Hence,

$$(\nabla F_*)(X, Z) = 0 \Leftrightarrow \omega(\widehat{\nabla}_X BZ + \mathcal{T}_X CZ) + C(\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ) = 0.$$

Since $(\nabla F_*)(X, Z) = (\nabla F_*)(Z, X)$, we get the result. □

Let F be a semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Assume that $\mathcal{D}_1$ is integrable. Choose a local orthonormal frame $\{v_1, \dots, v_l\}$ of $\mathcal{D}_2$ and a local orthonormal frame $\{e_1, \dots, e_{2k}\}$ of $\mathcal{D}_1$ such that $e_{2i} = Je_{2i-1}$ for $1 \le i \le k$. Since

$$F_*(\nabla_{Je_{2i-1}} Je_{2i-1}) = -F_*(\nabla_{e_{2i-1}} e_{2i-1})$$

for $1 \le i \le k$, we have

$$trace(\nabla F_*) = 0 \Leftrightarrow \sum_{j=1}^{l} F_*(\nabla_{v_j} v_j) = 0.$$

Theorem 2.18. *Let F be a semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) such that $\mathcal{D}_1$ is integrable. Then F is a harmonic map if and only if*

$$trace(\nabla F_*) = 0 \quad on\ \mathcal{D}_2.$$

Let $F : (M, g_M) \mapsto (N, g_N)$ be a Riemannian submersion. The map F is called a Riemannian submersion *with totally umbilical fibers* if

$$\mathcal{T}_X Y = g_M(X, Y)H \quad \text{for } X, Y \in \Gamma(\ker F_*), \tag{2}$$

where H is the mean curvature vector field of the fiber.

In a similar way with Lemma 4.2 of [18], we obtain:

Lemma 2.19. *Let F be a semi-slant submersion with totally umbilical fibers from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then we have*

$$H \in \Gamma(\omega\mathcal{D}_2).$$

Proof. For $X, Y \in \Gamma(\mathcal{D}_1)$ and $W \in \Gamma(\mu)$, we get

$$\mathcal{T}_X JY + \widehat{\nabla}_X JY = \nabla_X JY = J\nabla_X Y = B\mathcal{T}_X Y + C\mathcal{T}_X Y + \phi\widehat{\nabla}_X Y + \omega\widehat{\nabla}_X Y$$

so that

$$g_M(\mathcal{T}_X JY, W) = g_M(C\mathcal{T}_X Y, W).$$

By (2), with a simple calculation we obtain

$$g_M(X, JY)g_M(H, W) = -g_M(X, Y)g_M(H, JW).$$

Interchanging the role of X and Y, we get

$$g_M(Y, JX)g_M(H, W) = -g_M(Y, X)g_M(H, JW)$$

so that combining the above two equations, we have

$$g_M(X, Y)g_M(H, JW) = 0$$

which means $H \in \Gamma(\omega\mathcal{D}_2)$, since $J\mu = \mu$. Therefore, we obtain the result. □

Remark 2.20. Let F be a semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$. Furthermore,

$$\phi\mathcal{D}_2 \subset \mathcal{D}_2,\ \omega\mathcal{D}_2 \subset (\ker F_*)^\perp,\ (\ker F_*)^\perp = \omega\mathcal{D}_2 \oplus \mu,$$

where μ is the orthogonal complement of $\omega\mathcal{D}_2$ in $(\ker F_*)^\perp$ and is invariant under J. As we know, the holomorphic sectional curvatures determine the Riemannian curvature tensor in a Kähler manifold.

Given a plane P being invariant by J in T_pM, $p \in M$, there is an orthonormal basis $\{X, JX\}$ of P. Denote by $K(P)$, $K_*(P)$, and $\widehat{K}(P)$ the sectional curvatures of the plane P in M, N, and the fiber $F^{-1}(F(p))$, respectively, where $K_*(P)$ denotes the sectional curvature of the plane $P_* = \langle F_*X, F_*JX\rangle$ in N. Let $K(X \wedge Y)$ be the sectional curvature of the plane spanned by the tangent vectors $X, Y \in T_pM$, $p \in M$. Using both Corollary 1 of [14, p. 465] and (1.27) of [7, p. 12], we obtain the following:

(1) If $P \subset (\mathcal{D}_1)_p$, then with some computations we have

$$K(P) = \widehat{K}(P) + |\mathcal{T}_X X|^2 - |\mathcal{T}_X JX|^2 - g_M(\mathcal{T}_X X, J[JX, X]).$$

(2) If $P \subset (\mathcal{D}_2 \oplus \omega\mathcal{D}_2)_p$ with $X \in (\mathcal{D}_2)_p$, then we get

$$\begin{aligned} K(P) = {} & \cos^2\theta \cdot K(X \wedge \phi X) + 2g_M((\nabla_{\phi X} T)(X, X) \\ & - (\nabla_X T)(\phi X, X), \omega X) + \sin^2\theta \cdot K(X \wedge \omega X). \end{aligned}$$

(3) If $P \subset (\mu)_p$, then we obtain

$$K(P) = K_*(P) - 3|\mathcal{V}J\nabla_X X|^2.$$

3. Examples

Example 3.1. Let F be a slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) [17]. Then the map F is a semi-slant submersion with $\mathcal{D}_2 = \ker F_*$.

Example 3.2. Let F be a semi-invariant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) [18]. Then the map F is a semi-slant submersion with the semi-slant angle $\theta = \frac{\pi}{2}$.

Example 3.3. Let F be an almost h-slant submersion from a hyperkähler manifold (M, g_M, I, J, K) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-slant basis [15]. Then the map $F : (M, g_M, R) \mapsto (N, g_N)$ is a semi-slant submersion with $\mathcal{D}_2 = \ker F_*$ for $R \in \{I, J, K\}$.

Example 3.4. Let F be an almost h-semi-invariant submersion from a hyperkähler manifold (M, g_M, I, J, K) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-invariant basis [16]. Then the map $F : (M, g_M, R) \mapsto (N, g_N)$ is a semi-slant submersion with the semi-slant angle $\theta = \frac{\pi}{2}$ for $R \in \{I, J, K\}$.

Example 3.5. Define a map $F : \mathbb{R}^6 \mapsto \mathbb{R}^2$ by

$$F(x_1, x_2, \ldots, x_6) = (x_3 \sin\alpha - x_5 \cos\alpha, x_6),$$

where $\alpha \in (0, \frac{\pi}{2})$. Then the map F is a semi-slant submersion such that

$$\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle \text{ and } \mathcal{D}_2 = \left\langle \frac{\partial}{\partial x_4}, \cos\alpha\frac{\partial}{\partial x_3} + \sin\alpha\frac{\partial}{\partial x_5} \right\rangle$$

with the semi-slant angle $\theta = \alpha$.

Example 3.6. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^2$ by

$$F(x_1, x_2, \ldots, x_8) = \left(\frac{x_5 - x_8}{\sqrt{2}}, x_6 \right).$$

Then the map F is a semi-slant submersion such that

$$\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\rangle \text{ and } \mathcal{D}_2 = \left\langle \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_8} \right\rangle$$

with the semi-slant angle $\theta = \frac{\pi}{4}$.

Example 3.7. Define a map $F : \mathbb{R}^{10} \mapsto \mathbb{R}^5$ by

$$F(x_1, x_2, \ldots, x_{10}) = \left(x_2, x_1, \frac{x_5 + x_6}{\sqrt{2}}, \frac{x_7 + x_9}{\sqrt{2}}, \frac{x_8 + x_{10}}{\sqrt{2}} \right).$$

Then the map F is a semi-slant submersion such that

$$\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, -\frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}, -\frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}} \right\rangle \text{ and } \mathcal{D}_2 = \left\langle -\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \right\rangle$$

with the semi-slant angle $\theta = \frac{\pi}{2}$.

Example 3.8. Define a map $F : \mathbb{R}^{10} \mapsto \mathbb{R}^4$ by

$$F(x_1, x_2, \ldots, x_{10}) = \left(\frac{x_3 - x_5}{\sqrt{2}}, x_6, \frac{x_7 - x_9}{\sqrt{2}}, x_8 \right).$$

Then the map F is a semi-slant submersion such that

$$\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle \text{ and } \mathcal{D}_2 = \left\langle \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_{10}} \right\rangle$$

with the semi-slant angle $\theta = \frac{\pi}{4}$.

Example 3.9. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$ by

$$F(x_1, x_2, \ldots, x_8) = (x_1, x_2, x_3 \cos\alpha - x_5 \sin\alpha, x_4 \sin\beta - x_6 \cos\beta),$$

where α and β are constant. Then the map F is a semi-slant submersion such that

$$\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle \text{ and } \mathcal{D}_2 = \left\langle \sin\alpha \frac{\partial}{\partial x_3} + \cos\alpha \frac{\partial}{\partial x_5}, \cos\beta \frac{\partial}{\partial x_4} + \sin\beta \frac{\partial}{\partial x_6} \right\rangle$$

with the semi-slant angle θ with $\cos\theta = |\sin(\alpha + \beta)|$.

Example 3.10. Let G be a slant submersion from an almost Hermitian manifold (M_1, g_{M_1}, J_1) onto a Riemannian manifold (N, g_N) with the slant angle θ and (M_2, g_{M_2}, J_2) an almost Hermitian manifold. Denote by (M, g, J) the warped product of (M_1, g_{M_1}, J_1) and (M_2, g_{M_2}, J_2) by a positive function f on M_1 [7], where $J = J_1 \times J_2$. Define a map $F : (M, g, J) \mapsto (N, g_N)$ by

$$F(x, y) = G(x) \quad \text{for } x \in M_1 \text{ and } y \in M_2.$$

Then the map F is a semi-slant submersion such that

$$\mathcal{D}_1 = TM_2 \text{ and } \mathcal{D}_2 = \ker G_*$$

with the semi-slant angle θ.

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