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SURFACES IN $\mathbb{E}^3$ WITH L_1-POINTWISE 1-TYPE GAUSS MAP

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Abstract. In this paper, we study surfaces in $\mathbb{E}^3$ whose Gauss map G satisfies the equation $\square G = f(G + C)$ for a smooth function f and a constant vector C, where $\square$ stands for the Cheng-Yau operator. We focus on surfaces with constant Gaussian curvature, constant mean curvature and constant principal curvature with such a property. We obtain some classification and characterization theorems for these kinds of surfaces. Finally, we give a characterization of surfaces whose Gauss map G satisfies the equation $\square G = \lambda(G + C)$ for a constant λ and a constant vector C.

1. Introduction

The notion of finite type submanifolds and finite type mappings has been extremely studied since they were introduced by B.-Y. Chen in the late 1970's. Let M be a submanifold of Euclidean space $\mathbb{E}^m$. A smooth mapping $\phi : M \to \mathbb{E}^N$ is said to be of finite type if it can be expressed as a finite sum of eigenvectors of Laplace operator Δ, that is, $\phi = \phi_0 + \sum_{i=1}^{k} \phi_i$, where ϕ_0 is a constant map, $\phi_1, \phi_2, \ldots, \phi_k$ non-constant maps such that $\Delta\phi_i = \lambda_i\phi_i,\ \lambda_i \in \mathbb{R},\ i = 1, \ldots, k$. More precisely, if the eigenvalues $\lambda_1,\ \lambda_2, \ldots,\ \lambda_k$ are distinct, then ϕ is said to be of k-type ([12]). Several results on the study of finite type mappings were summed up in a report by B.-Y. Chen ([10]).

The Gauss map of oriented submanifolds and isometric immersions from submanifolds into Euclidean space has been studied in several works in terms of being finite type (cf. [6, 8, 9, 13]). For example, in [8, 9], finite type isometric immersions were studied. Note that if an isometric immersion x from M into $\mathbb{E}^m$ is of k-type, then the submanifold M itself is said to be of k-type.

On the other hand, in [13], B.-Y. Chen and P. Piccinni studied compact submanifolds of Euclidean spaces with finite type Gauss map. If an oriented submanifold M of a Euclidean space has 1-type Gauss map G, then G satisfies

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$\Delta G = \lambda(G + C)$ for a constant $\lambda \in \mathbb{R}$ and a constant vector C. In time, it has been seen that the equation

$$\Delta G = f(G + C) \tag{1.1}$$

is satisfied for some function $f \in C^\infty(M)$ and constant vector C by the Gauss map of some interesting submanifolds, such as a helicoid and a catenoid in $\mathbb{E}^3$ and several rotational surfaces in $\mathbb{E}^4$ ([11, 17]). These submanifolds whose Gauss map G satisfying (1.1) are said to have pointwise 1-type Gauss map. Several articles also appeared on submanifolds with such Gauss map (cf. [2, 3, 4, 5, 11, 16, 19, 20]).

Recently, in [1, 18] the definition of finite type submanifolds is extended in a natural way, by replacing the Laplace operator Δ with a sequence of operators $L_0,\ L_1,\ L_2,\ \ldots,\ L_k$ such that $L_0 = -\Delta$. Let L_k, $k = 1, 2, \ldots, n-1$, denote the linearized operators of the first variation of the $(k+1)$th mean curvature arising from normal variations of an hypersurface M of the Euclidean space $\mathbb{E}^{n+1}$. Note that sometimes the letter $\Box$ is used to denote the operator L_1 which is the Cheng-Yau operator introduced in [14]. By replacing the operator $\Delta = -L_0$ with the operator L_k in the definition of finite type submanifold, the definition of being L_k-finite type has been given.

In this paper, we aim to extend the notion of pointwise 1-type Gauss map for the surfaces of Euclidean 3-space $\mathbb{E}^3$ in terms of the Chen-Yau operator $\Box$. In Section 2, we give the definition of having $\Box$-pointwise 1-type Gauss map with the basic definitions and the facts on the theory of surfaces in $\mathbb{E}^3$. In Section 3, we focus on the surfaces with constant Gaussian curvature, constant mean curvature and constant principal curvature. We obtain some classification and characterization theorems for such surfaces with $\Box$-pointwise 1-type Gauss map. In Section 4, we give a characterization theorem for the surfaces of $\mathbb{E}^3$ with $\Box$-(global) 1-type Gauss map.

The surfaces M we are dealing with are smooth and connected unless otherwise stated.

2. Preliminaries

Let M be an oriented surface in $\mathbb{E}^3$. We denote the Levi-Civita connections of $\mathbb{E}^3$ and M by $\widetilde{\nabla}$ and ∇, respectively and D stands for the normal connection of M. Let $\{e_1, e_2\}$ be an orthonormal frame of tangent space of M and N the unit normal frame associated with the orientation of M. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + h_{ij} N, \tag{2.1}$$

$$\widetilde{\nabla}_{e_i} N = -h_{i1} e_1 - h_{i2} e_2, \quad i, j = 1, 2, \tag{2.2}$$

where h_{ij} are the components of the second fundamental form h of M, i.e., $h_{ij} = h(e_i, e_j)$.

The mapping

$$\begin{array}{rcl} G: M & \rightarrow & S^2 \\ p & \mapsto & N(p) \end{array}$$

is called the Gauss map of M and the shape operator (or Weingarten map) S of M is defined as the differential of mapping G. Since S is a linear self-adjoint operator, it has two real eigenvalues, called the principal curvatures of M and the eigenvectors, called the principal directions of M, corresponding to these eigenvalues are orthogonal. Note that S satisfies $S(X) = -\widetilde{\nabla}_X N$ for a tangent vector field X of M.

Let M be an oriented surface in $\mathbb{E}^3$ and e_1, e_2 the principal directions corresponding to the principal curvatures k_1, k_2, respectively. Then, we have $\widetilde{\nabla}_{e_i} N = -k_i e_i$ and $S = \text{diag }(k_1, k_2)$. The (first) mean curvature H and Gaussian (or the second mean) curvature K of M is defined as

$$H = \frac{k_1 + k_2}{2}, \tag{2.3}$$

$$K = k_1 k_2. \tag{2.4}$$

If $H \equiv 0$ (resp., $K \equiv 0$), then M is called minimal (resp., flat). We define the functions ω_1 and ω_2 as $\omega_1 = \langle \nabla_{e_1} e_1, e_2 \rangle$ and $\omega_2 = -\langle \nabla_{e_2} e_2, e_1 \rangle$, where $\langle , \rangle$ denotes the inner product in $\mathbb{E}^3$.

On the other hand, the well-known Gauss and Codazzi equations become, respectively,

$$e_1(\omega_2) - e_2(\omega_1) + \omega_1^2 + \omega_2^2 = -K, \tag{2.5}$$

$$e_i(k_j) = \omega_j (k_1 - k_2) \tag{2.6}$$

and the first and second structural equations are, respectively,

$$d\theta_i = \theta_{ij} \wedge \theta_j, \tag{2.7}$$

$$d\theta_{12} = -K \theta_1 \wedge \theta_2 \tag{2.8}$$

for $i, j = 1, 2$ and $i \neq j$ where θ_1, θ_2 and θ_{ij} are the 1-forms such that $\theta_i(e_j) = \delta_{ij}$ and $\theta_{12} = \omega_1 \theta_1 + \omega_2 \theta_2$ satisfying $\theta_{12} = -\theta_{21}$ and $\theta_{ii} = 0$.

We will use $C^\infty(M, \mathbb{E}^3)$ to denote the space of all smooth functions from M into the 3-dimensional Euclidean space $\mathbb{E}^3$ and $C^\infty(M)$ will stand for the space of all smooth functions defined on M.

If $X \in C^\infty(M, \mathbb{E}^3)$ is tangent to M, its divergence div$X$ is defined as the mapping which assigns every point p at M into trace of the linear mapping $Y(p) \mapsto (\nabla_Y X)(p)$. On the other hand, the gradient of a function $f \in C^\infty(M)$ is defined by $\nabla f = e_1(f) e_1 + e_2(f) e_2$ and the Laplace operator acting on M is $\Delta = \nabla_{e_1} e_1 + \nabla_{e_2} e_2 - e_1 e_1 - e_2 e_2$ ([15]).

2.1. Cheng-Yau operator $\Box$

Let $L_k : C^\infty(M) \to C^\infty(M)$, $L_k(f) = tr(P_k \circ \nabla^2 f)$, $k = 1, 2$ denote the linearized operator of the first variation of the $(k+1)$th mean curvature arising from normal variations of the surface, where $P_0 = I$, $P_1 = (k_1 + k_2)I - S$ and

I is the identity operator acting on the tangent bundle of M. Note that the operator L_0 is nothing but the Laplace operator acting on M, i.e., $L_0 = -\Delta$ and $L_1 = \Box$ is called the Cheng-Yau operator introduced in [14].

As a matter of fact, it turns out to be $L_k f = \operatorname{div}(P_k(\nabla f))$ for $f \in C^\infty(M)$ ([1]). By putting $k = 1$ in this equation, we obtain

$$\begin{aligned}\Box f &= \operatorname{div}(P_1(\nabla f)) \\ &= \operatorname{div}(k_2 e_1(f) e_1 + k_1 e_2(f) e_2) \\ &= e_1(k_2 e_1(f)) + e_2(k_1 e_2(f)) - k_1 \nabla_{e_1} e_1(f) - k_2 \nabla_{e_2} e_2(f).\end{aligned}$$

Hence, we have

$$\Box = e_1(k_2)\, e_1 + e_2(k_1)\, e_2 + k_2\left(e_1 e_1 - \nabla_{e_2} e_2\right) + k_1\left(e_2 e_2 - \nabla_{e_1} e_1\right). \tag{2.9}$$

2.2. Definitions

First, we extend the definition of operator $\Box$ to the vector valued mappings defined on M into Euclidean space $\mathbb{E}^3$ in a natural way below:

Definition 1. The operator $\Box : C^\infty(M, \mathbb{E}^3) \to C^\infty(M, \mathbb{E}^3)$ is defined by

$$\Box Z = (\Box z_1, \Box z_2, \Box z_3),$$

where $z_1, z_2, z_3 \in C^\infty(M)$ are the components of Z, i.e., $Z = (z_1, z_2, z_3)$.

Then, we have

$$\Box = e_1(k_2)\widetilde{\nabla}_{e_1} + e_2(k_1)\widetilde{\nabla}_{e_2} + k_2\left(\widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_1} - \widetilde{\nabla}_{\nabla_{e_2} e_2}\right) + k_1\left(\widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_2} - \widetilde{\nabla}_{\nabla_{e_1} e_1}\right). \tag{2.10}$$

By a direct calculation, we obtain

$$\begin{aligned}\Box G = {} & e_1(k_2)\widetilde{\nabla}_{e_1} G + e_2(k_1)\widetilde{\nabla}_{e_2} G + k_2\left(\widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_1} G - \widetilde{\nabla}_{\nabla_{e_2} e_2} G\right) \\ & + k_1\left(\widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_2} G - \widetilde{\nabla}_{\nabla_{e_1} e_1} G\right) \\ = {} & -k_1 e_1(k_2) e_1 - k_2 e_2(k_1) e_2 + k_2\left(-\widetilde{\nabla}_{e_1}(k_1 e_1) - \widetilde{\nabla}_{-\omega_2 e_1} G\right) \\ & + k_1\left(-\widetilde{\nabla}_{e_2}(k_2 e_2) - \widetilde{\nabla}_{\omega_1 e_2} G\right) \\ = {} & -k_1 e_1(k_2) e_1 - k_2 e_2(k_1) e_2 + k_2\left(-e_1(k_1) e_1 - k_1\omega_1 e_2 - k_1^2 G - \omega_2 k_1 e_1\right) \\ & + k_1\left(-e_2(k_2) e_2 + k_2\omega_2 e_1 - k_2^2 G + \omega_1 k_2 e_2\right).\end{aligned}$$

By re-arranging the last equation, we state:

Lemma 2.1. *Let M be an oriented surface in $\mathbb{E}^3$ with Gaussian curvature K and mean curvature H. Then, the Gauss map G of M satisfies*

$$\Box G = -\nabla K - 2HKG. \tag{2.11}$$

Next we will give the definition concerning $\Box$-pointwise 1-type Gauss map.

Definition 2. An oriented surface M of Euclidean space $\mathbb{E}^3$ is said to have $\Box$-pointwise 1-type Gauss map if its Gauss map satisfies

$$\Box G = f(G + C) \tag{2.12}$$

for a smooth function $f \in C^\infty(M)$ and a constant vector $C \in \mathbb{E}^3$. More precisely, a $\Box$-pointwise 1-type Gauss map is said to be of the first kind if (2.12) is satisfied for $C = 0$; otherwise, it is said to be of the second kind. Moreover, if (2.12) is satisfied for a constant function f, then we say M has $\Box$-(global) 1-type Gauss map.

We will use following lemma later:

Lemma 2.2. *Let M be an oriented surface in $\mathbb{E}^3$ and e_1, e_2 its principal directions corresponding to the principal curvatures k_1, k_2, respectively. A vector field $C \in C^\infty(M, \mathbb{E}^3)$ is constant if and only if*

$$e_1(C_1) = \omega_1 C_2 + k_1 C_3, \tag{2.13}$$
$$e_1(C_2) = -\omega_1 C_1, \tag{2.14}$$
$$e_1(C_3) = -k_1 C_1, \tag{2.15}$$
$$e_2(C_1) = \omega_2 C_2, \tag{2.16}$$
$$e_2(C_2) = -\omega_2 C_1 + k_2 C_3, \tag{2.17}$$
$$e_2(C_3) = -k_2 C_2, \tag{2.18}$$

where $C_1 = \langle C, e_1\rangle$, $C_2 = \langle C, e_2\rangle$, $C_3 = \langle C, G\rangle$, $\omega_1 = \langle \nabla_{e_1} e_1, e_2\rangle$ and $\omega_2 = -\langle \nabla_{e_2} e_2, e_1\rangle$.

Proof. With the definitions above, we have $C = C_1 e_1 + C_2 e_2 + C_3 G$. By a direct calculation we have

$$\begin{aligned}\widetilde{\nabla}_{e_1} C &= e_1(C_1)e_1 + e_1(C_2)e_2 + e_1(C_3)G + C_1 \widetilde{\nabla}_{e_1} e_1 + C_2 \widetilde{\nabla}_{e_1} e_2 + C_3 \widetilde{\nabla}_{e_1} G \\ &= e_1(C_1)e_1 + e_1(C_2)e_2 + e_1(C_3)G + C_1(\omega_1 e_2 + k_1 G) - C_2\omega_1 e_1 - C_3 k_1 e_1 \\ &= \Big(e_1(C_1) - \omega_1 C_2 - k_1 C_3\Big)e_1 + \Big(e_1(C_2) + \omega_1 C_1\Big)e_2 + \Big(e_1(C_3) + k_1 C_1\Big)G.\end{aligned}$$

Thus, $\widetilde{\nabla}_{e_1} C = 0$ if and only if (2.13)-(2.15) hold. By a similar way, one can check that $\widetilde{\nabla}_{e_2} C = 0$ is equivalent to (2.16)-(2.18). □

A function (or mapping) ϕ defined on M is said to be harmonic if its Laplacian vanishes identically, i.e., $\Delta\phi = 0$. By changing the Laplace operator Δ with Cheng-Yau operator $\Box$, we give following definition:

Definition 3. An oriented surface M of Euclidean space $\mathbb{E}^3$ is said to have $\Box$-harmonic Gauss map if its Gauss map satisfies

$$\Box G = 0. \tag{2.19}$$

The following theorem can be directly obtained from [7, p. 116, Proposition 3.1] by putting $c = 0$.

Theorem 2.3 ([7]). *Let M be a surface in $\mathbb{E}^3$. M has constant Gaussian curvature and constant mean curvature if and only if it is an open part of a sphere, a right circular cylinder $S^1(r) \times \mathbb{E}^1$ or a plane, where $S^1(r)$ denotes the circle with radius r.*

3. Some characterization theorems on surfaces with $\Box$-pointwise 1-type Gauss map

In this section, we will give some characterization theorems on the surfaces of $\mathbb{E}^3$ in terms of their Gauss map. We focus on the surfaces with constant mean curvature, constant Gaussian curvature or constant principal curvature.

3.1. Surfaces with constant Gaussian curvature

Theorem 3.1. *An oriented surface M in $\mathbb{E}^3$ has $\Box$-harmonic Gauss map if and only if it is flat, i.e., its Gaussian curvature vanishes identically.*

Proof. Because of Lemma 2.1, M has $\Box$-harmonic Gauss map if and only if K is constant and $KH = 0$. If $H = 0$ and K is constant, then Theorem 2.3 implies that M is an open part of a plane. Hence, the proof is completed. $\Box$

Theorem 3.2. *An oriented surface M in $\mathbb{E}^3$ has $\Box$-pointwise 1-type Gauss map of the first kind if and only if it has constant Gaussian curvature.*

Proof. The proof directly follows from Lemma 2.1. $\Box$

Theorem 3.3. *An oriented surface M in $\mathbb{E}^3$ has $\Box$-(global) 1-type Gauss map of the first kind if and only if it is either a flat surface or an open part of a sphere.*

Proof. Because of Lemma 2.1 the Gauss map of M satisfies $\Box G = \lambda G$ for a constant λ if and only if

$$\nabla K = 0 \quad \text{and} \quad -2KH = \lambda. \tag{3.1}$$

If $K = 0$, then $\lambda = 0$ and M has $\Box$-harmonic Gauss map. Thus, M is flat because of Theorem 3.1. If $K \neq 0$, then K and H are constants. Therefore, Theorem 2.3 yields that M is an open part of a sphere. $\Box$

3.2. Surfaces with constant mean curvature

In this section, we first show that the only minimal surface with $\Box$-pointwise 1-type Gauss map is plane. First, we state:

Corollary 3.4. *An oriented surface M in $\mathbb{E}^3$ has $\Box$-pointwise 1-type Gauss map of the first kind if and only if it is an open part of a plane.*

Proof. Proof directly follows from Theorem 2.3 and Theorem 3.2. $\Box$

Next, we prove following proposition:

Proposition 3.5. *An oriented connected minimal surface M in $\mathbb{E}^3$ has $\Box$-pointwise 1-type Gauss map of the second kind if and only if it is an open part of a plane.*

Proof. If M has pointwise 1-type Gauss map of the second kind, then (2.12) is satisfied for a constant vector C and a smooth function f. Let $\mathcal{O} = \{p \in M \mid f(p) \neq 0\}$. We now suppose $\mathcal{O} \neq \emptyset$. Since M is minimal, (2.11) and (2.12) imply $f(G + C) = -\nabla K$. Therefore, we have $C_3 = \langle C, G\rangle = -1$ on $\mathcal{O}$. Thus, from (2.15) and (2.18) we obtain

$$k_1 C_1 = k_2 C_2 = 0 \tag{3.2}$$

on $\mathcal{O}$. We note that minimality of M implies $k_1 = -k_2$. Let $\mathcal{O}_1 = \{p \in \mathcal{O} \mid k_1(p) \neq 0\}$. Then, $C_1 = C_2 = 0$ on $\mathcal{O}_1$. Thus, the constant vector $C = -G$ on $\mathcal{O}_1$ and thus $\mathcal{O}_1$ is part of a plane which is a contradiction. Therefore, we have $\mathcal{O}_1 = \emptyset$ which implies $k_1 = k_2 \equiv 0$ on $\mathcal{O}$. Hence, $\mathcal{O}$ is an open part of a plane. Moreover, by continuity, we have $M = \mathcal{O}$.

Conversely, suppose M is an open part of a plane. Then, its Gauss map G is a non-zero constant vector which implies $\Box G = 0$. Therefore, (2.12) is satisfied for $C = -G \neq 0$ and an arbitrary smooth function f. Hence, M has $\Box$-pointwise 1-type Gauss map of the second kind. □

By combining Corollary 3.4 and Proposition 3.5, we obtain:

Theorem 3.6. *An oriented minimal surface M in $\mathbb{E}^3$ has $\Box$-pointwise 1-type Gauss map if and only if it is an open part of a plane.*

Next, we give a complete classification of surfaces with constant mean curvature whose Gauss map satisfying $\Box G = \lambda(G+C)$ for a constant λ and constant vector C.

Theorem 3.7. *Let M be a surface with constant mean curvature in $\mathbb{E}^3$. Then M has $\Box$-(global) 1-type Gauss map if and only if it is an open part of a sphere, a right circular cylinder or a plane.*

Proof. Let k_1 and k_2 be the principal curvatures of M and e_1, e_2 the corresponding principal directions. As M has constant mean curvature, we have $k_1 + k_2 = h_0$ for a constant h_0 which implies

$$e_i(k_1) = -e_i(k_2), \quad i = 1, 2. \tag{3.3}$$

Now, we suppose that M has $\Box$-(global) 1-type Gauss map. Then equation (2.12) is satisfied for $f = \lambda$ and a constant vector C, where λ is a constant. Therefore, from (2.11) and (2.12) we obtain $-\nabla K - h_0 K G = \lambda(G + C)$. From this equation, Codazzi equation (2.6) and (3.3) we get

$$\lambda C_1 = -e_1(K) = e_1(k_1)(k_1 - k_2), \tag{3.4}$$

$$\lambda C_2 = -e_2(K) = e_2(k_1)(k_1 - k_2), \tag{3.5}$$

$$\lambda(C_3 + 1) = -h_0 K. \tag{3.6}$$

Moreover, since C is a constant vector, equations (2.13)-(2.18) are satisfied because of Lemma 2.2.

By using (3.4) and (3.6) we obtain

$$\lambda e_1(C_3+1) = -h_0 e_1(K) = \lambda h_0 C_1. \tag{3.7}$$

Note that if $\lambda = 0$, then we have $\Box G = 0$ and it implies that M is flat because of Theorem 3.1. Thus M is an open part of either a plane or a right circular cylinder because of Theorem 2.3. If $\lambda \neq 0$, then (3.7) implies $e_1(C_3) = h_0 C_1$. This equation and (2.15) imply

$$C_1(h_0 + k_1) = 0. \tag{3.8}$$

By a similar way, one can obtain

$$C_2(h_0 + k_2) = 0. \tag{3.9}$$

Consider the open subset $\mathcal{O} = \{p \in M \mid k_1 \neq -h_0, k_2 \neq -h_0\}$ of M. Then, $C_1 = C_2 = 0$ on $\mathcal{O}$. Thus, (3.4) and (3.5) imply that K is constant on each component of $\mathcal{O}$. Thus, we have C_3 is a constant because of (3.6). If $C_3 = 0$, then $C = 0$ which implies M has constant Gaussian curvature because of Theorem 3.2. Therefore, $\mathcal{O}$ is an open part of a plane, a right circular cylinder or a sphere because of Theorem 2.3. On the other hand, if $C_3 \neq 0$, then we have $G = C/C_3$ is a constant vector, from which, it follows that $\mathcal{O}$ is an open part of a plane. Moreover, in both cases continuity imply $\mathcal{O} = M$.

Now, we suppose that at every point of M, $C_1 \neq 0$ or $C_2 \neq 0$. Then, (3.8) and (3.9) imply that one of k_1 and k_2 is constant. Since $k_1 + k_2$ is constant, k_1 and k_2 are constants and thus M has constant Gaussian curvature. Hence, Theorem 2.3 implies it is an open part of a sphere, a right circular cylinder or a plane. □

3.3. Surfaces with constant principal curvature

A surface in $\mathbb{E}^3$ is said to have a constant principal curvature if one of its principal curvatures is constant. In this subsection, we will give some characterization theorems for that kind of surfaces with □-pointwise 1-type Gauss map.

Let M be a surface in M with the principal directions $e_1,\ e_2$ associated with the principal curvatures $k_1,\ k_2$, respectively. We suppose k_2 is constant. Then the Codazzi equation (2.6) become

$$e_2(k_1) = \omega_1(k_1 - k_2), \tag{3.10}$$

$$0 = \omega_2(k_1 - k_2). \tag{3.11}$$

(3.11) implies $k_1 = k_2$ or $\omega_2 = 0$. Therefore, if $k_1 - k_2$ does not vanish on an open set $\mathcal{O}$ of M, then from the Gauss equation (2.5) we have

$$e_2(\omega_1) = \omega_1^2 + K \tag{3.12}$$

on $\mathcal{O}$.

If M has a constant principal curvature k_2 and $K = k_1k_2$ is a constant, then we have either $k_2 = 0$ or k_1 and k_2 are constants. Therefore, by using Theorem 2.3 and Theorem 3.2 we state:

Theorem 3.8. *Let M be a surface in $\mathbb{E}^3$ with a constant principal curvature. Then, M has $\Box$-pointwise 1-type Gauss map of the first kind if and only if it is either a flat surface or an open part of a sphere.*

Next, we give the following theorem to classify surfaces with a constant principal curvature whose Gauss map satisfies $\Box G = \lambda(G + C)$ for a constant λ and a constant vector C:

Theorem 3.9. *Let M be a surface with a constant principal curvature in $\mathbb{E}^3$. Then M has $\Box$-(global) 1-type Gauss map if and only if it is either a flat surface or an open part of a sphere.*

Proof. Let M be a surface with a constant principal curvature k_2 in $\mathbb{E}^3$ with $\Box$-(global) 1-type Gauss map. Then equation (2.12) is satisfied for $f = \lambda$ and a constant vector C, where λ is a constant. Therefore, (2.11) and (2.12) imply $-\nabla K - 2HKG = \lambda(G + C)$ from which and (3.10) we get

$$\lambda C_1 = -k_2 e_1(k_1), \tag{3.13}$$

$$\lambda C_2 = -k_2\omega_1(k_1 - k_2), \tag{3.14}$$

$$\lambda(C_3 + 1) = -k_1k_2(k_1 + k_2) \tag{3.15}$$

as k_2 is constant.

Consider the open subset $\mathcal{U} = \{p \mid k_1(p) \neq 0 \text{ and } k_1(p) \neq k_2\}$ and let $\mathcal{O}$ be a component of $\mathcal{U}$. Note that (3.11) implies $\omega_2 = 0$ on $\mathcal{O}$. Now, we assume that $k_2 \neq 0$ and $\mathcal{O} \neq \emptyset$. If ω_1 vanishes identically on an open part $\mathcal{O}_2 \subset \mathcal{O}$, then we have $\theta_{12} = 0$ and the second structural equation (2.8) implies that $\mathcal{O}_2$ is flat which is a contradiction. Therefore, without loss of generality, we may, locally, assume that ω_1 does not vanish on $\mathcal{O}$. From (3.10) and (3.15) we get

$$\lambda e_2(C_3) = -k_2\omega_1(k_1 - k_2)(2k_1 + k_2).$$

Next, we use this equation and (3.14) in (2.18) to obtain

$$k_2\omega_1(k_1 - k_2)(2k_1 + k_2) = -k_2^2\omega_1(k_1 - k_2),$$

from which, we get $k_1 = -k_2$ that implies $\mathcal{O}$ is minimal. Therefore, Theorem 3.6 implies that $\mathcal{O}$ is an open part of a plane, but this is a contradiction. Hence, we have either $\mathcal{O} = \emptyset$ or $k_2 = 0$. If $k_2 = 0$, then M is flat. On the other hand, if $\mathcal{O} = \emptyset$, then M is either flat or an open of a sphere.

The converse follows from Theorem 3.2. $\Box$

Now suppose that M is a non-flat surface which is not an open part of a sphere, which implies $k_1 \neq k_2$ and $k_2 \neq 0$. Thus from (3.11) we get $\omega_2 = 0$. Therefore, we have $d\theta_2 = 0$ from the first structural equation (2.7) for $i = 2$, $j = 1$. Poincaré Lemma implies that θ_2 is exact. Thus, there exist local coordinates s, t such that the metric tensor of M is of the form of $g = E^2ds^2 + dt^2$ for a nowhere vanishing smooth function $E = E(s,t)$ which implies $e_1 = \frac{1}{E}\partial_s$ and $e_2 = \partial_t$. By a direct calculation together with the first structural equation (2.7) for $i = 1$, $j = 2$ we see that $\omega_1 = -\frac{E_t}{E}$. Thus, we have $\theta_{12} = -E_t ds$. By using the second structural equation (2.8), we have $d\theta_{12} = E_{tt} ds \wedge dt = -KE ds \wedge dt$, from which we get $K = -\frac{E_{tt}}{E}$. On the other hand, from Codazzi equation (3.10) we obtain $k_1 = \frac{k_2\phi_1}{E} + k_2$ for a smooth function $\phi_1 = \phi_1(s)$. Thus, we have $K = -\frac{E_{tt}}{E} = \frac{k_2^2\phi_1}{E} + k_2^2$. Hence, we state following lemma:

Lemma 3.10. *Let M be a non-flat surface in $\mathbb{E}^3$ with a constant principal curvature k_2. Suppose that M is not an open part of a sphere. Then there exists a local coordinate system $\{s,t\}$ such that the principal directions of M are $e_1 = \frac{1}{E}\partial_s$, $e_2 = \partial_t$ and the corresponding principal curvatures satisfy*

$$k_1 = \frac{k_2\phi_1}{E} + k_2 \tag{3.16}$$

for a smooth function $E = E(s,t)$ which is of the form of

$$E = \phi_2 \cos\theta - \phi_1, \quad \theta = k_2 t + \phi_3, \tag{3.17}$$

where $\phi_1 = \phi_1(s)$, $\phi_2 = \phi_2(s)$ and $\phi_3 = \phi_3(s)$ are some smooth functions. Moreover, the Gaussian curvature of M and the functions ω_1 and ω_2 are

$$K = \frac{k_2^2\phi_1}{E} + k_2^2, \tag{3.18}$$

$$\omega_1 = -\frac{E_t}{E} \tag{3.19}$$

and $\omega_2 = 0$.

By using Lemma 3.10, we obtain:

Theorem 3.11. *Let M be a surface with a constant principal curvature in $\mathbb{E}^3$. Then M has $\Box$-pointwise 1-type Gauss map if and only if it is either a flat surface or an open part of a sphere.*

Proof. Let M be a surface in $\mathbb{E}^3$ with $\Box$-pointwise 1-type Gauss map and the principal curvatures k_1, k_2. We assume that k_2 is a constant. We now suppose M is non-flat. Thus, we have $k_2 \neq 0$. Let $\{s,t\}$ be the local coordinate system satisfying the conditions given in Lemma 3.10.

Since M has $\Box$-pointwise 1-type Gauss map, equation (2.12) is satisfied for a smooth map f and a constant vector C. Therefore, (2.11) and (2.12) imply $-\nabla K - 2HKG = f(G + C)$ from which we obtain

$$fC_1 = \frac{1}{E}K_s, \tag{3.20}$$

$$fC_2 = -K_t, \tag{3.21}$$

$$f(C_3 + 1) = -k_1k_2(k_1 + k_2). \tag{3.22}$$

If f is a constant function, Theorem 3.9 implies M is either a flat surface or an open part of a sphere. Now, we assume that f is not a constant function.

On the other hand, as C is a constant vector, Lemma 2.2 implies that C_1, C_2 and C_3 satisfy (2.13)-(2.18). As $\omega_2 = 0$ and $k_2 \neq 0$, from (2.16)-(2.18) we obtain

$$C_1 = C_1(s), \tag{3.23}$$

$$C_2 = \psi_1 \cos\rho, \tag{3.24}$$

$$C_3 = -\psi_1 \sin\rho, \quad \rho = k_2 t + \psi_2, \tag{3.25}$$

where $\psi_1 = \psi_1(s)$ and $\psi_2 = \psi_2(s)$ are some smooth functions.

On the other hand, by using (3.16), (3.18), (3.24) and (3.25) in (3.21) and (3.22) we obtain

$$f\psi_1 \cos\rho = -\tfrac{k_2^3\phi_2}{E^2}\phi_1 \sin\theta, \tag{3.26}$$

$$f(1-\psi_1\sin\rho) = -\tfrac{k_2^3\phi_2}{E^2}\cos\theta\,(\phi_2\cos\theta - \phi_1). \tag{3.27}$$

Next, we multiply both side of (3.26) and (3.27) by, respectively, $(1-\psi_1\sin\rho)$ and $\psi_1\cos\rho$ to get

$$\psi_1\cos\rho\cos\theta\,(\phi_2\cos\theta-\phi_1) = \phi_1\sin\theta(1-\psi_1\sin\rho).$$

By a direct calculation, we see that this equation implies

$$\frac{\psi_1\phi_2}{4}\Big(\cos(2\phi_3+\psi_2)\cos(3k_2t) - \sin(2\phi_3+\psi_2)\sin(3k_2t)\Big) + \mathcal{P} = 0, \tag{3.28}$$

where $\mathcal{P} = \mathcal{P}(t,s)$ is of the form of $\mathcal{P} = A_1\cos(2k_2t)+A_2\sin(2k_2t)+A_3\cos(k_2t)$ $+A_4\sin(k_2t)$ for some smooth functions $A_i = A_i(s)$, $i = 1, 2, 3, 4$. Since $\{\cos(mkt), \sin(mkt) \mid m \in \mathbb{N}\}$ is a set of linearly independent functions, (3.28) implies

$$\psi_1\phi_2 = 0. \tag{3.29}$$

Consider the subset $\mathcal{U} = \{p \in M \mid \psi_1(p) = 0\}$ of M and let $\mathcal{O}$ be its interior. First, we assume that $\mathcal{O} = \emptyset$, that is $\psi_1 = 0$ on an open subset $M - \mathcal{U}$. Then, from (3.29) we have $\phi_2 = 0$ on $M - \mathcal{U}$. Thus, (3.16) and (3.17) imply $k_1 = 0$ on $M - \mathcal{U}$. Since k_1 is a continuous function, we have $k_1 = 0$ on M which implies M is flat, a contradiction.

Therefore, we have $\mathcal{O} = \mathrm{Int}(\mathcal{U}) \neq \emptyset$. On $\mathcal{O}$, $C_2 = 0$ from (3.24). Together with (3.21), we get $K_t = 0$. It is obvious that $K_t = 0$ implies K is constant because of (3.17) and (3.18). Hence, Proposition 3.8 implies that $\mathcal{O}$ is an open part of a sphere. By continuity, $\mathcal{O}$ must be M.

Converse is obvious. Hence the proof is completed. □

4. Surfaces with □-1-type Gauss map

In this section, we will obtain a necessary and sufficient condition for a surface to have □-(global) 1-type Gauss map. Let M be a non-flat surface in $\mathbb{E}^3$ with Gaussian curvature K, mean curvature H, the principal directions e_1, e_2 and the corresponding principal curvatures k_1, k_2. We assume that M has □-(global) 1-type Gauss map of the second kind. Then, Theorem 3.5 implies that M is not minimal. Therefore, there exists an open set $\mathcal{U}$ of M on which H and K are not vanishing. We may assume that $M = \mathcal{U}$.

Since M has $\Box$-(global) 1-type Gauss map, there exist a constant $\lambda \in \mathbb{R}$ and a constant vector $C \in \mathbb{E}^3$ such that $\lambda(G + C) = -\nabla K - 2KHG$ which implies

$$\lambda C_1 = -e_1(K), \tag{4.1}$$
$$\lambda C_2 = -e_2(K), \tag{4.2}$$
$$\lambda(C_3 + 1) = -2KH. \tag{4.3}$$

Note that if $\lambda = 0$, then we have $\Box G = 0$ which implies M is flat because of Theorem (3.1). Thus, we have $\lambda \neq 0$.

Therefore, the vector C is of the form of

$$C = -\frac{e_1(K)}{\lambda}e_1 - \frac{e_2(K)}{\lambda}e_2 - \left(\frac{2KH}{\lambda} + 1\right)G. \tag{4.4}$$

Since C is a constant vector, its components C_1, C_2 and C_3 satisfy (2.13)-(2.18) because of Lemma 2.2. We put (3.4)-(3.6) on these equations to obtain

$$e_1e_1(K) = \omega_1 e_2(K) + k_1(2KH + \lambda), \tag{4.5}$$
$$e_1e_2(K) = -\omega_1 e_1(K), \tag{4.6}$$
$$2Ke_1(H) + 2He_1(K) = -k_1e_1(K), \tag{4.7}$$
$$e_2e_1(K) = \omega_2 e_2(K), \tag{4.8}$$
$$e_2e_2(K) = -\omega_2 e_1(K) + k_2(2KH + \lambda), \tag{4.9}$$
$$2Ke_2(H) + 2He_2(K) = -k_2e_2(K). \tag{4.10}$$

Note that a vector field C given by (4.4) is constant if and only if (4.5)-(4.10) are satisfied.

After a direct calculation by using (2.6), (4.5) and (4.9), we obtain

$$-\Delta K = 2H(2KH + \lambda),$$
$$\Box K = 2K(2KH + \lambda).$$

Therefore, we have

$$\lambda = -\frac{\Delta K}{2H} - 2KH, \tag{4.11}$$
$$K\Delta K + H\Box K = 0. \tag{4.12}$$

On the other hand, from (4.6) we obtain

$$e_1e_2(K) = (\nabla_{e_1}e_2)\,K. \tag{4.13}$$

Next, from (4.7) and (4.10) we get

$$e_1(k_1)(3k_1k_2 + k_2^2) + e_1(k_2)(2k_1^2 + 2k_1k_2) = 0, \tag{4.14}$$
$$e_2(k_1)(2k_2^2 + 2k_1k_2) + e_2(k_2)(k_1^2 + 3k_1k_2) = 0, \tag{4.15}$$

respectively. By dividing both sides of (4.14) by k_2^2 and putting $k_1 = \phi k_2$ we obtain

$$\frac{(3\phi+1)e_1(\phi)}{5\phi^2+3\phi} + \frac{e_1(k_2)}{k_2} = 0,$$

from which, we obtain

$$e_1\left(k_1^{1/3}k_2^{2/5}(5k_1+3k_2)^{4/15}\right) = 0. \tag{4.16}$$

In a similar way, we get

$$e_2\left(k_1^{2/5}k_2^{1/3}(3k_1+5k_2)^{4/15}\right) = 0 \tag{4.17}$$

from (4.15). Hence, we have shown that if M has $\square$-1-type Gauss map, then (4.12), (4.13), (4.16) and (4.17) are satisfied.

Conversely, let us assume that (4.12), (4.13), (4.16) and (4.17) are satisfied and λ given by (4.11) is a constant. One can check that (2.12) is satisfied for $f = \lambda$ and the vector C given by (4.4). We will show that C is a constant vector.

It is obvious that (4.13) implies (4.6). Moreover, (4.6) implies (4.8) because of the identity $e_1e_2 - e_2e_1 = \nabla_{e_1}e_2 - \nabla_{e_2}e_1$. On the other hand, (4.11) and (4.12) imply

$$-\Delta K = (e_1e_1(K) - \omega_1e_2(K)) + (e_2e_2(K) + \omega_2e_1(K)) = 2H(2KH + \lambda),$$
$$\square K = k_1\left(e_1e_1(K) - \omega_1e_2(K)\right) + k_2\left(e_2e_2(K) + \omega_2e_1(K)\right) = 2K(2KH + \lambda).$$

From these equations, one can obtain (4.5) and (4.9) under the condition $k_1 \neq k_2$. Note that if $k_1 = k_2$ on an open subset of M, then M becomes an open part of a sphere. Therefore, K is constant and from (4.11) we have $\lambda = -2KH$. Thus, equations (4.5) and (4.9) are still satisfied. On the other hand, by a direct calculation we see that (4.16) and (4.17) imply (4.7) and (4.10), respectively. Therefore, we have shown that if (4.5)-(4.10) are satisfied which implies the vector field C given by (4.4) is constant. Hence, we state:

Proposition 4.1. *Let M be a surface in $\mathbb{E}^3$ with non-vanishing mean curvature H and Gaussian curvature K. Then M has $\square$-(global) 1-type Gauss map if and only if equations* (4.12), (4.13), (4.16) *and* (4.17) *are satisfied and λ given by* (4.11) *is constant. In this case,* (2.12) *is satisfied for the constant function $f = \lambda$ and the constant vector $C \in \mathbb{E}^3$ given by* (4.11) *and* (4.4).

Finally, we propose a problem.

Open Problem. Classify surfaces in $\mathbb{E}^3$ with $\square$-1-type Gauss map.

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References

[1] L. J. Alias and N. Gürbüz, *An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures*, Geom. Dedicata **121** (2006), 113–127.

[2] C. Baikoussis, *Ruled submanifolds with finite type Gauss map*, J. Geom. **49** (1994), no. 1-2, 42–45.

[3] C. Baikoussis and D. E. Blair, *On the Gauss map of ruled surfaces*, Glasgow Math. J. **34** (1992), no. 3, 355–359.

[4] C. Baikoussis, B.-Y. Chen, and L. Verstraelen, *Ruled surfaces and tubes with finite type Gauss map*, Tokyo J. Math. **16** (1993), no. 2, 341–349.

[5] C. Baikoussis and L. Verstralen, *The Chen-type of the spiral surfaces*, Results Math. **28** (1995), no. 3-4, 214–223.

[6] D. D. Bleecker and J. L. Weiner, *Extrinsic bounds on λ_1 of Δ on a compact manifold*, Comment. Math. Helv. **51** (1976), no. 4, 601–609.

[7] B.-Y. Chen, *Geometry of Submanifolds*, Pure and Applied Mathematics, No. 22. Marcel Dekker, Inc., New York, 1973.

[8] ———, *On the total curvature of immersed manifolds. VI. submanifolds of finite type and their applications*, Bull. Inst. Math. Acad. Sinica **11** (1983), no. 3, 309–328.

[9] ———, *Total Mean Curvature and Submanifold of Finite Type*, World Scientific, 1984.

[10] ———, *A report on submanifolds of finite type*, Soochow J. Math. **22** (1996), no. 2, 117–337.

[11] B.-Y. Chen, M. Choi, and Y. H. Kim, *Surfaces of revolution with pointwise 1-type Gauss map*, J. Korean Math. Soc. **42** (2005), no. 3, 447–455.

[12] B.-Y. Chen, J. M. Morvan, and T. Nore, *Energy, tension and finite type maps*, Kodai Math. J. **9** (1986), no. 3, 406–418.

[13] B.-Y. Chen and P. Piccinni, *Submanifolds with finite type Gauss map*, Bull. Austral. Math. Soc. **35** (1987), no. 2, 161–186.

[14] S. Y. Cheng and S. T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann. **225** (1977), no. 3, 195–204.

[15] M. P. do Carmo, *Riemannian Geometry*, Translated from the second Portuguese edition by Francis Flaherty. Mathematics: Theory & Applications. Birkhauser Boston, Inc., Boston, MA, 1992.

[16] U. Dursun, *Hypersurfaces with pointwise 1-type Gauss map*, Taiwanese J. Math. **11** (2007), no. 5, 1407–1416.

[17] U. Dursun and N. C. Turgay, *General rotational surfaces in Euclidean space $\mathbb{E}^4$ with pointwise 1-type Gauss map*, Math. Commun. (accepted).

[18] S. M. B. Kashani, *On some L_1-finite type (hyper)surfaces in $\mathbb{R}^{n+1}$*, Bull. Korean Math. Soc. **46** (2009), no. 1, 35–43.

[19] U. H. Ki, D. S. Kim, Y. H. Kim, and Y. M. Roh, *Surfaces of revolution with pointwise 1-type Gauss map in Minkowski 3-space*, Taiwanese J. Math. **13** (2009), no. 1, 317–338.

[20] Y. H. Kim and D. W. Yoon, *Ruled surfaces with pointwise 1-type Gauss map*, J. Geom. Phys. **34** (2000), no. 3-4, 191–205.

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