Bull. Korean Math. Soc. **50** (2013), No. 3, pp. 929–934
http://dx.doi.org/10.4134/BKMS.2013.50.3.929

A NOTE ON DEGREES OF TWISTED ALEXANDER POLYNOMIALS II

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Abstract. In this short note we discuss degrees of twisted Alexander polynomials and demonstrate an explicit example of a closed 3-manifold which is related to the degree formula due to Friedl, Kim and Kitayama.

1. Introduction

The twisted Alexander polynomial was introduced by Lin [7] for knots in the 3-sphere and by Wada [12] for finitely presentable groups. It is a generalization of the classical Alexander polynomial of a knot and has been developped for these twenty years (see the survey paper [2] for details).

Let N be a 3-manifold with empty or toroidal boundary, $\rho : \pi_1 N \to GL(d, \mathbb{F})$ a representation over a field $\mathbb{F}$ and $\alpha : \pi_1 N \to F$ an epimorphism onto a free abelian group F (we consider F as a multiplicative group). Then one can define the twisted Alexander polynomial $\tau(N, \alpha \otimes \rho)$ which in general lives in the quotient field $\mathbb{F}(F)$ of the group ring $\mathbb{F}[F]$. In this paper we consider degrees of twisted Alexander polynomials for knots and 3-manifolds. Recently Friedl, Kim and Kitayama showed in [1] the following theorem.

Theorem 1.1 (Friedl-Kim-Kitayama [1, Theorem 1.4]). *Let N be an irreducible 3-manifold with empty or toroidal boundary such that $N \neq S^1 \times D^2$. Let $\rho : \pi_1 N \to GL(d, \mathbb{F})$ be a representation over a field $\mathbb{F}$ with involution and let $\alpha : \pi_1 N \to \mathbb{Z} \cong \langle t \rangle$ be an admissible epimorphism (namely α is non-trivial if it is restricted to any boundary component). If ρ is conjugate to its dual $\rho^\dagger$ and if $\tau(N, \alpha \otimes \rho) \neq 0$, then*

$$\deg \tau(N, \alpha \otimes \rho) \equiv d\|\alpha\| \mod 2$$

holds, where $\rho^\dagger(g) = \overline{\rho(g^{-1})}^t$ for $g \in \pi_1 N$ and $\|\alpha\|$ denotes the Thurston norm of $\alpha \in H^1(N, \mathbb{Z}) = \mathrm{Hom}(\pi_1 N, \mathbb{Z})$.

Received April 16, 2012.
2010 *Mathematics Subject Classification.* 57M27.
Key words and phrases. twisted Alexander polynomial, knot group, zero-surgery.

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Remark 1.2. When $d = 2$ the above theorem implies that $\tau(N, \alpha \otimes \rho) \in \mathbb{F}(t)$ is of even degree (see Section 2 for the precise definition of the degree of a rational function).

The purpose of this note is to give a pair of a closed 3-manifold and a representation of the fundamental group such that the degree of the corresponding twisted Alexander polynomial drops by 2. It means Theorem 1.1 is optimal in the sense that the formula holds modulo 2 but not modulo 4. In [10] we constructed such a pair of a knot and an $SL(2, \mathbb{C})$-representation (namely in a toroidal boundary case) by using Riley's description [11] of the representation space of a 2-bridge knot group (see also [3], [9]). In this note we consider a representation over a prime field $\mathbb{F}_p$ to construct an example of a closed 3-manifold. The point is to find a representation $\rho : G(K) \to SL(2, \mathbb{F}_p)$ of a knot group $G(K) = \pi_1(S^3 \backslash \nu K)$ which can be extended to the fundamental group of the zero-framed surgery along K and has the twisted Alexander polynomial with dropped degree.

In the next section we quickly review a basic property of the twisted Alexander polynomial. An explicit example of a representation of a closed 3-manifold will be given in Section 3. In the last section, we remark a relationship among the twisted Alexander polynomials associated to surjective $SL(2, \mathbb{F}_p)$-representations.

2. Twisted Alexander polynomials of zero-surgeries

Let N be a 3-manifold with empty or toroidal boundary and $\tilde{N}$ its universal cover. We endow N with a finite CW-structure. For a tensor representation $\alpha \otimes \rho : \pi_1 N \to GL(d, \mathbb{F}[F])$, we can define the twisted Alexander polynomial $\tau(N, \alpha \otimes \rho) \in \mathbb{F}(F)$ as the Reidemeister torsion of the $\mathbb{F}(F)$-complex

$$C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1 N]} \mathbb{F}(F)^d$$

if it is acyclic. We do not recall the precise definition of the Reidemeister torsion here (see [8] for example), but we explain a method to calculate it for the knot exterior in Section 3. If a representation is $\rho : \pi_1 N \to SL(2, \mathbb{F})$, then $\tau(N, \alpha \otimes \rho)$ is well-defined up to multiplication by a unit in F (see [2, Section 2]) and becomes a Laurent polynomial under mild assumptions (see [5], [12]).

Given a knot $K \subset S^3$ the zero-framed surgery N_K of S^3 along K is defined to be $N_K = S^3 \backslash \nu K \cup_T S^1 \times D^2$, where $T = \partial(S^3 \backslash \nu K)$ is glued to $S^1 \times D^2$ by gluing the meridian to $S^1 \times \mathrm{pt}$. The inclusion map induces an isomorphism $\mathbb{Z} \cong H_1(S^3 \backslash \nu K; \mathbb{Z}) \to H_1(N_k; \mathbb{Z})$. We then obtain the following lemma (see [2, Lemma 5.1]).

Lemma 2.1. *Let $K \subset S^3$ be a knot and μ its meridian. Let $\alpha \otimes \rho : \pi_1 N_K \to GL(d, \mathbb{F}[t^{\pm 1}])$ be the tensor representation induced by the abelianization $\alpha : \pi_1 N_K \to H_1(N_K; \mathbb{Z}) \cong \langle t \rangle$ and a representation $\rho : \pi_1 N_K \to GL(d, \mathbb{F})$ over a field $\mathbb{F}$. We denote the representation $G(K) \to \pi_1 N_K \to GL(d, \mathbb{F}[t^{\pm 1}])$ induced* *by the inclusion by $\alpha \otimes \rho$ as well. Then*

$$\tau(S^3 \backslash \nu K, \alpha \otimes \rho) = \tau(N_K, \alpha \otimes \rho) \cdot \det\left((\alpha \otimes \rho)(\mu) - I\right),$$

where I denotes the identity matrix.

Now the degree of a rational function $f_1(t)/f_2(t) \in \mathbb{F}(t)$ is defined as follows.

Definition 2.2. For a given $f(t) = \sum_{i=k}^{l} c_i t^i \in \mathbb{F}[t^{\pm 1}]$ with $c_k \neq 0$ and $c_l \neq 0$, $\deg f(t)$ is defined to be $l - k$. For $f_1(t)/f_2(t)$ ($f_j(t) \in \mathbb{F}[t^{\pm 1}]$), we define $\deg f_1(t)/f_2(t) = \deg f_1(t) - \deg f_2(t)$.

By Lemma 2.1 and Definition 2.2, if $\deg \tau(S^3 \backslash \nu K, \alpha \otimes \rho)$ drops by n, then $\deg \tau(N_K, \alpha \otimes \rho)$ does so. In the next section we show the existence of such a knot and a representation over a prime field.

3. Examples

In [10] we gave a way to find an $SL(2, \mathbb{C})$-representation of a knot group such that the degree of the corresponding twisted Alexander polynomial is less by 2 than that for the trivial 2-dimensional representation. The argument was based on the calculation of the twisted Alexander polynomial of a 2-bridge knot in [3] and [9].

In this section we consider an example of a closed 3-manifold. Let K be the knot 7_3 in Rolfsen's table, which is the 2-bridge knot $K(13, 9)$. Its Alexander polynomial is $\Delta_K(t) = 2 - 3t + 3t^2 - 3t^3 + 2t^4$ and thus the genus of K is two. We fix a presentation of the knot group $G(K)$:

$$G(K) = \langle x, y \,|\, wx = yw \rangle, \quad w = \bar{x}y\bar{x}\bar{y}x\bar{y}\bar{x}y\bar{x}\bar{y}x\bar{y},$$

where $\bar{x}$ means the inverse element x^{-1} in the group $G(K)$. We note that the total exponent sum of w is -4.

Let $\mathbb{F}_{19}$ be a prime field with characteristic 19 and consider a representation $\rho : G(K) \to SL(2, \mathbb{F}_{19})$ defined by the correspondence

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ 18 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & 18 \\ 1 & 1 \end{pmatrix}.$$

It is easy to see that ρ is conjugate to its dual representation $\rho^\dagger(g) = \overline{\rho(g^{-1})}^t$ (that is, ρ satisfies the assumption of Theorem 1.1). For the preferred longitude $\lambda = ww^* y^8$ of the knot K (the total exponent sum of λ is zero), where w^* is w written backwards, it holds that $\rho(\lambda) = I$. Namely ρ induces the representation of the zero-framed surgery N_K of S^3 along K.

By a result of Kitano [4, Theorem A], $\tau(S^3 \backslash \nu K, \alpha \otimes \rho)$ coincides with the Wada invariant $\Delta_{K,\rho}(t)$ (see [12] for the precise definition), so we calculate $\Delta_{K,\rho}(t)$ for $K = 7_3$. Here we remark that the Wada invariant is defined up to a factor t^{2n} ($n \in \mathbb{Z}$).

For the relation $r = wx - yw$, we apply Fox derivation by x:

$$\frac{\partial r}{\partial x} = \frac{\partial w}{\partial x} + w - y\frac{\partial w}{\partial x},$$

where

$$\frac{\partial w}{\partial x} = -\bar{x} - \bar{x}y\bar{x} + \bar{x}y\bar{x}\bar{y} - \bar{x}y\bar{x}\bar{y}x\bar{y}\bar{x} - \bar{x}y\bar{x}\bar{y}x\bar{y}\bar{x}y\bar{x} + \bar{x}y\bar{x}\bar{y}x\bar{y}\bar{x}y\bar{x}\bar{y}.$$

By definition, the numerator of the Wada invariant $\Delta_{K,\rho}(t)$ is given by

$$\det \Phi \left(\frac{\partial r}{\partial x} \right) \doteq 17t + 3t^2 + 13t^3 + 5t^4 + 13t^5 + 3t^6 + 17t^7 \in \mathbb{F}_{19}[t^{\pm 1}],$$

where $\Phi : \mathbb{Z}[F_2] \to M(2, \mathbb{F}_{19}[t^{\pm 1}])$ is the composition of the epimorphism $\mathbb{Z}[F_2] \to \mathbb{Z}[G(K)]$ induced by the presentation of $G(K)$ and the homomorphism $\mathbb{Z}[G(K)] \to M(2, \mathbb{F}_{19}[t^{\pm 1}])$ induced by the tensor representation $\alpha \otimes \rho$. Here F_2 denotes the free group of rank 2 and $M(2, \mathbb{F}_{19}[t^{\pm 1}])$ is the matrix algebra of degree 2 over $\mathbb{F}_{19}[t^{\pm 1}]$.

On the one hand, the denominator of $\Delta_{K,\rho}(t)$ is given by

$$\det \Phi(1 - y) = 1 + 18t + t^2.$$

Therefore we obtain

$$\begin{aligned} \Delta_{K,\rho}(t) &= \frac{\det \Phi \left(\frac{\partial r}{\partial x} \right)}{\det \Phi(1 - y)} \\ &= \frac{17t + 3t^2 + 13t^3 + 5t^4 + 13t^5 + 3t^6 + 17t^7}{1 + 18t + t^2} \\ &= 17t + t^2 + 16t^3 + t^4 + 17t^5 \end{aligned}$$

and $\deg \Delta_{K,\rho}(t) = 5 - 1 = 4$.

Next let us consider the trivial representation $\rho_0 : G(K) \to SL(2, \mathbb{F}_{19})$. Namely $\rho_0(x) = \rho_0(y) = I$ holds. Then the Wada invariant is given by

$$\Delta_{K,\rho_0}(t) = \frac{(\Delta_K(t))^2}{(1 - t)^2} = \frac{(2 + 16t + 3t^2 + 16t^3 + 2t^4)^2}{(1 + 18t)^2} \in \mathbb{F}_{19}(t)$$

and $\deg \Delta_{K,\rho_0}(t) = 8 - 2 = 6$.

By Lemma 2.1, we see that there is a representation $\rho : \pi_1 N_K \to SL(2, \mathbb{F}_{19})$ such that the degree of the corresponding twisted Alexander polynomial $\tau(N_K, \alpha \otimes \rho)$ drops by 2. This is a desired example of a closed 3-manifold.

While we consider knots satisfying that the degree drops by 2 and prime numbers $p \leq 23$ in $SL(2, \mathbb{F}_p)$-representation of knot groups, 7_3 is of the smallest crossing number. Furthermore, 19 is the smallest prime number for 7_3, that is, there is no such $SL(2, \mathbb{F}_p)$-representation with smaller prime number p such that the degree drops by 2. The above observation proposes us a question: is 7_3 of the smallest crossing number with such a property for any prime number p in $SL(2, \mathbb{F}_p)$-representation?

4. Remark

In Section 3, we gave an example whose degree of the twisted Alexander polynomial drops by 2. In this section, we show that there exist such other examples associated to the original example.

Suppose that there exists a representation $\rho : G(K) \to SL(2, \mathbb{F}_p)$. Then we can define two representations $\rho_\flat : G(K) \to SL(2, \mathbb{F}_p)$ and $\rho_\sharp : G(K) \to SL(2, \mathbb{F}_p)$ by using the representation ρ. First we fix the Wirtinger presentation of $G(K)$ and then $\rho_\flat$ is defined by

$$\rho_\flat(x_i) = -(\rho(x_i)) \in SL(2, \mathbb{F}_p)$$

for each generator $x_i \in G(K)$. Next let r be a primitive root of the prime number p in the cyclic group $(\mathbb{F}_p)^* = \mathbb{F}_p \setminus \{0\}$. We define $\rho_\sharp$ by

$$\rho_\sharp(x) = S^{-1}\rho(x)S,$$

where

$$S = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} & \text{if } p \equiv 1 \mod 4, \\ \\ \begin{pmatrix} 1 & 0 \\ 0 & p-1 \end{pmatrix} & \text{if } p \equiv 3 \mod 4. \end{cases}$$

By a result of [6], if ρ is surjective, then $\rho_\flat, \rho_\sharp, (\rho_\flat)_\sharp = (\rho_\sharp)_\flat$ are also surjective and these four representations are not conjugate with one another. The following gives us the relationship among these Wada invariants.

Proposition 4.1. *Let K be a knot and $\rho : G(K) \to SL(2, \mathbb{F}_p)$ a representation of the knot group $G(K)$. Then*

(1) $\Delta_{K,\rho_\flat}(t) = \Delta_{K,\rho}(-t)$,
(2) $\Delta_{K,\rho_\sharp}(t) = \Delta_{K,\rho}(t)$.

Proof. (1) In order to get $\Delta_{K,\rho}(t)$ and $\Delta_{K,\rho_\flat}(t)$, we replace x_k with $t\cdot\rho(x_k)$ and $t\cdot\rho_\flat(x_k) = -t\cdot\rho(x_k)$ in $\Phi(\frac{\partial r_i}{\partial x_j})$, respectively. This implies that $\Delta_{K,\rho_\flat}(t)$ is obtained by substituting $-t$ for t in $\Delta_{K,\rho}(t)$.

(2) Two representations ρ and $\rho_\sharp$ are conjugate as $GL(2, \mathbb{F}_p)$-representations (not as $SL(2, \mathbb{F}_p)$-representations), then $\Delta_{K,\rho_\sharp}(t)$ is same as $\Delta_{K,\rho}(t)$. □

Therefore if the degree of $\Delta_{K,\rho}(t)$ associated to a representation $\rho : G(K) \to SL(2, \mathbb{F}_p)$ drops by 2, then those of $\Delta_{K,\rho_\flat}(t), \Delta_{K,\rho_\sharp}(t), \Delta_{K,(\rho_\flat)_\sharp}(t)$ also drop by 2.

Acknowledgement. The authors would like to thank the referee for some useful comments. This research was supported in part by the Grant-in-Aid for Scientific Research (No. 23540076 and 24740035), the Ministry of Education, Culture, Sports, Science and Technology, Japan.

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