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CHARACTERIZATION OF THE MULTIPLIERS FROM $\dot{H}^r$ TO $\dot{H}^{-r}$

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Abstract. In this paper, we will provide an alternative proof to characterize the pointwise multipliers which maps a Sobolev space $\dot{H}^r(\mathbb{R}^d)$ to its dual $\dot{H}^{-r}(\mathbb{R}^d)$ in the case $0 < r < \frac{d}{2}$ by a simple application of the definition of fractional Sobolev space. The proof relies on a method introduced by Maz'ya-Verbitsky [9] to prove the same result.

1. Introduction

In the present paper we aim to prove the following result on the multiplier space $\mathcal{M}(\dot{H}^r(\mathbb{R}^d) \to \dot{H}^{-r}(\mathbb{R}^d))$, where $\dot{H}^r(\mathbb{R}^d)$ and $\dot{H}^{-r}(\mathbb{R}^d)$ are the (standard) homogeneous Sobolev spaces, $d \geq 3$. Based upon [4] and [7], we recall the definition of the multiplier space $\mathcal{M}(\dot{H}^r(\mathbb{R}^d) \to \dot{H}^{-r}(\mathbb{R}^d))$: Recall that this is defined by

$$\mathcal{M}\left(\dot{H}^r(\mathbb{R}^d) \to \dot{H}^{-r}(\mathbb{R}^d)\right) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \sup_{g \in \mathcal{S}(\mathbb{R}^d)} \frac{\|fg\|_{\dot{H}^{-r}}}{\|g\|_{\dot{H}^r}} < \infty \right\},$$

where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class space of rapidly decreasing smooth functions, so that $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions.

Hereafter, $\dot{X}_r(\mathbb{R}^d)$, $0 \leq r < \frac{d}{2}$, denotes the space of functions, which are locally square integrable on $\mathbb{R}^d$ and such that pointwise multiplication with these functions maps boundedly $\dot{H}^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, i.e.,

$$\dot{X}_r(\mathbb{R}^d) = \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}^d) : \forall g \in \dot{H}^r(\mathbb{R}^d) \quad fg \in L^2(\mathbb{R}^d) \right\}.$$

The norm of $\dot{X}_r$ is given by the operator norm of pointwise multiplication:

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{H^r} \leq 1} \|fg\|_{L^2}.$$

Our main theorem reads as follows:

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Theorem 1. *Let* $d \geq 3$ *and* $0 < r < \frac{d}{2}$. *Assume* $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$ *satisfies* $F := (-\Delta)^{-\frac{r}{2}} f \in \dot{X}_r\left(\mathbb{R}^d\right)$. *Then*

$$f \in \dot{\mathcal{Z}}_r\left(\mathbb{R}^d\right) = \mathcal{M}\left(\dot{H}^r\left(\mathbb{R}^d\right) \to \dot{H}^{-r}\left(\mathbb{R}^d\right)\right)$$

and the estimate

$$\|f\|_{\dot{\mathcal{Z}}_r} \leq C \|F\|_{\dot{X}_r}$$

holds.

The precise definition of $(-\Delta)^{\frac{r}{2}}$ is given in Section 2. As an immediate consequence we have by setting $\Lambda = (-\Delta)^{\frac{1}{2}}$.

Corollary 1. *Let* $d \geq 3$ *and* $0 < r < \frac{d}{2}$. *Suppose that* $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$ *is such that* $g = (-\Delta)^{-\frac{r}{2}} f \in \dot{X}_r\left(\mathbb{R}^d\right)$. *Then*

$$\Lambda^r g \in \dot{\mathcal{Z}}_r\left(\mathbb{R}^d\right)$$

and

$$\|\Lambda^r g\|_{\dot{\mathcal{Z}}_r} \leq C \|g\|_{\dot{X}_r},$$

where the constant C *is independent of* g.

Remark 1. The proof for $r = 1$ relies on potential theory and uses some fine properties of equilibrium measures (see [8]). As far as we know, the case of $r \neq 0$ and $r \neq 1$ is still open (see e.g. [4]).

The motivation of investigating $\dot{\mathcal{Z}}_r$ lies in investigating the partial differential equations. Especially, when it comes to partial differential equations with non-constant coefficients, such spaces arise naturally. Another motivation is in potential analysis. Let $\mathcal{M}^+\left(\mathbb{R}^d\right)$ be the class of positive Borel measures on $\mathbb{R}^d$, finite on compact sets and $\mu \in \mathcal{M}^+\left(\mathbb{R}^d\right)$. The inequality

$$\int_{\mathbb{R}^d} |h(x)|^2 \, d\mu(x) \leq C \|\nabla h\|_{L^2(\mathbb{R}^d)}^2, \qquad h \in C_0^\infty\left(\mathbb{R}^d\right)$$

is called the trace inequality (see [3], [7]). It is by now well known that inequalities of this type become extremely important in various areas of analysis including harmonic analysis and partial differential equations since it is closely connected with spectral properties of Schrödinger operators (see, e.g., [1], [3], [9]) and lead to deep applications in partial differential equations (see, e.g., [7]), in studying eigenvalues of Schrödinger operators [11] and in the theory of Sobolev spaces [5], [6].

2. The function space $\dot{\mathcal{Z}}_r(\mathbb{R}^d)$

In what follows $\mathcal{F}f$ denotes the Fourier transform of a function f in the Schwartz class $\mathcal{S}\left(\mathbb{R}^d\right)$ defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx, \qquad \xi \in \mathbb{R}^d$$

and $\mathcal{F}^{-1}$ the inverse Fourier transform

$$\mathcal{F}^{-1}f(\xi) = (2\pi)^{-d}\,\mathcal{F}f(-\xi).$$

Sometimes $\mathcal{F}f$ will be denoted by $\widehat{f}$.

Definition 1. For $0 < r < \frac{d}{2}$, we define the homogeneous Sobolev space $\dot{H}^r\left(\mathbb{R}^d\right)$ as the closure of $\mathcal{S}\left(\mathbb{R}^d\right)$ for the norm

$$\|f\|_{\dot{H}^r} = (2\pi)^{-\frac{d}{2}} \left\| |\xi|^r \widehat{f}(\xi) \right\|_{L^2}.$$

We then have the following dense embeddings

$$\mathcal{S}\left(\mathbb{R}^d\right) \subset \dot{H}^r\left(\mathbb{R}^d\right) \subset \mathcal{S}'\left(\mathbb{R}^d\right).$$

Recall that $\dot{H}^r\left(\mathbb{R}^d\right)$ is a Hilbert space with inner product

$$\langle f, g\rangle_{\dot{H}^r} = \int_{\mathbb{R}^d} |\xi|^{2r}\,\widehat{f}(\xi)\overline{\widehat{g}}(\xi)d\xi.$$

Moreover, the scalar product in $L^2\left(\mathbb{R}^d\right)$ allows one to identify $\dot{H}^{-r}\left(\mathbb{R}^d\right)$ to the dual space of $\dot{H}^r\left(\mathbb{R}^d\right)$: using the Plancherel formula

$$\int_{\mathbb{R}^d} f(x)\overline{g}(x)dx = (2\pi)^{-d}\int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{g}}(\xi)d\xi,$$

we get that

$$\dot{H}^{-r}\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{S}'\left(\mathbb{R}^d\right) \mid \exists C \geq 0, \forall g \in \mathcal{S}\left(\mathbb{R}^d\right) : |\langle f, g\rangle| \leq C\,\|g\|_{\dot{H}^r} \right\}$$

and

$$\|f\|_{\dot{H}^{-r}} = \sup_{g\in\mathcal{S}} \frac{|\langle f, g\rangle|}{\|g\|_{\dot{H}^r}}.$$

Finally, we quote the following well-known Sobolev inequalities: for $r \in [0, \frac{d}{2}[$ and $\frac{1}{p} = \frac{1}{2} - \frac{r}{d}$, there exists a constant $C_r > 0$ such that for all $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$,

$$\|\varphi\|_{L^p} \leq C_r\,\|\varphi\|_{\dot{H}^r}.$$

We now check that the product between a distribution in $\dot{H}^r\left(\mathbb{R}^d\right)$ and a distribution in $\dot{H}^{-r}\left(\mathbb{R}^d\right)$ is well defined as distribution in $\mathcal{S}'\left(\mathbb{R}^d\right)$.

Lemma 1 ([4]). *Let $0 < r < \frac{d}{2}$. Then there exists a constant $C_r \geq 0$ such that, for all φ, ψ and ω in $\mathcal{S}\left(\mathbb{R}^d\right)$, we have*

$$\left|\int_{\mathbb{R}^d} \varphi(x)\psi(x)\omega(x)dx\right| \leq C_r \|\varphi\|_{\dot{H}^r} \|\psi\|_{\dot{H}^{-r}} \left(\left\||\xi|^{\frac{d}{2}}\, \widehat{\omega}\right\|_{L^2} + \|\widehat{\omega}\|_{L^1}\right).$$

A direct consequence of Lemma 1 is the following result [4].

Corollary 2. *Let $0 < r < \frac{d}{2}$. Then the pointwise product $(f,g) \mapsto fg$ can be extended as a bounded bilinear map from $\dot{H}^r\left(\mathbb{R}^d\right) \times \dot{H}^{-r}\left(\mathbb{R}^d\right)$ to $\mathcal{S}'\left(\mathbb{R}^d\right)$.*

It is well-known that for $s > 0$, $u \in \dot{H}^s\left(\mathbb{R}^d\right)$ if and only if u can be represented in the form

$$u = (-\Delta)^{-\frac{s}{2}}\, g,$$

where $g \in L^2\left(\mathbb{R}^d\right)$. Here the operator $I_s = (-\Delta)^{-\frac{s}{2}}$ admits the representation

$$(-\Delta)^{-\frac{s}{2}} f = K_s * f,$$

where K_s is the function with the Fourier transform $|\xi|^{-s}$. The operator $I_s f = K_s * f$ is called the Riesz potential of order $s > 0$. For properties of Riesz kernel see e.g. [12].

Example 1. Due to the well-known Hardy inequality (see e.g. [10], p. 92)

$$\left\|\frac{u}{|x|}\right\|_{L^2} \leq \frac{2}{d-2}\|\nabla u\|_{L^2},$$

we see that $|x|^{-2} \in \mathcal{M}(\dot{H}^1(\mathbb{R}^d) \to \dot{H}^{-1}(\mathbb{R}^d))$.

Proof. Indeed, since the functions of class $C_0^\infty\left(\mathbb{R}^d\right)$ are dense in $\dot{H}^1\left(\mathbb{R}^d\right)$ in the norm $\|.\|_{\dot{H}^r(\mathbb{R}^d)}$, suppose $u, v \in C_0^\infty\left(\mathbb{R}^d\right)$. Then by virtue of the Cauchy-Schwarz inequality we obtain

$$\left|\left\langle |x|^{-2} u, v\right\rangle\right| \leq \left\|\frac{u}{|x|}\right\|_{L^2} \left\|\frac{v}{|x|}\right\|_{L^2} \leq 4\|\nabla u\|_{L^2}\|\nabla v\|_{L^2},$$

and thus

$$\begin{aligned}\left\||x|^{-2}\right\|_{\mathcal{M}\left(\dot{H}^r \to \dot{H}^{-r}\right)} &= \sup_{u \in \mathcal{D}} \left\||x|^{-2} u\right\|_{\dot{H}^{-r}} \\ &= \sup_{u,v \in \mathcal{D}} \left|\left\langle |x|^{-2} u, v\right\rangle\right| \leq 4 < \infty.\end{aligned}$$

□

Additionally, we have the following inclusion:

Lemma 2. *Let $0 \leq r < \frac{d}{2}$. Then the following embedding*

$$L^{\frac{d}{2r}}\left(\mathbb{R}^d\right) \subset \mathcal{M}\left(\dot{H}^r\left(\mathbb{R}^d\right) \to \dot{H}^{-r}\left(\mathbb{R}^d\right)\right) \tag{2.1}$$

and

$$L^{\frac{d}{r}}\left(\mathbb{R}^d\right) \subset \dot{X}_r\left(\mathbb{R}^d\right) \tag{2.2}$$

hold.

Proof. We shall prove only (2.1); (2.2) is proved similarly. Let $f \in L^{\frac{d}{2r}}\left(\mathbb{R}^d\right)$. By using the following well-known Sobolev embedding:

$$L^q\left(\mathbb{R}^d\right) \subset \dot{H}^{-r}\left(\mathbb{R}^d\right)$$

with $\frac{d}{q} = \frac{d}{2} + r$, we have by Hölder' s inequality

$$\begin{aligned}\|fg\|_{\dot{H}^{-r}} &\leq C\|fg\|_{L^q} \leq C\|f\|_{L^{\frac{d}{2r}}}\|g\|_{L^\sigma} \\ &\leq C\|f\|_{L^{\frac{d}{2r}}}\|g\|_{\dot{H}^r} \quad \left(\dot{H}^r\left(\mathbb{R}^d\right) \subset L^\sigma\left(\mathbb{R}^d\right)\right),\end{aligned}$$

where $\frac{1}{\sigma} = \frac{1}{2} - \frac{r}{d}$. Then, it follows that

$$\|f\|_{\mathcal{M}\left(\dot{H}^r \to \dot{H}^{-r}\right)} = \sup_{\|g\|_{\dot{H}^r} \leq 1} \|fg\|_{H^{-r}} \leq C\|f\|_{L^{\frac{d}{2r}}}.$$

This proves (2.1). □

The spaces $\mathcal{M}(\dot{H}^r(\mathbb{R}^d) \to \dot{H}^{-r}(\mathbb{R}^d))$ for the range $r \in [0, \frac{d}{2}[$ were introduced by Maz'ya and his co-workers [5], [6], [7] in order to study regularity problems for non-linear estimates in PDEs. For instance, the space $\mathcal{M}(\dot{H}^1(\mathbb{R}^d) \to \dot{H}^{-1}(\mathbb{R}^d))$ has been considered as the space of potentials V such that the Schrödinger operator $-\Delta + V$ is bounded from $\dot{H}^1\left(\mathbb{R}^d\right)$ to $\dot{H}^{-1}\left(\mathbb{R}^d\right)$ [9]. Recently, Maz'ya and Verbitsky established a necessary and sufficient conditions for the boundedness of the relativistic Schrödinger operator $(-\Delta)^{\frac{1}{2}} + V$ is bounded from $\dot{H}^{\frac{1}{2}}\left(\mathbb{R}^d\right)$ to $\dot{H}^{-\frac{1}{2}}\left(\mathbb{R}^d\right)$ [8]. Further result can be found in [4] and references therein.

3. Auxiliary estimates

In this section, we always assume that $d \geq 3$. Now, we give another equivalent norm on $\dot{H}^s\left(\mathbb{R}^d\right)$. Their proofs can be found in Strichartz [15] (see also [13]).

Lemma 3. *Let $u \in \mathcal{S}\left(\mathbb{R}^d\right)$ and let $0 < s < 1$. Then the following two statements are equivalent*:

(i) $u \in \dot{H}^s\left(\mathbb{R}^d\right)$.

(ii) $d_s u(x) = C(d,s)\left(\int_{\mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}} dy\right)^{\frac{1}{2}} < +\infty.$

We can also define the fractional derivative operators

$$(-\Delta)^{\frac{s}{2}} u(x) = c(d,s) \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+s}} dy \tag{3.1}$$

for $0 < s < 2$.

Another equivalence we need is:

Lemma 4. *Let $0 < s < 2$. Then, for all $f \in \dot{H}^s(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, if we set*

$$d_s f(x) = \begin{cases} \left\{ \int_{\mathbb{R}^d} \frac{|f(x+y)+f(x-y)-2f(x)|^2}{|y|^{d+2s}} dy \right\}^{\frac{1}{2}} & \text{for } 1 < s < 2, \\ \left\{ \int_{\mathbb{R}^d} \frac{|f(x+y)-f(x)|^2}{|y|^{d+2s}} dy \right\}^{\frac{1}{2}} & \text{for } 0 < s < 1. \end{cases}$$

we have

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \sim \|d_s f\|_{L^2(\mathbb{R}^d)}.$$

Proof. Indeed, by the Fubini theorem and the Plancherel theorem,

$$\begin{aligned} \|d_s f\|_{L^2(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|f(x+y)-2f(x)+f(x-y)|^2}{|y|^{d+2s}} dy \right) dx \right)^{1/2} \\ &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|f(x+y)-2f(x)+f(x-y)|^2}{|y|^{d+2s}} dx \right) dy \right)^{1/2} \\ &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|(\exp(-i\xi \cdot y)-1|^4 |\mathcal{F}f(\xi)|^2}{|y|^{d+2s}} d\xi \right) dy \right)^{1/2} \\ &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|(\exp(-i\xi \cdot y)-1|^4}{|y|^{d+2s}} dy \right) |\mathcal{F}f(\xi)|^2 \, d\xi \right)^{1/2}. \end{aligned}$$

Now observe that the function

$$\xi \mapsto \int_{\mathbb{R}^d} \frac{|(\exp(-i\xi \cdot y)-1|^4}{|y|^{d+2s}} dy$$

is radial and that the integral defining this function converges. Therefore, by the homogeneity, we have

$$\int_{\mathbb{R}^d} \frac{|(\exp(-i\xi \cdot y)-1|^4}{|y|^{d+2s}} dy = C(d,s)|\xi|^{-2s}.$$

Consequently,

$$\begin{aligned} \|d_s f\|_{L^2(\mathbb{R}^d)} &= \sqrt{C(d,s)} \left(\int_{\mathbb{R}^d} |\xi|^{-2s} |\mathcal{F}f(\xi)|^2 \, d\xi \right)^{1/2} \\ &= \sqrt{C(d,s)} \|f\|_{\dot{H}^s(\mathbb{R}^d)}. \end{aligned}$$

□

We conclude this section by showing that one may write the singular integral in (3.1) as a weighted second order differential quotient.

Lemma 5. *Let $0 < s < 2$ and let $(-\Delta)^{\frac{s}{2}}$ be the fractional Laplacian operator defined by* (3.1)*. Then, for any $u \in \mathcal{S}(\mathbb{R}^d)$,*

$$(-\Delta)^{\frac{s}{2}} u(x) = c(d,s) \int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|x-y|^{d+s}} dy. \tag{3.2}$$

Proof. The equivalence of the definitions in (3.1) and (3.2) immediately follows by the standard changing variable formula. Indeed, by choosing $z = y - x$, we have

$$(-\Delta)^{\frac{s}{2}} u(x) = c(d,s) \int_{\mathbb{R}^d} \frac{u(y) - u(x)}{|y-x|^{d+s}} dy = c(d,s) \int_{\mathbb{R}^d} \frac{u(z+x) - u(x)}{|z|^{d+s}} dz.$$

Moreover, by substituting $t = -z$ in last term of the above equality, we have

$$\int_{\mathbb{R}^d} \frac{u(z+x) - u(x)}{|z|^{d+s}} dz = \int_{\mathbb{R}^d} \frac{u(x-t) - u(x)}{|t|^{d+s}} dt$$

and so after relabeling t as z

$$\begin{aligned} 2\int_{\mathbb{R}^d} \frac{u(z+x) - u(x)}{|z|^{d+s}} dz &= \int_{\mathbb{R}^d} \frac{u(z+x) - u(x)}{|z|^{d+s}} dz \\ &\quad + \int_{\mathbb{R}^d} \frac{u(x-z) - u(x)}{|z|^{d+s}} dz \\ &= \int_{\mathbb{R}^d} \frac{u(z+x) - 2u(x) + u(x-z)}{|z|^{d+s}} dz. \end{aligned}$$

Hence

$$(-\Delta)^{\frac{s}{2}} u(x) = c(d,s) \int_{\mathbb{R}^d} \frac{u(z+x) - 2u(x) + u(x-z)}{|z|^{d+s}} dz. \qquad \square$$

Remark 2. The above representation is useful to remove the singularity of the integral at the origin. Indeed, for any smooth function u, a second order Taylor expansion yields

$$\frac{|u(x+z) - 2u(x) + u(x-z)|}{|z|^{d+s}} \leq \frac{\|D^2 u\|_{L^\infty}}{|z|^{d+s-2}},$$

which is integrable near 0 (for any fixed $0 < s < 2$). Therefore, since $u \in \mathcal{S}(\mathbb{R}^d)$, one can get (3.2).

Our main goal in this paper is to extend the Maz'ya-Verbitsky result to a more general case. Namely, we want to give a much more direct and simplified proof on the result due to Lemarié and Gala [4] on the characterization of the multipliers from $\dot{H}^r(\mathbb{R}^d)$ to $\dot{H}^{-r}(\mathbb{R}^d)$ without use of paradifferential calculus (see [4, Theorem 3, p. 1053]).

We introduce the Hardy-Littlewood maximal operator $\mathcal{M}$ defined by

$$\mathcal{M}f(x) = \sup_{0<R<\infty} R^{-d} \int_{B_R(x)} |f(y)| \, dy.$$

It is known (see e.g. [14]) that the operator $\mathcal{M}$ is bounded on $L^p\left(\mathbb{R}^d\right)$ for all $1<p<\infty$.

To prove Theorem 1, we make use of:

Lemma 6 ([2])**.** *Suppose $\lambda \in (0,1)$, $0<r<d$ and $I_r f$ be the Riesz potential of order r with a non-negative density f. Then for almost all $x \in \mathbb{R}^d$*

$$\left(I_{r\lambda} f\right)(x) \leq C\left[\left(I_r f\right)(x)\right]^{\lambda}\left[(Mf)(x)\right]^{1-\lambda},$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator.

Before we continue, we make note of:

Lemma 7. *For any $\alpha>0$ and $\beta>0$, we have*

$$\left(d_\alpha u\right)(x) \leq C(d,\alpha)\ \left[I_\beta d_\alpha\left(-\Delta\right)^{\frac{1}{2}\beta} u\right](x) \tag{3.3}$$

for all $u \in \mathcal{D}\left(\mathbb{R}^d\right)$.

Proof. The result is due to [6], but we give here a detailed proof for the readers' convenience. Suppose $u \in \mathcal{D}\left(\mathbb{R}^d\right)$ such that

$$u=(-\Delta)^{-\frac{1}{2}\beta} f=I_\beta f.$$

In view of the identity

$$u(x+2h)-2u(x+h)+u(x)=[u(x+2h)-u(x)]-2\left[u(x+h)-u(x)\right],$$

one has

$$\begin{aligned}
\left[d_\alpha u(x)\right]^2 &= c(d,\alpha)\int_{\mathbb{R}^d}\frac{|u(x+2h)-2u(x+h)+u(x)|^2}{|h|^{d+2\alpha}}dh\\
&= c(d,\alpha)\int_{\mathbb{R}^d}\frac{|(u(x+2h)-u(x))-2\left(u(x+h)-u(x)\right)|^2}{|h|^{d+2\alpha}}dh\\
&\leq c(d,\alpha)\left(\int_{\mathbb{R}^d}\frac{|(u(x+2h)-u(x))|^2}{|h|^{d+2\alpha}}dh+\int_{\mathbb{R}^d}\frac{|(u(x+h)-u(x))|^2}{|h|^{d+2\alpha}}dh\right)\\
&\leq c(d,\alpha)\left(\int_{\mathbb{R}^d}\frac{|(u(x+h')-u(x))|^2}{|h'|^{d+2\alpha}}dh'+\int_{\mathbb{R}^d}\frac{|(u(x+h)-u(x))|^2}{|h|^{d+2\alpha}}dh\right)\\
&\leq c(d,\alpha)\left(\int_{\mathbb{R}^d}\frac{1}{|h|^{d+2\alpha}}\left|I_\beta f(x+h)-I_\beta f(x)\right|^2 dh\right)^{\frac{1}{2}}.
\end{aligned}$$

We can write

$$\begin{aligned}
B &= I_\beta f(x+h)-I_\beta f(x)\\
&= \left(K_\beta * f\right)(x+h)-\left(K_\beta * f\right)(x)\\
&= \int_{\mathbb{R}^d} K_\beta\left(x+h-\zeta\right) f(\zeta)d\zeta-\int_{\mathbb{R}^d} K_\beta\left(x-\zeta\right) f(\zeta)d\zeta.
\end{aligned}$$

Now, by setting $\zeta = \zeta' + h$, it follows that

$$\begin{aligned} B &= \int_{\mathbb{R}^d} K_\beta (x-\zeta') f(\zeta'+h) d\zeta' - \int_{\mathbb{R}^d} K_\beta (x-\zeta') f(\zeta') d\zeta' \\ &= \int_{\mathbb{R}^d} K_\beta (x-\zeta) \left[f(\zeta+h) - f(\zeta) \right] d\zeta. \end{aligned}$$

Then, we have

$$\begin{aligned} d_\alpha u(x) &= c(d,\alpha) \left(\int_{\mathbb{R}^d} \frac{|B|^2}{|h|^{d+2\alpha}} dh \right)^{\frac{1}{2}} \\ &= c(d,\alpha) \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} K_\beta (x-\zeta) \left[\frac{f(\zeta+h) - f(\zeta)}{|h|^{\frac{d}{2}+\alpha}} \right] d\zeta \right|^2 dh \right)^{\frac{1}{2}} \\ &\le c(d,\alpha) \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K_\beta (x-\zeta) \left| \frac{f(\zeta+h) - f(\zeta)}{|h|^{\frac{d}{2}+\alpha}} \right| d\zeta \right)^2 dh \right)^{\frac{1}{2}}. \end{aligned}$$

By Minkowski's integral inequality, it follows that

$$\begin{aligned} d_\alpha u(x) &\le c(d,\alpha) \int_{\mathbb{R}^d} \left(|K_\beta (x-\zeta)|^2 \int_{\mathbb{R}^d} \frac{|f(\zeta+h) - f(\zeta)|^2}{|h|^{d+2\alpha}} dh \right)^{\frac{1}{2}} d\zeta \\ &= c(d,\alpha) \int_{\mathbb{R}^d} K_\beta (x-\zeta) \left(\int_{\mathbb{R}^d} \frac{|f(\zeta+h) - f(\zeta)|^2}{|h|^{d+2\alpha}} dh \right)^{\frac{1}{2}} d\zeta \\ &= c(d,\alpha) \int_{\mathbb{R}^d} K_\beta (x-\zeta) \left(d_\alpha f \right)(\zeta) d\zeta \\ &= c(d,\alpha) I_\beta d_\alpha \left(-\Delta \right)^{\frac{\beta}{2}} u(x). \end{aligned}$$

The proof is complete. □

4. Proof of Theorem 1

Proof. Let F be in $\dot{X}_r\left(\mathbb{R}^d\right)$ and $f = (-\Delta)^{\frac{r}{2}} F$. Assume that every function takes real value $u \in \mathcal{D}\left(\mathbb{R}^d\right)$,

$$\left| \int_{\mathbb{R}^d} (u(x))^2 f(x) dx \right| = \left| \int_{\mathbb{R}^d} (u(x))^2 (-\Delta)^{\frac{r}{2}} F(x) dx \right|.$$

By duality, we have

$$\left| \int_{\mathbb{R}^d} (u(x))^2 (-\Delta)^{\frac{r}{2}} F(x) dx \right| = \left| \int_{\mathbb{R}^d} \left((-\Delta)^{\frac{r}{2}} (u(x))^2 \right) F(x) dx \right|,$$

where $F \in L^2_{loc}\left(\mathbb{R}^d\right)$ and the integral on the right-hand side is well defined.

Firstly suppose that $0<r<2$. Recall that for $u \in \mathcal{D}\left(\mathbb{R}^d\right)$,

$$(-\Delta)^{\frac{r}{2}}\left(u(x)\right)^2 = C\left(d,r\right)\int_{\mathbb{R}^d}\frac{u(x+2y)^2-2u(x+y)^2+u(x)^2}{|x-y|^{d+r}}dy.$$

In view of the identity

$$\begin{aligned}&u(x+2y)^2-2u(x+y)^2+u(x)^2\\ =\ &2u(x)(u(x+2y)-2u(x+y)+u(x))\\ &+2(u(x+y)-u(x))(u(x+2y)-2u(x+y)+u(x))\\ &+(u(x+y)-u(x))^2+(u(x+2y)-u(x+y))^2,\end{aligned}$$

one has

$$\begin{aligned}&(-\Delta)^{\frac{r}{2}}\left(u(x)\right)^2\\ =\ &2C\left(d,r\right)u(x)\int_{\mathbb{R}^d}\frac{u(x+y)-2u(x)+u(x-y)}{|x-y|^{d+r}}dy\\ &+2C\left(d,r\right)\int_{\mathbb{R}^d}\frac{(u(x+y)-u(x))(u(x+2y)-2u(x+y)+u(x))}{|x-y|^{d+r}}dy\\ &+C\left(d,r\right)\int_{\mathbb{R}^d}\frac{\left(u(x+2y)-u(x+y)\right)^2+\left(u(x)-u(x-y)\right)^2}{|x-y|^{d+r}}dy\end{aligned}$$

for any $u \in \mathcal{D}\left(\mathbb{R}^d\right)$. By virtue of the Hölder inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\frac{(u(x+y)-u(x))(u(x+2y)-2u(x+y)+u(x))}{|x-y|^{d+r}}\right|dy\\ \le\ &\left(\int_{\mathbb{R}^d}\frac{(u(x+y)-u(x))^2}{|x-y|^{d+r}}dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}\frac{(u(x+2y)-2u(x+y)+u(x))^2}{|x-y|^{d+r}}dy\right)^{\frac{1}{2}}\\ \le\ &\left(\int_{\mathbb{R}^d}\frac{(u(x+y)-u(x))^2}{|x-y|^{d+r}}dy\right)^{\frac{1}{2}}\\ &\times\left[\left(\int_{\mathbb{R}^d}\frac{(u(x+2y)-u(x+y))^2}{|x-y|^{d+r}}dy\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^d}\frac{(u(x+y)-u(x))^2}{|x-y|^{d+r}}dy\right)^{\frac{1}{2}}\right]\\ \le\ &C\left|d_{\frac{r}{2}}u(x)\right|^2,\end{aligned}$$

where

$$d_\alpha u(x) = \begin{cases}\left\{\int_{\mathbb{R}^d}\frac{|u(x+y)+u(x-y)-2u(x)|^2}{|x-y|^{d+2\alpha}}dy\right\}^{\frac{1}{2}} & \text{for } 1<\alpha<2,\\ \left\{\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2\alpha}}dy\right\}^{\frac{1}{2}} & \text{for } 0<\alpha<1.\end{cases}$$

Hence,

$$\begin{aligned}&\left|(-\Delta)^{\frac{r}{2}}\left(u(x)\right)^2\right|\\ &\leq C\left\{\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+r}}dy+2\,|u(x)|\left|\int_{\mathbb{R}^d}\frac{u(x+y)-2u(x)+u(x-y)}{|x-y|^{d+r}}dy\right|\right\}\\ &=2C\,|u(x)|\left|(-\Delta)^{\frac{r}{2}}u(x)\right|+C\left|d_{\frac{r}{2}}u(x)\right|^2.\end{aligned}$$

Using the preceding inequality, we estimate

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\left((-\Delta)^{\frac{r}{2}}\left(u(x)\right)^2\right)F(x)dx\right|\\ &\leq C\,\|Fu\|_{L^2}\left\|(-\Delta)^{\frac{r}{2}}u\right\|_{L^2}+C\int_{\mathbb{R}^d}|F(x)|\left|d_{\frac{r}{2}}u(x)\right|^2dx\\ &\leq C\,\|F\|_{\dot{X}_r}\|u\|_{\dot{H}^r}\left\|(-\Delta)^{\frac{r}{2}}u\right\|_{L^2}+C\int_{\mathbb{R}^d}|F(x)|\left|d_{\frac{r}{2}}u(x)\right|^2dx\\ &\leq C\,\|F\|_{X_r}\|u\|_{\dot{H}^r}^2+C\int_{\mathbb{R}^d}|F(x)|\left|d_{\frac{r}{2}}u(x)\right|^2dx.\end{aligned}$$

We set $g=(-\Delta)^{\frac{r}{2}}u$, that is, $u=I_rg$. By virtue to Lemma 7, the last integral is bounded by:

$$\int_{\mathbb{R}^d}|F(x)|\left|I_{\frac{r}{2}}d_{\frac{r}{2}}I_{\frac{r}{2}}g(x)\right|^2dx.$$

Using Hedberg' s inequality (Lemma 6):

$$\left|I_{\frac{r}{2}}v(x)\right|\leq C\left(Mv\right)^{\frac{1}{2}}(x)\left(I_rv\right)^{\frac{1}{2}}(x),$$

we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d}|F(x)|\left|I_{\frac{r}{2}}d_{\frac{r}{2}}I_{\frac{r}{2}}g(x)\right|^2dx\\ &\leq C\int_{\mathbb{R}^d}|F(x)|\,M\left(d_{\frac{r}{2}}I_{\frac{r}{2}}g\right)(x)\left|I_rd_{\frac{r}{2}}I_{\frac{r}{2}}g(x)\right|dx\\ &\leq C\left\|M\left(d_{\frac{r}{2}}I_{\frac{r}{2}}g\right)\right\|_{L^2}\left\|\left(I_rd_{\frac{r}{2}}I_{\frac{r}{2}}g\right)\cdot F\right\|_{L^2}.\end{aligned}\tag{4.1}$$

The boundedness of $\mathcal{M}$ in $L^2\left(\mathbb{R}^d\right)$ implies that the left-hand side of (4.1) is dominated by

$$\begin{aligned}&\leq C\left\|d_{\frac{r}{2}}I_{\frac{r}{2}}g\right\|_{L^2}\|F\|_{\dot{X}_r}\left\|I_rd_{\frac{r}{2}}I_{\frac{r}{2}}g\right\|_{\dot{H}^r}\\ &\leq C\,\|g\|_{L^2}^2\|F\|_{\dot{X}_r}\\ &=C\,\|F\|_{\dot{X}_r}\|u\|_{\dot{H}^r}^2.\end{aligned}$$

When $r=2$, then we just use

$$(-\Delta)^{r/2}(u^2)=(-\Delta)(u^2)$$

$$= -2u\Delta u - 2\sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}\right)^2.$$

The treatment of the first term is identical to the case when $0 < r < 2$. As for the second term, we use the interpolation inequality and argue as we did in Lemma 6.

Let us finish the proof of the case when $r > 2$. We let $m = [r/2] \geq 1$, where $[\cdot]$ denotes the step function. Then for any $u, v \in \mathcal{D}\left(\mathbb{R}^d\right)$, one has

$$\begin{aligned}
&(-\Delta)^{r/2-m}(uv) \\
&= c(d,r)\int_{\mathbb{R}^d} \frac{u(x+y)v(x+y) - 2u(x)v(x) + u(x-y)v(x-y)}{|y|^{d+r-2m}}\,dy \\
&= c(d,r)v(x)\int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{d+r-2m}}\,dy \\
&\qquad + c(d,r)\int_{\mathbb{R}^d} \frac{u(x+y)(v(x+y)-v(x)) + u(x-y)(v(x)-v(x-y))}{|y|^{d+r-2m}}\,dy \\
&= c(d,r)v(x)\int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{d+r-2m}}\,dy \\
&\qquad + c(d,r)u(x)\int_{\mathbb{R}^d} \frac{v(x+y) - 2v(x) + v(x-y)}{|y|^{d+r-2m}}\,dy \\
&\qquad + c(d,r)\int_{\mathbb{R}^d} \frac{(u(x-y)-u(x))(v(x)-v(x-y))}{|y|^{d+r-2m}}\,dy \\
&\qquad + c(d,r)\int_{\mathbb{R}^d} \frac{(u(x+y)-u(x))(v(x+y)-v(x))}{|y|^{d+r-2m}}\,dy.
\end{aligned}$$

Consequently, we have

$$|(-\Delta)^{r/2-m}(uv)| \leq C(|u(-\Delta)^{r/2-m}v| + |v(-\Delta)^{r/2-m}u| + d_{r/2-m}u \cdot d_{r/2-m}v).$$

Hence

$$\begin{aligned}
|(-\Delta)^{r/2}(u^2)| &\leq |(-\Delta)^{r/2-m}(u\Delta^m u)| + |(-\Delta)^{r/2-m}\text{other terms (1)}| \\
&\leq |u(-\Delta)^{r/2}u| + \text{other terms (2)}.
\end{aligned}$$

Here "the other terms" is a linear combination of

$$d_{r/2-m}[\partial^\alpha u]d_{r/2-m}[\partial^\beta u],$$

where α, β are multiindices such that $|\alpha| + |\beta| = 2m$. The treatment of $|u(-\Delta)^{r/2}u|$ is the same as the case when $0 < r \leq 2$ and the control of other terms can be achieved similarly by using Lemmas 6 and 7. This completes the proof of Theorem 1. □

As a corollary of Theorem 1, we want to study coercivity property of the Poisson equation

$$-\Delta u = f, \tag{4.2}$$

on $\mathbb{R}^d$, $d \geq 3$ where f be a complex-valued distribution from $\dot{\mathcal{Z}}_1\left(\mathbb{R}^d\right)$.

Corollary 3. *Let* $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$ *and let* u *be a solution of* (4.2) *such that* $\nabla u \in \dot{X}_1\left(\mathbb{R}^d\right)$. *Then* $f \in \dot{\mathcal{Z}}_1\left(\mathbb{R}^d\right)$.

Proof. Indeed, by virtue of the integration by parts and the Schwarz inequality, we obtain for all $v, w \in \mathcal{D}\left(\mathbb{R}^d\right)$,

$$\begin{aligned}|\langle fv, w\rangle| &= |\langle f, \overline{v}w\rangle| = |\langle \mathrm{div}\ \nabla u, \overline{v}w\rangle| \\ &= |\langle \nabla u, w\nabla\overline{v}\rangle + \langle \nabla u, \overline{v}\nabla w\rangle| \\ &\leq \|\overline{w}\nabla u\|_{L^2}\|\nabla\overline{v}\|_{L^2} + \|v\nabla u\|_{L^2}\|\nabla w\|_{L^2} \\ &\leq 2\|\nabla u\|_{\dot{X}_1}\|\nabla v\|_{L^2}\|\nabla w\|_{L^2}\end{aligned}$$

and thus

$$\|f\|_{\dot{\mathcal{Z}}_1} = \sup_{v\in\mathcal{D}}\|fv\|_{\dot{H}^{-1}} = \sup_{v,w\in\mathcal{D}}|\langle fv, w\rangle| \leq 2\|\nabla u\|_{\dot{X}_1}.$$

The proof of the Corollary 3 is thus complete. □

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