Bull. Korean Math. Soc. **50** (2013), No. 3, pp. 901–914
http://dx.doi.org/10.4134/BKMS.2013.50.3.901

INTERPOLATIONS FOR HÖLDER'S INEQUALITY

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Abstract. Kwon and Bae gave an interpolation for a continuous form of Hölder's inequality for a real-valued bounded measurable function on a product of measure spaces. It is given some generalizations of their result.

1. Introduction

For real numbers A and B satisfying $A \leq B$, a continuous and increasing function h on the open interval $(0,1)$ is said to be an interpolation for $A \leq B$ if

$$\lim_{t\to 0} h(t) = A \quad \text{and} \quad \lim_{t\to 1} h(t) = B.$$

Let (X,μ) be a measure space with $\mu(X) = 1$ and $f \in L^1(\mu)$ satisfy $f > 0$ on X. By [4, p. 71],

$$h(t) := \Big(\int_X f^t \, d\mu \Big)^{\frac{1}{t}}, \quad 0 < t \leq 1,$$

is an interpolation for the inequality

$$\exp \int_X \log f \, d\mu \leq \int_X f \, d\mu.$$

Let X_{ij} and p_j, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, be positive numbers satisfying

$$\sum_{j=1}^{n} \frac{1}{p_j} = 1.$$

The inequality

$$\sum_{i=1}^{m} \prod_{j=1}^{n} X_{ij} \leq \prod_{j=1}^{n} \Big(\sum_{i=1}^{m} X_{ij}^{p_j} \Big)^{\frac{1}{p_j}} \tag{1.1}$$

Received April 10, 2012.
2010 *Mathematics Subject Classification.* Primary 26D15.
Key words and phrases. interpolation, Hölder's inequality, continuous form of Hölder's inequality.
The first author is partially supported by Grant-in-Aid for Scientific Research (No. 21540166), Japan Society for the Promotion of Science.

is well known as the classical Hölder inequality [1]. In [5], Yang gave an interpolation for Hölder's inequality as follows.

Theorem A. *A function*

$$h(t) := \prod_{k=1}^{n} \Big[\sum_{i=1}^{m} \Big(\prod_{j=1}^{n} X_{ij} \Big)^{1-t} X_{ik}^{tp_k} \Big]^{\frac{1}{p_k}}, \quad 0 \le t \le 1,$$

is an interpolation for Hölder's inequality given in (1.1)*, and h is continuous on* $[0,1]$*.*

Let (X, μ) and (Y, ν) be σ-finite measure spaces with positive measures μ, ν and $\mu(X) = 1$. Let $(X \times Y, \mu \times \nu)$ be their product measure space. In [2], Kwon gave a continuous form of Hölder's inequality as follows.

Theorem B. *Let $\mu(X) = 1$ and $f \in L^1(\mu \times \nu)$ satisfy $f(x,y) > 0$ on $X \times Y$. Then*

$$\begin{aligned} &\int_Y \exp\Big(\int_X \log f(x,y)\, d\mu(x) \Big)\, d\nu(y) \\ &\le \exp\Big[\int_X \log \Big(\int_Y f(x,y)\, d\nu(y) \Big)\, d\mu(x) \Big]. \end{aligned} \tag{1.2}$$

Equality holds in (1.2) *as a nonzero valued if and only if $f(x,y) = g(x)h(y)$ for $\mu \times \nu$-a.e. on $X \times Y$ with $-\infty < \int_X \log g\, d\mu$.*

It may happen that $\log f \notin L^1(\mu \times \nu)$; in that case, $\int_X \log f(x,y)\, d\mu(x)$ exists in the extended sense. Since $f \in L^1(\mu \times \nu)$,

$$-\infty \le \int_X \log f(x,y)\, d\mu(x) < \infty$$

for ν-a.e. on Y. We consider that $\exp(-\infty) = 0$ and $\log 0 = -\infty$.

Let $\mu(X) = 1$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$. Then

$$-\infty \le \int_X \log f(z,y)\, d\mu(z) \le \int_X f(z,y)\, d\mu(z) < \infty.$$

For $0 \le t < 1$, we define

$$g(t,x,y) := f^t(x,y) \exp\Big((1-t) \int_X \log f(z,y)\, d\mu(z) \Big) \tag{1.3}$$

and

$$h(t) := \exp\Big[\int_X \log \Big(\int_Y g(t,x,y)\, d\nu(y) \Big)\, d\mu(x) \Big]. \tag{1.4}$$

When $f = f_n$, we write $g_n(t,x,y)$ and $h_n(t)$. In [3], Kwon and Bae gave an interpolation for the inequality given in (1.2) as follows.

Theorem C. *Let $\mu(X) = 1$, $0 < \nu(Y) < \infty$ and f be a measurable function on $X \times Y$ satisfying $0 < \delta \leq f(x,y) \leq M < \infty$ for some positive numbers δ and M. In this case, we may define h on $[0,1]$. Then h is an interpolation for the inequality given in (1.2), and h is a continuous and convex function on $[0,1]$.*

In this paper, we shall study Theorem C and relax the assumption "$0 < \delta \leq f(x,y) \leq M < \infty$".

2. Generalizations of Kwon and Bae's result

Let $\mu(X) = 1$ and $f \in L^1(\mu)$ satisfy $f \geq 0$ a.e. on X. Then Jensen's inequality [4, Theorem 3.3] says that

$$\exp \int_X \log f \, d\mu \leq \int_X f \, d\mu.$$

Lemma 2.1. *For $a, b > 0$ and $0 \leq t \leq 1$, we have $a^t b^{1-t} \leq a + b$.*

Proof. We have

$$a^t b^{1-t} = \left(\frac{a}{b}\right)^t b \leq \max\{a, b\} \leq a + b. \qquad \square$$

Lemma 2.2. *Let $\mu(X) = 1$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$. Then for $0 \leq t < 1$, we have the following.*

(i) $0 \leq g(t,x,y) \leq f(x,y) + \int_X f(z,y)\, d\mu(z) \in L^1(\mu \times \nu)$.

(ii) $\log \int_Y g(t,x,y)\, d\nu(y) \leq \int_Y f(x,y)\, d\nu(y) + \int_{X\times Y} f \, d\mu \times \nu \in L^1(\mu)$.

Proof. (i) By Fubini's theorem, $\int_X f(z,y)\, d\mu(z) \in L^1(\nu)$. We have

$$\begin{aligned} 0 \leq g(t,x,y) &= f^t(x,y) \exp\Big((1-t) \int_X \log f(z,y)\, d\mu(z)\Big) \\ &\leq f^t(x,y) \exp\Big((1-t) \log \int_X f(z,y)\, d\mu(z)\Big) \quad \text{by Jensen's inequality} \\ &= f^t(x,y) \Big(\int_X f(z,y)\, d\mu(z)\Big)^{1-t} \\ &\leq f(x,y) + \int_X f(z,y)\, d\mu(z) \quad \text{by Lemma 2.1} \\ &\in L^1(\mu \times \nu). \end{aligned}$$

The second inequality follows from Jensen's inequality.

(ii) By (i), we have

$$\begin{aligned} \log \int_Y g(t,x,y)\, d\nu(y) &\leq \int_Y g(t,x,y)\, d\nu(y) \\ &\leq \int_Y f(x,y)\, d\nu(y) + \int_{X\times Y} f \, d\mu \times \nu \in L^1(\mu). \end{aligned} \qquad \square$$

Theorem 2.3. *Let $\mu(X) = \nu(Y) = 1$ and $f \in L^1(\mu \times \nu)$ satisfy $\delta \le f$ on $X \times Y$ for some positive number δ. Then h is an interpolation for the inequality given in* (1.2). *Moreover $h(1)$ is well defined and h is a continuous and convex function on* $[0,1]$.

Proof. We may assume that $0 < \delta < 1$. For each positive integer n, let

$$f_n(x,y) := \min\{f(x,y), n\}.$$

Then $0 < \delta \le f_n \le n$ and $f_n \uparrow f$ as $n \to \infty$ on $X \times Y$. By Theorem C, h_n is an interpolation for the inequality

$$\int_Y \exp\Big(\int_X \log f_n \, d\mu\Big)\, d\nu \le \exp\Big[\int_X \log\Big(\int_Y f_n \, d\nu\Big)\, d\mu\Big]$$

and, h_n is a continuous and convex function on $[0,1]$. Letting $n \to \infty$, by the Lebesgue monotone convergence theorem (we write LMCT in short) we have

$$\int_Y \exp\Big(\int_X \log f_n \, d\mu\Big)\, d\nu \to \int_Y \exp\Big(\int_X \log f \, d\mu\Big)\, d\nu,$$

$$\exp\Big[\int_X \log\Big(\int_Y f_n \, d\nu\Big)\, d\mu\Big] \to \exp\Big[\int_X \log\Big(\int_Y f \, d\nu\Big)\, d\mu\Big]$$

and

$$h_n(t) \to h(t), \quad n \to \infty$$

for $0 \le t \le 1$. Since h_n is a convex and increasing function, h is also a convex and increasing function on $[0,1]$. By [4, Theorem 3.2], h is a continuous function on $[0,1)$.

Since $\mu(X) = 1$, we have

$$\begin{aligned} h(0) &= \exp\Big[\int_X \log\Big(\int_Y \exp\Big(\int_X \log f(z,y)\, d\mu(z)\Big)\, d\nu(y)\Big)\, d\mu(x)\Big] \\ &= \int_Y \exp\Big(\int_X \log f(z,y)\, d\mu(z)\Big)\, d\nu(y). \end{aligned}$$

Since $\delta \le f \in L^1(\mu \times \nu)$, we have

$$-\infty < \log \delta \le \int_X \log f(z,y)\, d\mu(z) \le \int_X f(z,y)\, d\mu(z) < \infty$$

for ν-a.e. on Y. Hence

$$h(1) = \exp\Big[\int_X \log\Big(\int_Y f(x,y)\, d\nu(y)\Big)\, d\mu(x)\Big]$$

and

$$\begin{aligned} \int_Y \exp\Big(\int_X \log f \, d\mu\Big)\, d\nu &= h(0) \le h(t) \le \lim_{t \to 1} h(t) \le h(1) \\ &= \exp\Big[\int_X \log\Big(\int_Y f \, d\nu\Big)\, d\mu\Big]. \end{aligned}$$

To finish the proof, it is sufficient to show that

$$\lim_{t\to 1} h(t) = \exp\Big[\int_X \log\Big(\int_Y f\,d\nu\Big)\,d\mu\Big]. \tag{2.1}$$

Since

$$0 \le g(t,x,y) = f^t(x,y)\exp\Big((1-t)\int_X \log f(z,y)\,d\mu(z)\Big) \to f(x,y)$$

as $t \to 1$, by Lemma 2.2(i) and the Lebesgue dominated convergence theorem (we write LDCT in short) we have

$$\int_Y g(t,x,y)\,d\nu(y) \to \int_Y f(x,y)\,d\nu(y),$$

so

$$\log\int_Y g(t,x,y)\,d\nu(y) \to \log\int_Y f(x,y)\,d\nu(y)$$

as $t \to 1$ for μ-a.e. on X. Since $\delta = \delta^t\delta^{1-t} \le g(t,x,y)$,

$$-\infty < \log\delta \le \log\int_Y g(t,x,y)\,\nu(y).$$

Hence by Lemma 2.2(ii) and LDCT, we get (2.1). □

Theorem 2.4. *Let* $\mu(X) = \nu(Y) = 1$ *and* $f \in L^1(\mu\times\nu)$ *satisfy* $f > 0$ *on* $X\times Y$*. Then* h *is a continuous, convex and increasing function on* $[0,1)$*, and*

$$\int_Y \exp\Big(\int_X \log f d\mu\Big)d\nu = h(0) \le h(t) \le \lim_{t\to 1} h(t) \le \exp\Big[\int_X \log\Big(\int_Y f\,d\nu\Big)\,d\mu\Big].$$

Proof. For each positive integer N, let

$$f_N(x,y) = \max\Big\{f(x,y), \frac{1}{N}\Big\}.$$

Then $0 < 1/N \le f_N \in L^1(\mu\times\nu)$ and $f_N \downarrow f$ as $N\to\infty$ on $X\times Y$. By Theorem 2.3, h_N is an interpolation for the inequality

$$\int_Y \exp\Big(\int_X \log f_N\,d\mu\Big)\,d\nu \le \exp\Big[\int_X \log\Big(\int_Y f_N\,d\nu\Big)\,d\mu\Big],$$

and h_N is a continuous and convex function on $[0,1]$. Since $\log f_N \downarrow \log f$ and $\log f_N \le \log f_1 \le f_1 \in L^1(\mu\times\nu)$, by LMCT we have

$$\exp\Big(\int_X \log f_N\,d\mu\Big) \downarrow \exp\Big(\int_X \log f\,d\mu\Big).$$

By Jensen's inequality,

$$\begin{aligned}\exp\Big(\int_X \log f_N(x,y)\,d\mu(x)\Big) &\le \int_X f_N(x,y)\,d\mu(x)\\ &\le \int_X f_1(x,y)\,d\mu(x) \in L^1(\nu).\end{aligned}$$

Hence by LMCT, we have

$$\int_Y \exp\Big(\int_X \log f_N \, d\mu\Big)\, d\nu \to \int_Y \exp\Big(\int_X \log f \, d\mu\Big)\, d\nu.$$

Since

$$\log\Big(\int_Y f_N \, d\nu\Big) \downarrow \log\Big(\int_Y f \, d\nu\Big)$$

and

$$\log \int_Y f_N \, d\nu \le \int_Y f_N \, d\nu \le \int_Y f_1 \, d\nu \in L^1(\mu),$$

by LMCT we have

$$\int_X \log\Big(\int_Y f_N \, d\nu\Big)\, d\mu \to \int_X \log\Big(\int_Y f \, d\nu\Big)\, d\mu,$$

so

$$h_N(1) = \exp\Big[\int_X \log\Big(\int_Y f_N \, d\nu\Big)\, d\mu\Big] \to \exp\Big[\int_X \log\Big(\int_Y f \, d\nu\Big)\, d\mu\Big].$$

Let fix t with $0 \le t < 1$. By (1.3), we have

$$0 < g_N(t,x,y) \downarrow g(t,x,y), \quad N \to \infty.$$

By Lemma 2.2(i) and LMCT, we have

$$\log \int_Y g_N(t,x,y)\, d\nu(y) \downarrow \log \int_Y g(t,x,y)\, d\nu(y).$$

By Lemma 2.2(ii),

$$\begin{aligned}\log \int_Y g_N(t,x,y)\, d\nu(y) &\le \int_Y f_N(x,y)\, d\nu(y) + \int_{X\times Y} f_N \, d\mu\times\nu \\ &\le \int_Y f_1(x,y)\, d\nu(y) + \int_{X\times Y} f_1 \, d\mu\times\nu \\ &\in L^1(\mu).\end{aligned}$$

By (1.4) and LMCT, we have $h_N(t) \to h(t)$ as $N \to \infty$ for $0 \le t < 1$. Then h is a convex and increasing function on $[0,1)$, so h is continuous on $[0,1)$. We note that

$$h(0) = \int_Y \exp\Big(\int_X \log f \, d\mu\Big)\, d\nu.$$

Since

$$h_N(t) \le h_N(1) = \exp\Big[\int_X \log\Big(\int_Y f_N \, d\nu\Big)\, d\mu\Big],$$

we have

$$h(t) \le \exp\Big[\int_X \log\Big(\int_Y f \, d\nu\Big)\, d\mu\Big], \quad 0 \le t < 1.$$

Therefore we get our assertion. □

Suppose that $0 < \nu(Y) < \infty$. We have

$$\int_Y \exp\Big(\int_X \log f\,d\mu\Big)\,\frac{d\nu}{\nu(Y)} = \frac{1}{\nu(Y)}\int_Y \exp\Big(\int_X \log f\,d\mu\Big)\,d\nu,$$

$$\exp\Big[\int_X \log\Big(\int_Y f\,\frac{d\nu}{\nu(Y)}\Big)\,d\mu\Big] = \frac{1}{\nu(Y)}\exp\Big[\int_X \log\Big(\int_Y f\,d\nu\Big)\,d\mu\Big]$$

and

$$\exp\Big[\int_X \log\Big(\int_Y g(t,x,y)\,\frac{d\nu(y)}{\nu(Y)}\Big)\,d\mu(x)\Big] = \frac{1}{\nu(Y)}\exp\Big[\int_X \log\Big(\int_Y g(t,x,y)\,d\nu(y)\Big)\,d\mu(x)\Big].$$

By Theorem 2.4, we have the following.

Corollary 2.5. *Let $\mu(X) = 1$, $0 < \nu(Y) < \infty$ and $f \in L^1(\mu\times\nu)$ satisfy $f > 0$ on $X \times Y$. Then h is a continuous, convex and increasing function on $[0, 1)$, and*

$$\int_Y \exp\Big(\int_X \log f d\mu\Big)d\nu = h(0) \le h(t) \le \lim_{t\to 1} h(t) \le \exp\Big[\int_X \log\Big(\int_Y f\,d\nu\Big)\,d\mu\Big].$$

Next, we shall study the case of $\nu(Y) = \infty$.

Theorem 2.6. *Let $\mu(X) = 1$, $0 < \nu(Y) \le \infty$ and $f \in L^1(\mu\times\nu)$ satisfy $f > 0$ on $X \times Y$. Then h is a continuous, convex and increasing function on $[0, 1)$, and*

$$\int_Y \exp\Big(\int_X \log f d\mu\Big)d\nu = h(0) \le h(t) \le \lim_{t\to 1} h(t) \le \exp\Big[\int_X \log\Big(\int_Y f\,d\nu\Big)\,d\mu\Big].$$

Proof. By Corollary 2.5, it is sufficient to show the case of $\nu(Y) = \infty$.

First, suppose that

$$\int_Y \exp\Big(\int_X \log f\,d\mu\Big)\,d\nu = 0.$$

Hence

$$\int_X \log f(z, y)\,d\mu(z) = -\infty$$

ν-a.e. on Y. So by (1.3), $g(t, x, y) = 0$ for $\mu\times\nu$-a.e. on $X \times Y$ and $0 \le t < 1$. Thus by (1.4), we get $h(t) = 0$ for $0 \le t < 1$ and the assertion.

Next, suppose that

$$0 < \int_Y \exp\Big(\int_X \log f\,d\mu\Big)\,d\nu. \tag{2.2}$$

By Jensen's inequality,

$$0 < \int_Y \exp\Big(\int_X \log f\,d\mu\Big)\,d\nu \le \int_{X\times Y} f\,d\mu\times\nu < \infty.$$

Since ν is a σ-finite measure, there is a sequence of measurable subsets $\{Y_n\}_n$ of Y such that $Y_n \subset Y_{n+1}$, $\nu(Y_n) < \infty$ for every $n \geq 1$, $Y = \bigcup_{n=1}^{\infty} Y_n$ and by (2.2),

$$0 < \int_{Y_1} \exp\Big(\int_X \log f \, d\mu\Big) \, d\nu. \tag{2.3}$$

Let

$$h_n(t) := \exp\Big[\int_X \log\Big(\int_{Y_n} g(t,x,y) \, d\nu(y)\Big) \, d\mu(x)\Big], \quad 0 \leq t < 1.$$

By Corollary 2.5, h_n is a continuous, convex and increasing function on $[0,1)$, and

$$\begin{aligned} \int_{Y_n} \exp\Big(\int_X \log f \, d\mu\Big) \, d\nu = h_n(0) \leq h_n(t) \leq \lim_{t\to 1} h_n(t) \\ \leq \exp\Big[\int_X \log\Big(\int_{Y_n} f \, d\nu\Big) \, d\mu\Big] < \infty. \end{aligned} \tag{2.4}$$

For each $y \in Y$, we have

$$0 \leq \chi_{Y_n}(y) \exp\Big(\int_X \log f(z,y) \, d\mu(z)\Big) \uparrow \exp\Big(\int_X \log f(z,y) \, d\mu(z)\Big).$$

By LMCT,

$$\int_{Y_n} \exp\Big(\int_X \log f \, d\mu\Big) \, d\nu \uparrow \int_Y \exp\Big(\int_X \log f \, d\mu\Big) \, d\nu.$$

By (2.3) and (2.4),

$$\log \int_{Y_1} f \, d\nu \in L^1(\mu).$$

We have

$$\log \int_{Y_1} f \, d\nu \leq \log \int_{Y_n} f \, d\nu \uparrow \log \int_Y f \, d\nu$$

for μ-a.e. on X. By LMCT, we have

$$\exp\Big[\int_X \log\Big(\int_{Y_n} f \, d\nu\Big) \, d\mu\Big] \uparrow \exp\Big[\int_X \log\Big(\int_Y f \, d\nu\Big) \, d\mu\Big].$$

In the similar way, we obtain

$$h_n(t) \uparrow h(t), \quad 0 \leq t < 1.$$

By (2.4), h is a continuous, convex and increasing function on $[0,1)$, and

$$\begin{aligned} \int_Y \exp\Big(\int_X \log f \, d\mu\Big) \, d\nu = h(0) \leq h(t) \leq \lim_{t\to 1} h(t) \\ \leq \exp\Big[\int_X \log\Big(\int_Y f \, d\nu\Big) \, d\mu\Big]. \end{aligned}$$

□

3. Interpolations for Hölder's inequality

Let $\mu(X) = 1, 0 < \nu(Y) \leq \infty$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$. If

$$\exp\Big[\int_X \log\Big(\int_Y f\, d\nu\Big)\, d\mu\Big] = 0,$$

then by Theorem 2.6, trivially h is an interpolation for (1.2). Since

$$\log \int_Y f\, d\nu \leq \int_Y f\, d\nu \in L^1(\mu),$$

if $\log \int_Y f\, d\nu \notin L^1(\mu)$, then

$$\exp\Big[\int_X \log\Big(\int_Y f\, d\nu\Big)\, d\mu\Big] = 0,$$

so we assume that

$$\log \int_Y f\, d\nu \in L^1(\mu). \tag{3.1}$$

Let

$$Y_\infty := \Big\{y \in Y : \int_X \log f(z, y)\, d\mu(z) \neq -\infty\Big\}.$$

Suppose that $0 < \nu(Y \setminus Y_\infty)$. Then for $0 \leq t < 1$, we have

$$\int_Y \exp\Big(\int_X \log f\, d\mu\Big)\, d\nu = \int_{Y_\infty} \exp\Big(\int_X \log f\, d\mu\Big)\, d\nu,$$

$$\int_Y f^t \exp\Big((1-t)\int_X \log f\, d\mu\Big)\, d\nu = \int_{Y_\infty} f^t \exp\Big((1-t)\int_X \log f\, d\mu\Big)\, d\nu,$$

$$h(t) = \exp\Big[\int_X \log\Big(\int_{Y_\infty} g(t, x, y)\, d\nu\Big)\, d\mu\Big]$$

and by (3.1),

$$\exp\Big[\int_X \log\Big(\int_{Y_\infty} f\, d\nu\Big)\, d\mu\Big] < \exp\Big[\int_X \log\Big(\int_Y f\, d\nu\Big)\, d\mu\Big].$$

By Theorem 2.6, we have the following.

Corollary 3.1. *Let $\mu(X) = 1$, $0 < \nu(Y) \leq \infty$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$. If* (3.1) *holds and $0 < \nu(Y \setminus Y_\infty)$, then*

$$\begin{aligned}\int_Y \exp\Big(\int_X \log f\, d\mu\Big)\, d\nu &= h(0) \leq h(t) \leq \lim_{t\to 1} h(t)\\ \leq \exp\Big[\int_X \log\Big(\int_{Y_\infty} f\, d\nu\Big)\, d\mu\Big] &< \exp\Big[\int_X \log\Big(\int_Y f\, d\nu\Big)\, d\mu\Big].\end{aligned}$$

So h is not an interpolation for the inequality given in (1.2).

Hence for the study of interpolations, we may assume that

$$\nu(Y \setminus Y_\infty) = 0. \tag{3.2}$$

Lemma 3.2. *Let $\mu(X) = 1$, $0 < \nu(Y) \leq \infty$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$. If* (3.2) *holds and there is a function $F(x) \in L^1(\mu)$ such that*

$$F(x) \leq \log \int_Y g(t,x,y)\, d\nu(y)$$

for every $0 \leq t < 1$, then h is an interpolation for Hölder's inequality given in (1.2)*, and h is a convex function on* $[0,1)$*.*

Proof. By Theorem 2.6, it is sufficient to show that

$$\lim_{t\to 1} h(t) = \exp\Big[\int_X \log\Big(\int_Y f\, d\nu\Big)\, d\mu\Big].$$

By (3.2),

$$-\infty < \int_X \log f(z,y)\, d\mu(z) < \infty$$

for ν-a.e. on Y. Hence by (1.3),

$$0 \leq g(t,x,y) \to f(x,y), \quad t \to 1$$

for $\mu \times \nu$-a.e. on $X \times Y$. By Lemma 2.2(i) and LDCT, we have

$$\log \int_Y g(t,x,y)\, d\nu(y) \to \log \int_Y f(x,y)\, d\nu(y), \quad t \to 1.$$

By Lemma 2.2(ii), the assumption and LDCT, we get

$$\begin{aligned} h(t) &= \exp\Big[\int_X \log\Big(\int_Y g(t,x,y)\, d\nu(y)\Big)\, d\mu(x)\Big] \\ &\to \exp\Big[\int_X \log\Big(\int_Y f\, d\nu\Big)\, d\mu\Big], \qquad t \to 1. \end{aligned}$$

□

Theorem 3.3. *Let $\mu(X) = 1$ and $0 < \nu(Y) \leq \infty$. Let $\varphi_1, \varphi_2, \dots, \varphi_n$ be real valued functions in $L^1(\mu)$ and $\psi_1, \psi_2, \dots, \psi_n$ be real valued measurable functions on Y. Let*

$$f(x,y) = \exp \sum_{j=1}^n \varphi_j(x)\psi_j(y).$$

Suppose that $f \in L^1(\mu \times \nu)$. Then h is an interpolation for Hölder's inequality given in (1.2)*, and h is a convex function on* $[0,1)$*.*

Proof. We have

$$\int_X \log f(z,y)\, d\mu(z) = \sum_{j=1}^n \psi_j(y) \int_X \varphi_j(z)\, d\mu(z).$$

Hence (3.2) holds and by (1.3),

$$g(t,x,y) = \exp\Big(t \sum_{j=1}^n \varphi_j(x)\psi_j(y) + (1-t)\sum_{j=1}^n \psi_j(y) \int_X \varphi_j\, d\mu\Big)$$

$$= \exp\Big(\sum_{j=1}^{n}\psi_j(y)\Big(t\varphi_j(x)+(1-t)\int_X\varphi_j\,d\mu\Big)\Big)$$
$$\geq \exp\Big(-\sum_{j=1}^{n}|\psi_j(y)|\Big(|\varphi_j(x)|+\Big|\int_X\varphi_j\,d\mu\Big|\Big)\Big).$$

Hence for $0<\lambda<\infty$, we have

$$\begin{aligned}&\int_Y g(t,x,y)\,d\nu(y)\\ \geq\ &\int_Y \exp\Big(-\sum_{j=1}^{n}|\psi_j(y)|\Big(|\varphi_j(x)|+\Big|\int_X\varphi_j\,d\mu\Big|\Big)\Big)\,d\nu(y)\\ \geq\ &\int_{\bigcap_{j=1}^{n}\{|\psi_j|\leq\lambda\}}\exp\Big(-\sum_{j=1}^{n}|\psi_j(y)|\Big(|\varphi_j(x)|+\Big|\int_X\varphi_j\,d\mu\Big|\Big)\Big)\,d\nu(y)\\ \geq\ &\nu\Big(\bigcap_{j=1}^{n}\{|\psi_j|\leq\lambda\}\Big)\times\exp\Big(-\lambda\sum_{j=1}^{n}\Big(|\varphi_j(x)|+\Big|\int_X\varphi_j\,d\mu\Big|\Big)\Big).\end{aligned}$$

Since $f\in L^1(\mu\times\nu)$ and $\varphi_j\in L^1(\mu)$ for $1\leq j\leq n$, by Lemma 2.2(ii) we have

$$\nu\Big(\bigcap_{j=1}^{n}\{|\psi_j|\leq\lambda\}\Big)<\infty.$$

Since $\psi_1,\psi_2,\ldots,\psi_n$ are real valued measurable functions on Y,

$$0<\nu\Big(\bigcap_{j=1}^{n}\{|\psi_j|\leq\lambda\}\Big)$$

for some $0<\lambda<\infty$. Therefore

$$\begin{aligned}&\log\int_Y g(t,x,y)\,d\nu(y)\\ \geq\ &\log\nu\Big(\bigcap_{j=1}^{n}\{|\psi_j|\leq\lambda\}\Big)-\lambda\sum_{j=1}^{n}\Big(|\varphi_j(x)|+\Big|\int_X\varphi_j\,d\mu\Big|\Big)\\ \in\ &L^1(\mu)\qquad\text{by the assumption.}\end{aligned}$$

By Lemma 3.2, we get the assertion. □

Corollary 3.4. *Let $\mu(X)=1$, $0<\nu(Y)\leq\infty$ and $f\in L^1(\mu\times\nu)$ satisfy $f>0$ on $X\times Y$. Suppose that there are real valued functions $\varphi_1,\varphi_2,\ldots,\varphi_n\in L^1(\mu)$ and real valued measurable functions $\psi_1,\psi_2,\ldots,\ \psi_n$ on Y such that*

$$\sum_{j=1}^{n}\varphi_j(x)\psi_j(y)\leq\log f(x,y).$$

Then h is an interpolation for Hölder's inequality given in (1.2)*, and h is a convex function on* $[0,1)$.

Proof. We have

$$\exp \sum_{j=1}^{n} \varphi_j(x)\psi_j(y) \le f(x,y)$$

and

$$\sum_{j=1}^{n} \psi_j(y) \int_X \varphi_j(z)\, d\mu(z) \le \int_X \log f(z,y)\, d\mu(z).$$

Hence

$$\log \int_Y \exp\Big(\sum_{j=1}^{n} \psi_j(y)\Big(t\varphi_j(x) + (1-t)\int_X \varphi_j\, d\mu\Big)\Big)\, d\nu(y)$$
$$\le \log \int_Y g(t,x,y)\, d\nu(y).$$

By the proof of Theorem 3.3, there exists $F(x) \in L^1(\mu)$ satisfying

$$F(x) \le \log \int_Y \exp\Big(\sum_{j=1}^{n} \psi_j(y)\Big(t\varphi_j(x) + (1-t)\int_X \varphi_j\, d\mu\Big)\Big)\, d\nu(y)$$

for every $0 \le t < 1$. By Lemma 3.2, we get the assertion. □

Example 3.5. Let $X = \{1,2\}$ and $0 < \lambda < 1$. Let μ be the measure on X satisfying $\mu(\{1\}) = \lambda$ and $\mu(\{2\}) = 1-\lambda$. Let ν be a σ-finite positive measure on Y. Let

$$f(x,y) = e^{\chi_{\{1\}}\psi_1(y) + \chi_{\{2\}}\psi_2(y)}$$

for real valued measurable functions ψ_1, ψ_2 on Y. Then we have

$$\int_Y \exp\Big(\int_X \log f\, d\mu\Big)\, d\nu = \int_Y \big(e^{\psi_1}\big)^{\lambda}\big(e^{\psi_2}\big)^{1-\lambda}\, d\nu$$

and

$$\exp\Big[\int_X \log\Big(\int_Y f\, d\nu\Big)\, d\mu\Big] = \Big(\int_Y e^{\psi_1} d\nu\Big)^{\lambda}\Big(\int_Y e^{\psi_2} d\nu\Big)^{1-\lambda}.$$

We have also that $f \in L^1(\mu \times \nu)$ if and only if $e^{\psi_1}, e^{\psi_2} \in L^1(\nu)$. Under the condition that $e^{\psi_1}, e^{\psi_2} \in L^1(\nu)$, for $0 \le t < 1$ by Theorem 3.3

$$h(t) = \Big(\int_Y e^{t\psi_1 + (1-t)(\lambda\psi_1 + (1-\lambda)\psi_2)} d\nu\Big)^{\lambda}\Big(\int_Y e^{t\psi_2 + (1-t)(\lambda\psi_1 + (1-\lambda)\psi_2)} d\nu\Big)^{1-\lambda}$$

is an interpolation for Hölder's inequality;

$$\int_Y \big(e^{\psi_1}\big)^{\lambda}\big(e^{\psi_2}\big)^{1-\lambda}\, d\nu \le \Big(\int_Y e^{\psi_1}\, d\nu\Big)^{\lambda}\Big(\int_Y e^{\psi_2}\, d\nu\Big)^{1-\lambda}.$$

In the case of $0 < \nu(Y) < \infty$, we have another condition on f for the interpolation.

Proposition 3.6. *Let $\mu(X) = 1$, $0 < \nu(Y) < \infty$ and $f \in L^1(\mu \times \nu)$ satisfy $f > 0$ on $X \times Y$ and $\log f \in L^1(\mu \times \nu)$. Then h is an interpolation for the inequality given in* (1.2)*, and h is a convex function on* $[0,1)$.

Proof. By Fubini's theorem,

$$\int_X \log f(z,y)\,d\mu(z) \in L^1(\nu),$$

so (3.2) holds. By Jensen's inequality, we have

$$\begin{aligned}
&\log \int_Y g(t,x,y)\,\frac{d\nu(y)}{\nu(Y)} \\
\geq\ &\int_Y \log g(t,x,y)\,\frac{d\nu(y)}{\nu(Y)} \\
=\ &\int_Y \Bigl(t\log f(x,y) + (1-t)\int_X \log f(z,y)\,d\mu(z)\Bigr)\,\frac{d\nu(y)}{\nu(Y)} \\
=\ &t\int_Y \log f(x,y)\,\frac{d\nu(y)}{\nu(Y)} + (1-t)\int_{X\times Y} \log f\,\frac{d\mu\times\nu}{\nu(Y)} \\
\geq\ &-\int_Y |\log f(x,y)|\,\frac{d\nu(y)}{\nu(Y)} - \int_{X\times Y} |\log f|\,\frac{d\mu\times\nu}{\nu(Y)} \\
&\in L^1(\mu).
\end{aligned}$$

By Lemma 3.2, we get the assertion. □

In the case of $\nu(Y)=\infty$, Proposition 3.6 does not have meaning. For, there are no functions $f \in L^1(\mu\times\nu)$ satisfying $f>0$ on $X\times Y$ and $\log f \in L^1(\mu\times\nu)$. If $f \in L^1(\mu\times\nu)$, then

$$(\mu\times\nu)(\{f \geq 1/2\}) < \infty,$$

so

$$(\mu\times\nu)(\{f < 1/2\}) = \infty.$$

Hence

$$-\int_{\{f<1/2\}} \log f\,d\mu\times\nu > (\log 2)(\mu\times\nu)(\{f<1/2\}) = \infty.$$

Therefore $\log f \notin L^1(\mu\times\nu)$.

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