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ON THE GENERAL QUADRATIC GAUSS SUMS WEIGHTED BY CHARACTER SUMS OVER A SHORT INTERVAL

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Abstract. By using the analytic methods, the mean value of the general quadratic Gauss sums weighted by the first power mean of character sums over a short interval is investigated. Several sharp asymptotic formulae are obtained, which show that these sums enjoy good distributive properties. Moreover, interesting connections among them are established.

1. Introduction and main results

For any integer n, the general quadratic Gauss sums $G(n,\chi;q)$ is defined as

$$G(n,\chi;q)=\sum_{a=1}^{q}\chi(a)\mathrm{e}\left(\frac{na^2}{q}\right),$$

where $e(y)=e^{2\pi iy}$. This summation is very important, since it is the generalization of the classical quadratic Gauss sums. But we still know little about the properties of $G(n,\chi;q)$, we do not even know how large $G(n,\chi;q)$ is. Since the value of $|G(n,\chi;q)|$ is irregular as χ varies, one can only get some upper bound estimates. For example, for any integer n with $(n,q)=1$, from the general result of Cochrane and Zheng [1] we can deduce that

$$|G(n,\chi;q)|\le 2^{\omega(q)}q^{\frac{1}{2}},$$

where $\omega(q)$ denotes the number of distinct prime divisors of q. The case that q is prime is due to A. Weil [2].

However, weighted sums [4] involving $G(n,\chi;q)$ enjoys many good value distribution properties, through which interesting connections among them are established. Now we shall use analytic methods to study the mean value of the general quadratic Gauss sums weighted by the first power mean of character

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sums over the interval $\left[1, \frac{q}{4}\right)$, and give several sharp asymptotic formulae. That is, we have:

Theorem. *Let $p \geq 5$ be an odd prime. Then for any positive integer n with $(n,p)=1$, we have the asymptotic formulae that*

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |G(n,\chi;p)|^2 \left| \sum_{x < \frac{p}{4}} \chi(x) \right| = \frac{C \cdot p^{\frac{5}{2}}}{\pi} + O\left(p^{2+\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |G(n,\chi;p)|^4 \left| \sum_{x < \frac{p}{4}} \chi(x) \right| = \frac{3 \cdot C \cdot p^{\frac{7}{2}}}{\pi} + O\left(p^{3+\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |G(n,\chi;p)|^6 \left| \sum_{x < \frac{p}{4}} \chi(x) \right| = \frac{10 \cdot C \cdot p^{\frac{9}{2}}}{\pi} + O\left(p^{4+\epsilon}\right),$$

where $C = \sum_{\substack{n=1 \\ (n,2p)=1}}^{\infty} \frac{r^2(n)}{n^2}$, *$r(n)$ is a multiplicative function defined by*

$$r(1) = 1; \qquad r(p^\alpha) = \frac{\binom{2\alpha}{\alpha}}{4^\alpha},$$

and ϵ is any fixed positive number.

Remark. If we bound the left hand side by estimation of individual terms in the first part of the theorem, we can use the bounds for the general quadratic Gauss sums $|G(n,\chi;p)| \leq p^{\frac{1}{2}+\epsilon}$ and the character sums $\left| \sum_{x < \frac{p}{4}} \chi(x) \right| \leq p^{\frac{1}{2}} \ln p$. This will give us a bound for the left hand side $p^{\frac{5}{2}+\epsilon}$, which is about the same as the bound of the right hand side. This confirmed that the bound for the error term is best possible, and there is no major cancellation on the left side. Similar results can be obtained for other parts of the theorem.

2. Several auxiliary lemmas

To establish the main results of our theorem, we need the following several auxiliary lemmas.

Lemma 2.1. *Let $q \geq 5$ be an odd integer and χ be a primitive Dirichlet character modulo q such that $\chi(-1) = 1$. Then we have the identity that*

$$\sum_{x=1}^{\left[\frac{q}{4}\right]} \chi(x) = -\frac{i\overline{\chi}(4)}{2\pi} \tau(\chi\chi_4) L(1, \overline{\chi}\chi_4),$$

where χ_4 *is the primitive Dirichlet character modulo* 4*, and*

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right)$$

is the classical Gauss sums.

Proof. This is Lemma 2.2 of [5]. □

Lemma 2.2. *Let* p *be a prime,* n *be an integer with* $(n,p) = 1$*,* χ_1 *be the Legendre symbol and* $G(n;p)$ *be the classical quadratic Gauss sums. Then we have*

$$|G(n,\chi;p)|^2 = 2p + \chi_1(n) G(1;p) \sum_{a=1}^{p-1} \chi(a)\chi_1(a^2-1),$$

$$G(1;p) = \frac{1}{2}\sqrt{p}(1+i)\left(1+e^{-\frac{\pi i p}{2}}\right) = \begin{cases} \sqrt{p}, & \textit{if } p \equiv 1 \pmod 4, \\ i\sqrt{p}, & \textit{if } p \equiv 3 \pmod 4, \end{cases}$$

and

$$\left(\sum_{a=1}^{p-1} \chi(a)\chi_1(a^2-1)\right)^2 = 2\chi_1(-1)(p-3) + \sum_{a=2}^{p-2}\sum_{b=1}^{p-1} \chi(a)\chi_1(a^2-b^2)\chi_1(b^2-1).$$

Proof. See Lemmas 3 and 4 of [4]. □

Lemma 2.3. *Let* $q \geq 5$ *be an odd integer. Then we have*

$$\sideset{}{^*}\sum_{\chi \bmod q} \left| \sum_{x < \frac{q}{4}} \chi(x) \right| = \frac{J(q) q^{\frac{1}{2}}}{2\pi} \sum_{\substack{n=1 \\ (n,2q)=1}}^{\infty} \frac{r^2(n)}{n^2} + O\left(q^{1+\epsilon}\right),$$

where $J(q)$ *denotes the number of all primitive characters modulo* q*.*

Proof. This is Theorem 3.6 of [3]. □

Lemma 2.4. *Let* $q \geq 5$ *be an odd integer and* χ *be a primitive Dirichlet character modulo* q *such that* $\chi(-1) = 1$*. Then we have the estimate that*

$$\sideset{}{'}\sum_{a=1}^{q} \left| \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \chi(a) \left| L\left(1, \overline{\chi}\chi_4\right) \right| \right| \ll q^{1+\epsilon},$$

where $\sum'$ *denotes the summation over all* a *such that* $(a,q) = 1$*.*

Proof. Let $N = q^{\frac{3}{2}}$, χ be a primitive Dirichlet character modulo q and $A(\chi, y) = \sum_{N < n \leq y} \chi(n)$. Then by Abel identity and Pólya-Vinogradov inequality, we have

$$L(1, \overline{\chi}\chi_4) = \sum_{1 \leq n \leq N} \frac{\overline{\chi}\chi_4(n)}{n} + \int_N^{\infty} \frac{A(y, \overline{\chi}\chi_4)}{y^2} \mathrm{d}y$$

$$= \sum_{1\le n\le N} \frac{\overline{\chi}\chi_4(n)}{n} + O\left(\frac{\ln q}{q}\right).$$

Hence, we can write

$$|L(1,\overline{\chi}\chi_4)| = \left|\sum_{1\le n\le N} \frac{\overline{\chi}\chi_4(n)}{n}\right| + O\left(\frac{\ln q}{q}\right).$$

On the other hand, for the multiplicative function $r(n)$, we have

$$\sum_{d|n} r(d)\cdot r(\frac{n}{d}) = 1$$

and

$$\left(\sum_{n\le N}\frac{\chi(n)r(n)}{n}\right)^2 = \sum_{m\le N}\sum_{n\le N}\frac{\chi(nm)r(m)r(n)}{mn}$$
$$= \sum_{n\le N}\frac{\chi(n)}{n} + \sum_{N<n\le N^2}\frac{\chi(n)r(n,N)}{n},$$

where $r(n,N) = \sum_{\substack{d|n\\ d,\frac{n}{d}\le N}} r(d)\cdot r(\frac{n}{d})$.

Noting the identity that

$$\sum_{\chi(-1)=1}^{*}\chi(n) = \frac{1}{2}\sum_{\chi \bmod q}^{*}\chi(n) + \frac{1}{2}\sum_{\chi \bmod q}^{*}\chi(-n)$$
$$= \frac{1}{2}\sum_{d|(q,n-1)}\mu\left(\frac{q}{d}\right)\phi(d) + \frac{1}{2}\sum_{d|(q,n+1)}\mu\left(\frac{q}{d}\right)\phi(d),$$

we may have

$$\sum_{a=1}^{q}{}' \left|\sum_{\chi(-1)=1}^{*}\chi(a)\left|\sum_{n\le N}\frac{\overline{\chi}\chi_4(n)r(n)}{n}\right|^2\right|$$
$$= \sum_{a=1}^{q}{}' \left|\sum_{\chi(-1)=1}^{*}\chi(a)\left(\sum_{1\le n_1\le N}\frac{\overline{\chi}\chi_4(n_1)r(n_1)}{n_1}\right)\left(\sum_{1\le n_2\le N}\frac{\chi\chi_4(n_2)r(n_2)}{n_2}\right)\right|$$
$$= \sum_{a=1}^{q}{}' \left|\frac{1}{2}\sum_{1\le n_1\le N}{}'\sum_{1\le n_2\le N}{}'\frac{\chi_4(n_1n_2)r(n_1)r(n_2)}{n_1n_2}\sum_{d|(q,\overline{n}_1n_2a-1)}\mu\left(\frac{q}{d}\right)\phi(d)\right.$$
$$\left. + \frac{1}{2}\sum_{1\le n_1\le N}{}'\sum_{1\le n_2\le N}{}'\frac{\chi_4(n_1n_2)r(n_1)r(n_2)}{n_1n_2}\sum_{d|(q,\overline{n}_1n_2a+1)}\mu\left(\frac{q}{d}\right)\phi(d)\right|$$

$$= \sum_{a=1}^{q}{}' \left| \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{1 \le n_1 \le N \\ an_2 \equiv n_1 \pmod d}}{}' \sum_{1 \le n_2 \le N}{}' \frac{\chi_4(n_1 n_2) r(n_1) r(n_2)}{n_1 n_2} \right.$$

$$\left. + \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{1 \le n_1 \le N \\ an_2 \equiv -n_1 \pmod d}}{}' \sum_{1 \le n_2 \le N}{}' \frac{\chi_4(n_1 n_2) r(n_1) r(n_2)}{n_1 n_2} \right|,$$

where $\sum_{1 \le n \le N}{}'$ denotes the summation over n from 1 to N such that $(n, 2q) = 1$.

Noting that the solution of the congruence $an_2 \equiv n_1 \pmod d$ is $an_2 = n_1$ only in the case $1 \le n_1, an_2 \le d-1$, so we have

$$\frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{1 \le n_1 \le N \\ an_2 \equiv -n_1 \pmod d}}{}' \sum_{1 \le n_2 \le N}{}' \frac{\chi_4(n_1 n_2) r(n_1) r(n_2)}{n_1 n_2}$$

$$= \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{1 \le n_1 \le N \\ n_1 = an_2}}{}' \sum_{1 \le n_2 \le N}{}' \frac{\chi_4(n_1 n_2) r(n_1) r(n_2)}{n_1 n_2}$$

$$+ \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{1 \le n_1 \le N \\ n_1 - an_2 = d}}{}' \sum_{1 \le n_2 \le N}{}' \frac{\chi_4(n_1 n_2) r(n_1) r(n_2)}{n_1 n_2}$$

$$+ \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{1 \le n_1 \le N \\ n_1 - an_2 = ld,\ l \ge 2}}{}' \sum_{1 \le n_2 \le N}{}' \frac{\chi_4(n_1 n_2) r(n_1) r(n_2)}{n_1 n_2}$$

$$\ll \sum_{d|q} \phi(d) \sum_{1 \le n_2 \le \frac{d-1}{a}}{}' \frac{r(n_2) r(an_2)}{an_2^2} + \sum_{d|q} \phi(d) \sum_{1 \le n_2 \le N}{}' \frac{r(d + an_2) r(n_2)}{(d + an_2) n_2}$$

$$+ \sum_{d|q} \phi(d) \sum_{1 \le n_2 \le N}{}' \sum_{l = \left[\frac{-an_2}{d}\right] + 2}^{\left[\frac{N - an_2}{d}\right]} \frac{r(ld + an_2) r(n_2)}{(ld + an_2) n_2}$$

$$\ll \frac{r(a)}{a} \sum_{d|q} \phi(d) \sum_{1 \le n_2 \le \frac{d-1}{a}}{}' \frac{r^2(n_2)}{n_2^2} + \sum_{d|q} \frac{\phi(d)}{d} \sum_{1 \le n_2 \le N}{}' \frac{((d + an_2) n_2)^{\epsilon}}{n_2}$$

$$+ \sum_{d|q} \frac{\phi(d)}{d} \sum_{1 \le n_2 \le N}{}' \sum_{l = \left[\frac{-an_2}{d}\right] + 2}^{\left[\frac{N - an_2}{d}\right]} \frac{((ld + an_2) n_2)^{\epsilon}}{ln_2 + n_2^2/d}$$

$$\ll a^{\epsilon - 1} q^{1+\epsilon} + a^{\epsilon} q^{\epsilon},$$

where we have used the estimate $r(n) \ll n^{\epsilon}$.

Similarly, we can also get the estimate

$$\begin{aligned}
&\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sum_{\substack{1\le n_1\le N\\ an_2\equiv -n_1 \pmod d}}{}' \sum_{1\le n_2\le N}{}' \frac{\chi_4(n_1n_2)r(n_1)r(n_2)}{n_1n_2}\\
&= \frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sum_{\substack{1\le n_1\le N\\ an_2+n_1=d}}{}' \sum_{1\le n_2\le N}{}' \frac{\chi_4(n_1n_2)r(n_1)r(n_2)}{n_1n_2}\\
&\quad + \frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sum_{\substack{1\le n_1\le N\\ an_2+n_1=ld,\ l\ge 2}}{}' \sum_{1\le n_2\le N}{}' \frac{\chi_4(n_1n_2)r(n_1)r(n_2)}{n_1n_2}\\
&\ll \sum_{d|q}\phi(d)\sum_{1\le n_2\le N}{}' \frac{r(d-an_2)r(n_2)}{(d-an_2)n_2}\\
&\quad + \sum_{d|q}\phi(d)\sum_{1\le n_2\le N}{}' \sum_{l=\left[\frac{an_2}{d}\right]+2}^{\left[\frac{N+an_2}{d}\right]} \frac{r(ld-an_2)r(n_2)}{(ld-an_2)n_2}\\
&\ll \sum_{d|q}\frac{\phi(d)}{d}\sum_{1\le n_2\le N}{}' \frac{((d-an_2)n_2)^{\epsilon}}{n_2}\\
&\quad + \sum_{d|q}\frac{\phi(d)}{d}\sum_{1\le n_2\le N}{}' \sum_{l=\left[\frac{an_2}{d}\right]+2}^{\left[\frac{N+an_2}{d}\right]} \frac{((ld-an_2)n_2)^{\epsilon}}{ln_2-n_2^2/d}\\
&\ll a^{\epsilon}q^{\epsilon}.
\end{aligned}$$

So we have

$$\sum_{a=1}^{q}{}' \left|\sum_{\chi(-1)=1}^{*}\chi(a)\left|\sum_{n\le N}\frac{\overline{\chi}\chi_4(n)r(n)}{n}\right|^2\right| \ll q^{1+\epsilon}.$$

Applying Cauchy's inequality, we may have

$$\begin{aligned}
&\sum_{a=1}^{q}{}' \left|\sum_{\chi(-1)=1}^{*}\chi(a)\left(\left|\sum_{n\le N}\frac{\overline{\chi}\chi_4(n)}{n}\right| - \left|\sum_{n\le N}\frac{\overline{\chi}\chi_4(n)r(n)}{n}\right|^2\right)\right|\\
&\le q^{\frac{1}{2}}\left(\sum_{a=1}^{p-1}\left|\sum_{\chi(-1)=1}^{*}\chi(a)\left(\left|\sum_{n\le N}\frac{\overline{\chi}\chi_4(n)}{n}\right| - \left|\sum_{n\le N}\frac{\overline{\chi}\chi_4(n)r(n)}{n}\right|^2\right)\right|^2\right)^{\frac{1}{2}}\\
&\le q\left(\sum_{\chi(-1)=1}^{*}\left(\left|\sum_{n\le N}\frac{\overline{\chi}\chi_4(n)}{n}\right| - \left|\sum_{n\le N}\frac{\overline{\chi}\chi_4(n)}{n} + \sum_{N<n\le N^2}\frac{\overline{\chi}\chi_4(n)r(n,N)}{n}\right|\right)^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq q\left(\sideset{}{^*}\sum_{\chi(-1)=1}\left|\sum_{N<n\leq N^2}\frac{\overline{\chi}\chi_4(n)r(n,N)}{n}\right|^2\right)^{\frac{1}{2}}$$
$$\leq q^{\frac{3}{2}}\left(\sum_{\substack{N<m\leq N^2\\ m\equiv n \pmod q}}\sum_{N<n\leq N^2}\frac{r(m,N)\cdot r(n,N)}{mn}\right)^{\frac{1}{2}}$$
$$\ll q^{1+\epsilon}.$$

Combining the above we have

$$\sideset{}{'}\sum_{a=1}^{q}\left|\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\chi(a)\left|L\left(1,\overline{\chi}\chi_4\right)\right|\right|$$
$$=\sideset{}{'}\sum_{a=1}^{q}\left|\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\chi(a)\left|\sum_{n\leq N}\frac{\overline{\chi}\chi_4(n)}{n}\right|+O\left(\ln q\right)\right|$$
$$=\sideset{}{'}\sum_{a=1}^{q}\left|\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\chi(a)\left|\sum_{n\leq N}\frac{\overline{\chi}\chi_4(n)}{n}\right|\right|+O\left(q\ln q\right)$$
$$\ll\sideset{}{'}\sum_{a=1}^{q}\left|\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\chi(a)\left(\left|\sum_{n\leq N}\frac{\overline{\chi}\chi_4(n)}{n}\right|-\left|\sum_{n\leq N}\frac{\overline{\chi}\chi_4(n)r(n)}{n}\right|^2\right)\right|$$
$$+\sideset{}{'}\sum_{a=1}^{q}\left|\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\chi(a)\left|\sum_{n\leq N}\frac{\overline{\chi}\chi_4(n)r(n)}{n}\right|^2\right|+q\ln q$$
$$\ll q^{1+\epsilon}.$$

This completes the proof of Lemma 2.4. □

3. Proof of Theorem

In this section we only prove the second and third parts of the theorem, similarly we can prove the first part. Note that $G(n,\chi;p)=0$ for odd character

χ modulo p, then for any integer n with $(n,p)=1$, from Lemma 2.1 we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}} |G(n,\chi;p)|^4 \left|\sum_{x<\frac{p}{4}}\chi(x)\right| \\
&= \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} |G(n,\chi;p)|^4 \left|\sum_{x<\frac{p}{4}}\chi(x)\right| \\
&= \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \left(2p+\chi_1(n)G(1;p)\sum_{a=1}^{p-1}\chi(a)\chi_1(a^2-1)\right)^2 \left|\sum_{x<\frac{p}{4}}\chi(x)\right| \\
&= 4p^2 \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \left|\sum_{x<\frac{p}{4}}\chi(x)\right| \\
&\quad + 4pG(1;p)\chi_1(n)\sum_{a=1}^{p-1}\chi_1(a^2-1)\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}}\chi(a)\left|\sum_{x<\frac{p}{4}}\chi(x)\right| \\
&\quad + G^2(1;p)\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(ab)\chi_1(a^2-1)\chi_1(b^2-1)\left|\sum_{x<\frac{p}{4}}\chi(x)\right|.
\end{aligned}$$

Note the identity

$$\begin{aligned}
&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(ab)\chi_1(a^2-1)\chi_1(b^2-1) \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\chi_1(a^2\bar{b}^2-1)\chi_1(b^2-1) \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\chi_1(a^2-b^2)\chi_1(b^2-1) \\
&= 2\chi_1(-1)(p-3)+\sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\chi(a)\chi_1(a^2-b^2)\chi_1(b^2-1)
\end{aligned}$$

and the estimate of Weil

$$\sum_{b=1}^{p-1} \chi_1(b^2-a^2)\chi_1(b^2-1) \le 3\sqrt{p}, \qquad a^2 \not\equiv 1 \pmod{p}.$$

Then from Lemmas 2.2, 2.3 and 2.4 we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}} |G(n,\chi;p)|^4 \left|\sum_{x<\frac{p}{4}} \chi(x)\right| \\
=\ & \left(4p^2+2G^2(1;p)\chi_1(-1)(p-3)\right) \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \left|\sum_{x<\frac{p}{4}} \chi(x)\right| \\
&+ \frac{4p^{\frac{3}{2}}}{\pi} G(1;p)\chi_1(n) \sum_{a=1}^{p-1} \chi_1(a^2-1) \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \chi(a)\left|L(1,\overline{\chi}\chi_4)\right| \\
&+ \frac{p^{\frac{1}{2}}}{\pi} G^2(1;p) \sum_{a=2}^{p-2}\sum_{b=1}^{p-1} \chi_1(a^2-b^2)\chi_1(b^2-1) \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \chi(a)\left|L(1,\overline{\chi}\chi_4)\right| \\
=\ & (6p^2-6p) \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \left|\sum_{x<\frac{p}{4}} \chi(x)\right| + O\left(p^2 \sum_{a=2}^{p-2} \left| \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \chi(a)\left|L(1,\overline{\chi}\chi_4)\right| \right|\right) \\
=\ & \frac{3\cdot p^{\frac{7}{2}}}{\pi} \sum_{\substack{n=1\\ (n,2p)=1}}^{\infty} \frac{r^2(n)}{n^2} + O\left(p^{3+\epsilon}\right).
\end{aligned}$$

This proves the second part of the theorem.

For any odd prime $p \ge 5$ with $p \equiv 3 \pmod 4$, we have $\chi_1(-1) = -1$, thus

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}} \left|\overline{G(n,\chi;p)}\right|^6 \left|\overline{\sum_{x<\frac{p}{4}} \chi(x)}\right| \\
=\ & \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \left(2p+\chi_1(-n)G(1;p)\sum_{a=1}^{p-1}\chi(a)\chi_1(a^2-1)\right)^3 \left|\sum_{x<\frac{p}{4}} \chi(x)\right|.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}} |G(n,\chi;p)|^6 \left|\sum_{x<\frac{p}{4}} \chi(x)\right| \\
&= \frac{1}{2}\left(\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}} |G(n,\chi;p)|^6 \left|\sum_{x<\frac{p}{4}} \chi(x)\right| + \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}} \left|\overline{G(n,\chi;p)}\right|^6 \left|\overline{\sum_{x<\frac{p}{4}} \chi(x)}\right|\right) \\
&= \frac{1}{2}\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \left(2p+\chi_1(n)G(1;p)\sum_{a=1}^{p-1}\chi(a)\chi_1(a^2-1)\right)^3 \left|\sum_{x<\frac{p}{4}} \chi(x)\right| \\
&\quad + \frac{1}{2}\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \left(2p-\chi_1(n)G(1;p)\sum_{a=1}^{p-1}\chi(a)\chi_1(a^2-1)\right)^3 \left|\sum_{x<\frac{p}{4}} \chi(x)\right| \\
&= \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \left(8p^3+6pG^2(1;p)\left(\sum_{a=1}^{p-1}\chi(a)\chi_1(a^2-1)\right)^2\right) \left|\sum_{x<\frac{p}{4}} \chi(x)\right| \\
&= (20p^3-36p^2)\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \left|\sum_{x<\frac{p}{4}} \chi(x)\right| \\
&\quad + \frac{6p^{\frac{3}{2}}}{\pi}G^2(1;p)\sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\chi_1(a^2-b^2)\chi_1(b^2-1)\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \chi(a)\left|L(1,\overline{\chi}\chi_4)\right| \\
&= (20p^3-36p^2)\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \left|\sum_{x<\frac{p}{4}} \chi(x)\right| + O\left(p^3\sum_{a=2}^{p-2}\left|\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi(-1)=1}} \chi(a)\left|L(1,\overline{\chi}\chi_4)\right|\right|\right) \\
&= \frac{10\cdot p^{\frac{9}{2}}}{\pi}\sum_{\substack{n=1\\ (n,2p)=1}}^{\infty}\frac{r^2(n)}{n^2} + O\left(p^{4+\epsilon}\right).
\end{aligned}$$

This completes the proof of the third part of the theorem.

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