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THREE DIMENSIONAL CRITICAL POINT OF THE TOTAL SCALAR CURVATURE

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Abstract. It has been conjectured that, on a compact 3-dimensional orientable manifold, a critical point of the total scalar curvature restricted to the space of constant scalar curvature metrics of unit volume is Einstein. In this paper we prove this conjecture under a condition that $\ker s_g'^* \neq 0$, which generalizes the previous partial results.

1. Introduction

This paper is a continuation of the study of rigidity discussed in [4] and [6]. The result obtained here plays a significant role in proving the conjecture proposed by Besse ([1]).

Let M be a compact orientable 3-dimensional manifold. It is well known that for critical points of the total scalar curvature

$$g \longmapsto \int_M s_g \, dv_g$$

restricted to metrics of unit volume, are Einstein metrics. Here, s_g is the scalar curvature and dv_g is the volume form determined by the metric and orientation. Let $\mathcal{C}$ be the space of metrics with constant scalar curvature and unit volume. A metric g is a critical point of the total scalar curvature restricted to $\mathcal{C}$, if and only if there exists a function f such that

$$z_g = s_g'^*(f), \tag{1}$$

where z_g is the traceless Ricci tensor of g and the operator $s_g'^*$ is defined by

$$s_g'^*(f) = D_g df - g\Delta_g f - f r_g. \tag{2}$$

In 1987 Besse ([1]) proposed the following conjecture:

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Let (M, g) be a compact orientable 3-dimensional manifold with constant scalar curvature such that the equation (1) has a solution. Then g is Einstein.

Note that the geometric structure of an Einstein solution to (1) is known to be simple. We know this as Obata showed that such a solution is isometric to a standard 3-sphere ([7]).

The main purpose of this paper is to prove the following theorem, implying that Conjecture A holds on a compact orientable 3-dimensional manifold under a condition on $\ker s_g'^*$.

Theorem 1.1. *Let (g, f) be a non-trivial solution of (1) on a compact orientable 3-dimensional manifold M. If $\ker s_g'^* \neq 0$, then (M, g) is isometric to a standard sphere S^3.*

Theorem 1.1 is an improvment of the following two results.

Theorem 1.2 ([2])**.** *Let (g, f) be a non-trivial solution of (1) on a compact orientable 3-dimensional manifold M. If $\dim \ker s_g'^* \geq 2$, then (M, g) is isometric to a standard sphere S^3.*

Theorem 1.3 ([6])**.** *Let (g, f) be a non-trivial solution of (1) on a compact orientable 3-dimensional manifold M. If $\ker s_g'^* \neq 0$ and $H_2(M, \mathbb{Z}) = 0$, then (M, g) is isometric to a standard sphere S^3.*

2. Proof of Theorem 1.1

The differential operator d^D of $C^\infty(S^2M)$ into $\Lambda^2 M \otimes T^*M$ is defined as

$$d^D\eta(X, Y, Z) = (D_X\eta)(Y, Z) - (D_Y\eta)(X, Z)$$

for $\eta \in C^\infty(S^2M)$. Let φ be a non-trivial function in $\ker s_g'^*$. Then φ satisfies the following equation

$$\varphi\, z = D_g d\varphi + \frac{s}{6}\varphi\, g. \tag{3}$$

We begin by analyzing the structure of z.

Lemma 2.1. *If $\varphi \in \ker s_g'^*$, we have*

$$\varphi\, d^D z = -2\, d\varphi \wedge z - i_{\nabla\varphi} z \wedge g. \tag{4}$$

Here, i_X is the interior product.

Proof. By the Ricci identity, we have

$$d^D D d\varphi(X, Y, Z) = R(X, Y, Z, \nabla\varphi)$$

for any vector fields X, Y, Z on M. Thus, applying d^D to both sides of (3) gives

$$(d\varphi \wedge z + \varphi\, d^D z)(X, Y, Z) = R(X, Y, Z, \nabla\varphi) + \frac{s}{6}\,(d\varphi \wedge g)(X, Y, Z).$$

Here, $d\varphi \wedge \eta$ is defined as

$$(d\varphi \wedge \eta)(X, Y, Z) = d\varphi(X)\eta(Y, Z) - d\varphi(Y)\eta(X, Z)$$

for $\eta \in C^\infty(S^2M)$. From

$$R(X,Y,Z,W) = (g(X,Z)r(Y,W) + g(Y,W)r(X,Z) - g(Y,Z)r(X,W) - g(X,W)r(Y,Z)) - \frac{s}{2}(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)),$$

we obtain

$$R(X,Y,Z,\nabla\varphi) = -\left(i_{\nabla\varphi} r \wedge g + d\varphi \wedge r - \frac{s}{2}\, d\varphi \wedge g\right)(X,Y,Z). \tag{5}$$

By combining these results, we obtain (4). □

As a consequence of Lemma 2.1, we derive the following key lemma.

Lemma 2.2. *The function $z(d\varphi, d\varphi)$ is constant on each connected component of Γ.*

Proof. Let $\Gamma = \varphi^{-1}(0)$. It is proved in [3] that there are no critical points of φ in Γ. Taking the divergence of the equation (4) gives

$$\begin{aligned} -d^D z(d\varphi, \cdot, \cdot) + \varphi\, \delta d^D z &= -2\, \delta(df \wedge z) - \delta(i_{\nabla f} z \wedge g) \\ &= -s\varphi z + 2D_{d\varphi} z - 3Dd\varphi \circ z + \langle Dd\varphi, z\rangle g - Dz(d\varphi, \cdot), \end{aligned}$$

since

$$\delta(d\varphi \wedge z) = \frac{s}{2}\varphi\, z - D_{df} z + Dd\varphi \circ z$$

and

$$\delta(i_{\nabla\varphi} z \wedge g)(X,Y) = -\langle Dd\varphi, z\rangle g(X,Y) + (Dd\varphi \circ z)(X,Y) + D_Y z(d\varphi, X).$$

Thus on Γ

$$2D_{d\varphi} z(X,Y) = -d^D z(d\varphi, X, Y) + D_Y z(d\varphi, X). \tag{6}$$

Here, we use the fact that $Dd\varphi = \varphi\, z - \frac{s}{6}\varphi\, g = 0$ on Γ. Therefore, for $X = \nabla\varphi$ and $Y = \nabla f$, we have on Γ

$$2D_{d\varphi} z(d\varphi, df) = D_{df} z(d\varphi, d\varphi). \tag{7}$$

Similarly, with $X = Y = \nabla\varphi$ we have

$$D_{d\varphi} z(d\varphi, d\varphi) = 0. \tag{8}$$

Finally, with $X = Y = \nabla f$, we have

$$3D_{d\varphi} z(df, df) = 2D_{df} z(d\varphi, df). \tag{9}$$

Note that on Γ we have $D_X z(d\varphi, X) = 0$ for a tangent vector X to Γ, since $z(d\varphi, X) = 0$ and $Dd\varphi = 0$ on Γ. Therefore, if X is a tangential part of df, or $X = df - c\, d\varphi$ for the constant $c = \langle df, d\varphi\rangle / |d\varphi|^2$ (see [6]), by (7) and (8)

$$0 = D_X z(d\varphi, X) = D_{df} z(d\varphi, df) - \frac{3}{2} c\, D_{df} z(d\varphi, d\varphi). \tag{10}$$

By the fact that $h^3 z(d\varphi, d\varphi)$ is constant on Γ for a nontrivial solution f of (1) with $h = 1 + f$, as seen in the proof of Theorem 1.1, on Γ we have

$$D_X z(d\varphi, d\varphi) = X(z(d\varphi, d\varphi)) = -\frac{3}{h} df(X)\, z(d\varphi, d\varphi). \tag{11}$$

Thus, if X is orthogonal to df, it is clear that $z(d\varphi, d\varphi)$ is constant. When $X = df - c\, d\varphi$, by (8) we have

$$D_{df} z(d\varphi, d\varphi) = -\frac{3}{h}|X|^2 z(d\varphi, d\varphi), \tag{12}$$

where $|X|^2 = |df|^2 - c^2|d\varphi|^2$.

Now, for a nontrivial solution f to (1), the following formula holds;

$$h\, d^D z = -2df \wedge z - i_{\nabla f} z \wedge g,$$

where $h = 1 + f$, and so

$$h\, d^D z(d\varphi, df, df) = c|df|^2 z(d\varphi, d\varphi) - c|d\varphi|^2 z(df, df).$$

Combining these results with (11) gives

$$|X|^2 z(d\varphi, d\varphi) = hc\left(|df|^2 z(d\varphi, d\varphi) - |d\varphi|^2 z(df, df)\right).$$

Note that

$$|d\varphi|^2 z(df, df) = z(d\varphi, d\varphi)\left(c^2|d\varphi|^2 - \frac{1}{2}|X|^2\right),$$

we have either $\langle df, d\varphi\rangle = 0$ or $z(d\varphi, d\varphi) = 0$. Note that $\langle df, d\varphi\rangle \neq 0$ in general; in fact, if f satisfies $\langle df, d\varphi\rangle = 0$, we may consider the new function $\tilde{f} = f + \varphi$, which is also a solution of (1) with $\langle d\tilde{f}, d\varphi\rangle = |d\varphi|^2 \neq 0$. As a consequence, $z(d\varphi, d\varphi) = 0$ and so by (11) with a tangent vector X to Γ, $z(d\varphi, d\varphi)$ is constant on Γ. □

Now, we are ready to prove our main theorem.

Proof of Theorem 1.1. Let $\nu = d\varphi/|d\varphi|$ and Σ be a connected component of Γ. Since there is no critical point of φ on Γ, $|d\varphi| \neq 0$ on Σ. In fact, $|d\varphi|$ is constant on Σ, since for any tangent vector ξ to Σ,

$$\xi(|d\varphi|^2) = 2\langle D_\xi d\varphi, d\varphi\rangle = 0.$$

Thus $z(\nu, \nu)$ is constant on Σ by Lemma 2.2. It is also proved in [6] that $(1 + f)^3 z(\nu, \nu)$ is constant on Σ. Therefore, f should be constant on Σ unless $z(\nu, \nu) \equiv 0$ on Σ.

Note that if $z(\nu, \nu) \equiv 0$ on all connected components of Γ, the metric g should be Einstien since

$$0 = \int_\Gamma |d\varphi|\, z(\nu, \nu) = \int_{\{x\in M \mid \varphi(x)<0\}} \varphi|z|^2 = -\int_{\{x\in M \mid \varphi(x)>0\}} \varphi|z|^2,$$

implying that $|z| \equiv 0$ on all of M. This proves our theorem.

Now, suppose $z(\nu, \nu) \neq 0$ on some Σ. Then we may conclude that df is parallel to ν since f is constant on Σ.

Consider the following sets

$$\Sigma^A = \{x \in \Sigma \mid df \text{ is parallel to } \nu \text{ with } df \neq 0 \text{ at } x\},$$
$$\Sigma^B = \{x \in \Sigma \mid df \text{ is not parallel to } \nu \text{ with } df \neq 0 \text{ at } x\}, \text{ and}$$
$$\Sigma^C = \{x \in \Sigma \mid df(x) = 0\}.$$

Note $\langle df, \nu\rangle$ is constant by Lemma 5 in [4]. Thus, from the fact that $\langle df, \nu\rangle \neq 0$ on Σ^A and $\langle df, \nu\rangle = 0$ on Σ^C, and Σ is connected, either $\Sigma = \Sigma^A \cup \Sigma^B$ or $\Sigma = \Sigma^B \cup \Sigma^C$ holds. However, our previous conclusion that df is parallel to ν on Σ excludes the existence of Σ^B. Therefore, either $\Sigma = \Sigma^A$ or $\Sigma = \Sigma^C$. We shall derive a contradiction by showing that Σ cannot be either Σ^A or Σ^C, which proves our Theorem.

Note that it is proved in Lemma 2.3 of [5] that there is no open set in Σ^A. In other words, Σ^A has measure zero in Σ, and so Σ^A cannot be equal to all of Σ.

To prove that $\Sigma \neq \Sigma^C$, we consider a new function $\tilde{f} = f + \varphi$. Since $s_g'^*(\tilde{f}) = s_g'^*(f) = z_g$, $\tilde{f}$ is also a solution of (1). Then $d\tilde{f}$ is parallel to ν with $d\tilde{f} \neq 0$ at Σ since $df = 0$ on Σ. In other words, $\Sigma^C = \{x \in \Sigma \mid d\tilde{f}$ is parallel to ν with $d\tilde{f} \neq 0$ at $x\}$. Then, again by Lemma 2.3 of [5], Σ^C has measure zero, implying that Σ^C cannot be equal to all of Σ. This completes the proof of our Theorem. □

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